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CLASSICS ILLUSTRATED

GROUP THEORY

Exceptional Lie groups as invariance groups

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Abstract

We offer the ultimate birdtracker guide to exceptional Lie groups.

Keywords: exceptional Lie groups, invariant theory, Tits magic square

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Contents

1 Introduction **1**

2 A preview **5**
- 2.1 Basic concepts 5
- 2.2 First example: $SU(n)$ 9
- 2.3 Second example: E_6 family 12

3 Invariants and reducibility **15**
- 3.1 Preliminaries 15
  - 3.1.1 Groups 15
  - 3.1.2 Vector spaces 16
  - 3.1.3 Algebra 17
  - 3.1.4 Defining space, tensors, representations 18
- 3.2 Invariants 20
  - 3.2.1 Algebra of invariants 22
- 3.3 Invariance groups 23
- 3.4 Projection operators 24
- 3.5 Further invariants 25
- 3.6 Birdtracks 27
- 3.7 Clebsch-Gordan coefficients 29
- 3.8 Zero- and one-dimensional subspaces 31
- 3.9 Infinitesimal transformations 32
- 3.10 Lie algebra 36
- 3.11 Other forms of Lie algebra commutators 38
- 3.12 Irrelevancy of clebsches 38

4 Recouplings **41**
- 4.1 Couplings and recouplings 41
- 4.2 Wigner $3n-j$ coefficients 44
- 4.3 Wigner-Eckart theorem 45

5 Permutations **49**
- 5.1 Permutations in birdtracks 49
- 5.2 Symmetrization 50

5.3 Antisymmetrization . . . 52
5.4 Levi-Civita tensor . . . 53
5.5 Determinants . . . 55
5.6 Characteristic equations . . . 57
5.7 Fully (anti)symmetric tensors . . . 57
5.8 Young tableaux, Dynkin labels . . . 58

6 Casimir operators **59**
6.1 Casimirs and Lie algebra . . . 60
6.2 Independent casimirs . . . 60
6.3 Casimir operators . . . 60
6.4 Dynkin indices . . . 60
6.5 Quadratic, cubic casimirs . . . 60
6.6 Quartic casimirs . . . 60
6.7 Sundry relations between quartic casimirs . . . 60
6.8 Identically vanishing tensors . . . 60
6.9 Dynkin labels . . . 61

7 Group integrals **63**
7.1 Group integrals for arbitrary representations . . . 64
7.2 Characters . . . 64
7.3 Examples of group integrals . . . 64

8 Unitary groups **65**
8.1 Two-index tensors . . . 65
8.2 Three-index tensors . . . 66
8.3 Young tableaux . . . 68
  8.3.1 Definitions . . . 68
  8.3.2 $SU(n)$ Young tableaux . . . 69
  8.3.3 Reduction of direct products . . . 70
8.4 Young projection operators . . . 71
  8.4.1 A dimension formula . . . 72
  8.4.2 Dimension as the number of strand colorings . . . 73
8.5 Reduction of tensor products . . . 74
  8.5.1 Three- and four-index tensors . . . 74
  8.5.2 Basis vectors . . . 75
8.6 3-j symbols . . . 76
  8.6.1 Evaluation by direct expansion . . . 77
  8.6.2 Application of the negative dimension theorem . . . 77
  8.6.3 A sum rule for 3-j's . . . 78
8.7 Characters . . . 79
8.8 Mixed two-index tensors . . . 79
8.9 Mixed defining × adjoint tensors . . . 81
8.10 Two-index adjoint tensors . . . 83

8.11 Casimirs for the fully symmetric representations of $SU(n)$ 84
8.12 $SU(n)$, $U(n)$ equivalence in adjoint representation 84
8.13 Dynkin labels for $SU(n)$ representations 84

9 Orthogonal groups **85**
9.1 Two-index tensors 86
9.2 Three-index tensors 86
9.3 Mixed defining × adjoint tensors 86
9.4 Two-index adjoint tensors 86
9.5 Gravity tensors 86
9.6 Dynkin labels of $SO(n)$ representations 86

10 Spinors **89**
10.1 Spinograpy 90
10.2 Fierzing around 90
10.3 Fierz coefficients 90
10.4 $6j$ coefficients 90
10.5 Exemplary evaluations 90
10.6 Invariance of γ-matrices 90
10.7 Handedness 90
10.8 Kahane algorithm 90

11 Symplectic groups **91**
11.1 Two-index tensors 92
11.2 Mixed defining × adjoint tensors 93
11.3 Dynkin labels of $Sp(n)$ representations 93

12 Negative dimensions **95**
12.1 $SU(n) = SU(-n)$ 97
12.2 $SO(n) = Sp(-n)$ 98

13 Spinsters **101**

14 $SU(n)$ family of invariance groups **103**
14.1 Representations of $SU(2)$ 103
14.2 $SU(3)$ as invariance group of a cubic invariant 105
14.3 Levi-Civita tensors and $SU(n)$ 105
14.4 $SU(4)$ - $SO(6)$ isomorphism 105

15 G_2 family of invariance groups **107**
15.1 Jacobi relation 109
15.2 Alternativity and reduction of f-contractions 110
15.3 Primitivity implies alternativity 112
15.4 Casimirs for G_2 115
15.5 Hurwitz's theorem 116

15.6 Representations of G_2 118

16 E_8 family of invariance groups **119**
16.1 Two-index tensors 120
16.2 Decomposition of $\mathrm{Sym}^3 A$ 123
16.3 Decomposition of $|??| \otimes |??\|??|^*$ 125
16.4 Diophantine conditions 127
16.5 Generalized Young tableaux for E_8 127
16.6 Conjectures of Deligne 128

17 E_6 family of invariance groups **129**
17.1 Reduction of two-index tensors 129
17.2 Mixed two-index tensors 130
17.3 Diophantine conditions and the E_6 family 130
17.4 Three-index tensors 130
  17.4.1 Fully symmetric $\otimes V^3$ tensors 130
  17.4.2 Mixed symmetry $\otimes V^3$ tensors 130
  17.4.3 Fully antisymmetric $\otimes V^3$ tensors 130
17.5 Defining $\otimes$ adjoint tensors 130
17.6 Two-index adjoint tensors 130
  17.6.1 Reduction of antisymmetric 3-index tensors 131
17.7 Dynkin labels and Young tableaux for E_6 131
17.8 Casimirs for E_6 131
17.9 Subgroups of E_6 131
17.10Springer relation 131
  17.10.1Springer's construction of E_6 131

18 F_4 family of invariance groups **133**
18.1 Two-index tensors 133
18.2 Defining $\otimes$ adjoint tensors 136
  18.2.1 Two-index adjoint tensors 136
18.3 Jordan algebra and $F_4(26)$ 136

19 E_7 family of invariance groups **137**

20 Exceptional magic **139**
20.1 Magic triangle 139

21 Magic negative dimensions **143**
21.1 E_7 and $SO(4)$ 143
21.2 E_6 and $SU(3)$ 143

A Recursive decomposition **145**

B Properties of Young Projections **147**
B.1 Uniqueness of Young projection operators 147
B.2 Normalization 148
B.3 Orthogonality 149
B.4 The dimension formula 150
B.5 Literature 151

Bibliography **153**

Index **159**

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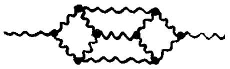
Chapter 1

Introduction

One simple field-theory question started this project; what is the group theoretic factor for the following QCD gluon self-energy diagram



$$= ? \tag{1.1}$$

I first computed the answer for $SU(n)$. There was a hard way of doing it, using Gell-Mann f_{ijk} and d_{ijk} coefficients. There was also an easy way, where one could doodle oneself to the answer in a few lines. This is the "birdtracks" method which will be described here. It works nicely for $SO(n)$ and $Sp(n)$ as well. Out of curiosity, I wanted the answer for the remaining five exceptional groups. This engendered further thought, and that which I learned can be better understood as the answer to a different question. Suppose someone came into your office and asked, "On planet Z, mesons consist of quarks and antiquarks, but baryons contain three quarks in a symmetric color combination. What is the color group?" The answer is neither trivial, nor without some beauty (planet Z quarks can come in 27 colors, and the color group can be E_6).

Once you know how to answer such group-theoretical questions, you can answer many others. This monograph tells you how. Like the brain, it is divided into two halves; the plodding half and the interesting half.

The plodding half describes how group theoretic calculations are carried out for unitary, orthogonal and symplectic groups. Probably none of that is new, but the methods are helpful in carrying out theorists' daily chores, such as evaluating Quantum Chromodynamics group theoretic weights, evaluating lattice gauge theory group integrals, computing $1/N$ corrections, evaluating spinor traces, evaluating casimirs, implementing evaluation algorithms on computers, and so on.

The interesting half describes the "exceptional magic" (a new construction of exceptional Lie algebras) and the "negative dimensions" (relations between bosonic and fermionic dimensions). The methods used are applicable to grand unified theories and supersymmetric theories. Regardless of their immediate utility, the results are sufficiently intriguing to have motivated this entire undertaking.

There are two complementary approaches to group theory. In the *canonical* approach one chooses the basis, or the Clebsch-Gordan coefficients, as simply as possible. This is the method which Killing [87] and Cartan [88] used to obtain the complete classification of semi-simple Lie algebras, and which has been brought to perfection by Dynkin [90]. There exist many excellent reviews of applications of Dynkin diagram methods to physics, such as the review by Slansky [71].

In the *tensorial* approach, the bases are arbitrary, and every statement is invariant under change of basis. Tensor calculus deals directly with the invariant blocks of the theory and gives the explicit forms of the invariants, Clebsch-Gordan series, evaluation algorithms for group theoretic weights, *etc.*

The canonical approach is often impractical for physicists' purposes, as a choice of basis requires a specific coordinatization of the representation space. Usually, nothing that we want to compute depends on such a coordinatization; physical predictions are pure scalar numbers ("color singlets"), with all tensorial indices summed. However, the canonical approach can be very useful in determining chains of subgroup embeddings. We refer reader to the Slansky review [71] for such applications; here we shall concentrate on tensorial methods, borrowing from Cartan and Dynkin only the nomenclature for identifying irreducible representations. Extensive listings of these are given by McKay and Patera [91] and Slansky [71].

To appreciate the sense in which canonical methods are impractical, let us consider using them to evaluate the group-theoretic factor (1.1) for the exceptional group E_8. This would involve summations over 8 structure constants. The Cartan-Dynkin construction enables us to construct them explicitly; an E_8 structure constant has about $248^3/6$ elements, and the direct evaluation of (1.1) is tedious even on a computer. An evaluation in terms of a canonical basis would be equally tedious for $SU(16)$; however, the tensorial approach (described in the example at the end of this section) yields the answer for all $SU(n)$ in a few steps.

This is one motivation for formulating a tensorial approach to exceptional groups. The other is the desire to understand their geometrical significance. The Killing-Cartan classification is based on a mapping of Lie algebras onto a Diophantine problem on the Cartan root lattice. This yields an exhaustive classification of simple Lie algebras, but gives no insight into the associated geometries. In the 19th century, the geometries, or the invariant theory was the central question and Cartan, in his 1894 thesis, made an attempt to identify the primitive invariants. Most of the entries in his classification were the classical groups $SU(n)$, $SO(n)$ and $Sp(n)$. Of the five exceptional algebras, Cartan [89] identified G_2 as the group of octonion isomorphisms, and noted already in his thesis that E_7 has a skew-symmetric quadratic and a symmetric quartic invariant. Dickinson [92] characterized E_6 as a 27-dimensional group with a cubic invariant[1]. The fact that the orthogonal, unitary and symplectic groups were invariance groups of real, complex and quaternion norms suggested that the exceptional groups were associated with octonions, but it took more than another fifty years to establish the connection. The remaining four exceptional Lie algebras emerged as rather complicated constructions from octonions and Jordan algebras, known as the Freudenthal-Tits construction. A mathematician's history of this subject is given in a delightful review by Freudenthal [93]. The subject has twice been taken up by physicists, first by Jordan, von Neumann and Wigner [63], and then in the 1970's by Gürsey and collaborators. Jordan *et al.'s* effort was a failed attempt at formulating a new quantum mechanics which would explain the neutron, discovered in 1932. However, it gave rise to the Jordan algebras, which became a mathematics field in itself. Gürsey *et al.* took up the subject again in the hope of formulating a quantum mechanics of quark confinement; the main applications so far, however, have been in building models of grand unification.

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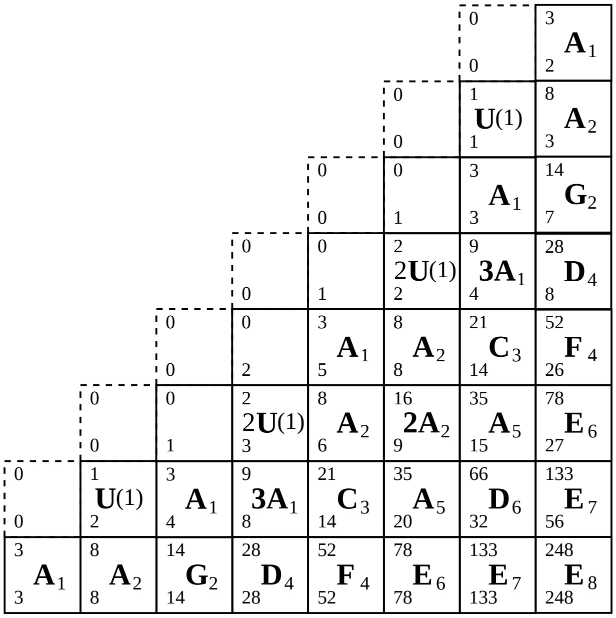
[1] I am indebted to G. Seligman for this reference.

Although beautiful, the Freudenthal-Tits construction is still not practical for the evaluation of group-theoretic weights. The reason is this; the construction involves $[3 \times 3]$ octonian matrices with octonian coefficients, and the 248 dimensional defining space of E_8 is written as a direct sum of various subspaces. This is convenient for studying subgroup embeddings [85], but awkward for group-theoretical computations.

The inspiration for the primitive invariants construction came from the axiomatic approach of Springer [94, 95] and Brown [96]: one treats the defining representation as a single vector space, and characterizes the primitive invariants by algebraic identities. This approach solves the problem of formulating efficient tensorial algorithms for evaluating group-theoretic weights, and also yields some intuition about the geometrical significance of the exceptional Lie groups. Such intuition might be of use to quark-model builders. For example, because $SU(3)$ has a cubic invariant $\epsilon^{abc}q_a q_b q_c$, QCD based on this color group can accommodate 3-quark baryons. Are there any other groups that could accommodate 3-quark singlets? As we shall show, the defining representations of G_2, F_4 and E_6 are some of the groups with such invariants.

Beyond being a mere computational aid, the primitive invariants construction of exceptional groups yields several unexpected results. First, it generates in a somewhat magical fashion a triangular array of Lie algebras, depicted in fig. 1.1. This is a classification of Lie algebras different from Cartan's classification; in particular, all exceptional Lie groups appear in the same series (the bottom line of fig. 1.1). The second unexpected result is that many groups and group representations are mutually related by interchanges of symmetrizations and antisymmetrizations, and replacement of the dimension parameter n by $-n$. I call this phenomenon "negative dimensions".

For me, the greatest surprise of all is that in spite of all the magic and the strange diagrammatic notation, the resulting manuscript is in essence not very different from Wigner's [2] classic group theory book. Regardless of whether one is doing atomic, nuclear or particle physics, all physical predictions ("spectroscopic levels") are expressed in terms of Wigner's $3n - j$ coefficients, which can be



						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $\mathbf{2A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $\mathbf{3A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Figure 1.1: The "magic triangle" for Lie algebras. The Freudenthal "magic square" is marked by the dotted line. The number in the lower left corner of each entry is the dimension of the defining representation. For more details consult chapter 20.

evaluated by means of recursive or combinatorial algorithms.

Chapter 2

A preview

This report on group theory had mutated greatly throughout its genesis. It arose from concrete calculations motivated by physical problems; but as it was written, the generalities were collected into introductory chapters, and the applications receded later and later into the text.

As a result, the first seven chapters are largely a compilation of definitions and general results which might appear unmotivated on the first reading. The reader is advised to work through the examples, sect. 2.2 and sect. 2.3 in this chapter, jump to the topic of possible interest (such as the unitary groups, chapter 8, or the E_8 family, chapter 16), and backtrack when necessary.

The goal of these notes is to provide the reader with a set of basic group-theoretic tools. They are not particularly sophisticated, and they rest on a few simple ideas. The text is long because various notational conventions, examples, special cases and applications have been laid out in detail, but the basic concepts can be stated in a few lines. We shall briefly state them in this chapter, together with several illustrative examples. This preview presumes that the reader has considerable prior exposure to group theory; if a concept is unfamiliar, the reader is referred to the appropriate section for a detailed discussion.

2.1 Basic concepts

An average quantum theory is constructed from a few building blocks which we shall refer to as the *defining representation.* They form the defining multiplet of the theory - for example, the "quark wave functions" q_a. The group-theoretical problem consists of determining the symmetry group, *ie.* the group of all linear transformations

$$q'_a = G_a{}^b q_b \qquad a, b = 1, 2, \ldots, n \,,$$

which leave invariant the predictions of the theory. The $[n \times n]$ matrices G form the *defining representation* of the invariance group $\mathcal{G}$. The conjugate multiplet (“antiquarks”) transforms as

$$q'^a = G^a{}_b q^b\,.$$

Combinations of quarks and antiquarks transform as *tensors*, such as

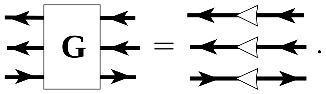
$$\begin{aligned} p'_a q'_b r'^c &= G_{ab}{}^{c},{}_{d}{}^{ef} p_f q_e r^d\,, \\ G_{ab}{}^{c},{}_{d}{}^{ef} &= G_a{}^f G_b{}^e G^c{}_d\,. \end{aligned}$$

(see sect. 3.1.4). Tensor representations are plagued by a proliferation of indices. These indices can either be replaced by a few collective indices

$$\alpha = \left\{ \begin{matrix} & c \\ ab & \end{matrix} \right\}\,, \quad \beta = \left\{ \begin{matrix} & ef \\ d & \end{matrix} \right\}\,,$$

$$q'_\alpha = G_\alpha{}^\beta q_\beta\,, \tag{2.1}$$

or represented diagrammatically



(Diagrammatic notation is explained in sect. 3.6). Collective indices are convenient for stating general theorems; diagrammatic notation speeds up explicit calculations.

A polynomial

$$H(\overline{q},\overline{r},s,\ldots) = h_{ab\ldots}{}^{\ldots c}\, q^a r^b \ldots s_c$$

is an invariant if (and only if) for any transformation $G \in \mathcal{G}$ and for any set of vectors $q, r, s, \ldots$ (see sect. 3.3)

$$H(\overline{Gq},\overline{Gr},Gs,\ldots) = H(\overline{q},\overline{r},s,\ldots)\,. \tag{2.2}$$

An invariance group is defined by its *primitive invariants*, *ie.* by a list of the elementary “singlets” of the theory. For example, the orthogonal group $O(n)$ is defined as the group of all transformations which leave the length of a vector invariant (see chapter 9). Another example is the color $SU(3)$ of QCD which leaves invariant the mesons ($q\overline{q}$) and the baryons (qqq) (see sect. 14.2). A complete list of primitive invariants *defines* the invariance group via the invariance conditions (2.2); only those transformations which respect them are allowed.

It is not necessary to list explicitly the components of primitive invariant tensors in order to define them. For example, the $O(n)$ group is defined by the requirement that it leaves invariant a symmetric and invertible tensor $g_{ab} = g_{ba}$, $\det(g) \neq 0$. Such definition is basis independent, while a component definition $g_{11} = 1, g_{12} = 0, g_{22} = 1, \ldots$ relies on a specific basis choice. We shall define all simple Lie groups in this manner, specifying the primitive invariants only by their symmetry, and by the basis-independent algebraic relations that they must satisfy.

These algebraic relations (which we shall call *primitiveness conditions*) are hard to describe without first giving some examples. In their essence they are statements of irreducibility: for example, if the primitive invariant tensors are δ_b^a, h_{abc} and h^{abc}, then $h_{abc}h^{cbe}$ must be proportional to δ_a^e, as otherwise the defining representation would be reducible. (Reducibility is discussed in sect. 3.4, sect. 3.5 and chapter 4).

The objective of physicist's group-theoretic calculations is a description of the spectroscopy of a given theory. This entails identifying the levels (irreducible multiplets), the degeneracy of a given level (dimension of the multiplet) and the level splittings (eigenvalues of various casimirs). The basic idea that enables us to carry this program through is extremely simple: a hermitian matrix can be diagonalized. This fact has many names: Schur's lemma, Wigner-Eckart theorem, full reducibility of unitary representations, and so on (see sect. 3.4 and sect. 4.3). We exploit it by constructing invariant hermitian matrices M from the primitive invariant tensors. M's have collective indices (2.1) and act on tensors. Being hermitian, they can be diagonalized

$$CMC^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & & \cdots \\ 0 & \lambda_1 & 0 & & \\ 0 & 0 & \lambda_1 & & \\ & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix},$$

and their eigenvalues can be used to construct projection operators which reduce multiparticle states into direct sums of lower-dimensional representations (see sect. 3.4):

$$P_i = \prod_{j\neq i} \frac{M-\lambda_j \mathbf{1}}{\lambda_i-\lambda_j} = C^\dagger \begin{pmatrix} \begin{array}{cc|} \ddots & \vdots \\ \cdots & 0 \\ \hline \end{array} & \cdots & 0 \\ \vdots & \boxed{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{matrix}} & \vdots \\ 0 & \cdots & \begin{array}{|cc} \hline 0 & \cdots \\ \vdots & \ddots \end{array} \end{pmatrix} C\,. \tag{2.3}$$

An explicit expression for the diagonalizing matrix C (Clebsch-Gordan coefficients, sect. 3.7) is unnecessary – it is in fact often more of an impediment than an aid, as it obscures the combinatorial nature of group theoretic computations (see sect. 3.12).

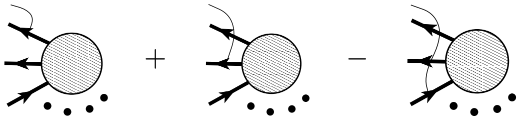
All that is needed in practice is knowledge of the characteristic equation for the invariant matrix M (see sect. 3.4). The characteristic equation is usually a simple consequence of the algebraic relations satisfied by the primitive invariants, and the eigenvalues λ_i are easily determined. λ_i' s determine the projection operators P_i, which in turn contain all relevant spectroscopic information: the representation dimension is given by $\mathrm{tr}\, P_i$, and the casimirs, $6j$'s, crossing matrices and recoupling coefficients (see chapter 4) are traces of various combinations of P_i's. All these numbers are *combinatoric*; they can often be interpreted as the number of different colorings of a graph, the number of singlets, and so on.

The invariance group is determined by considering infinitesimal transformations

$$G_a^b \simeq \delta_b^a + i\epsilon_i (T_i)_a^b \,.$$

The generators T_i are themselves clebsches, elements of the diagonalizing matrix C for the tensor product of the defining representation and its conjugate. They project out the adjoint representation, and are constrained to satisfy the *invariance conditions* (2.2) for infinitesimal transformations (see sect. 3.9 and sect. 3.10):

$$(T_i)_a^{a'} h_{a'b\ldots}{}^{c\ldots} + (T_i)_b^{b'} h_{ab'\ldots}{}^{c\ldots} - (T_i)_{c'}^{c} h_{ab\ldots}{}^{c'\ldots} + \ldots \quad = \quad 0$$



$$+ \ldots \quad = \quad 0 \,. \qquad (2.4)$$

As the corresponding projector operators are already known, we have an explicit construction of the symmetry group (at least infinitesimally – we will not consider discrete transformations).

If the primitive invariants are bilinear, the above procedure leads to the familiar tensor representations of classical groups. However, for trilinear or higher invariants the results are more surprising. In particular, all exceptional Lie groups emerge in a pattern of solutions which we will refer to as a “magic triangle”. The logic of the construction can be schematically indicated by the following chains of subgroups (see chapter 15):

primitive invariants			**invariance group**	
$q\overline{q}$			$SU(n)$	
qq		$SO(n)$		Sp(n)
qqq	$G_2 + \ldots$	$F_4 + \ldots$	$E_6 + \ldots$	
$qqqq$				$E_7 + \ldots$
higher order	$E_8 + \ldots$			

In the above diagram the arrows indicate the primitive invariants which characterize a particular group. For example, E_7 primitives are a sesquilinear invariant $q\bar{q}$, a skew symmetric qp invariant and a symmetric $qqqq$ (see chapter 19).

The strategy is to introduce the invariants one by one, and study the way in which they split up previously irreducible representations. The first invariant might be realizable in many dimensions. When the next invariant is added (sect. 3.5), the group of invariance transformations of the first invariant splits into two subsets; those transformations which preserve the new invariant, and those which do not. Such decompositions yield Diophantine conditions on representation dimensions. These conditions are so constraining that they limit the possibilities to a few which can be easily identified.

To summarize; in the primitive invariants approach, all simple Lie groups, classical as well as exceptional, are constructed by (see chapter 20):

i) defining a symmetry group by specifying a list of *primitive invariants*,

ii) using *primitiveness* and *invariance* conditions to obtain algebraic relations between primitive invariants,

iii) constructing *invariant matrices* acting on tensor product spaces,

iv) constructing *projection operators* for reduced representation from characteristic equations for invariant matrices.

Once the projection operators are known, all interesting spectroscopic numbers can be evaluated.

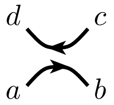
The foregoing run through the basic concepts was inevitably obscure. Perhaps working through the next two examples will make things clearer. The first example illustrates computations with classical groups. The second example is more interesting; it is a sketch of construction of irreducible representations of E_6.

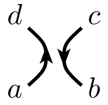
2.2 First example: $SU(n)$

How do we describe the invariance group that preserves the norm of a complex vector? The *list of primitives* consists of a single primitive invariant

$$m(p,q) = \delta^a_b p^b q_a = \sum_{a=1}^{n} (p_a)^* q_a \,.$$

The Kronecker δ^a_b is the only primitive invariant tensor. We can immediately write down the two *invariant matrices* on the tensor product of the defining space and its conjugate:

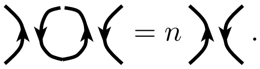
identity : $\mathbf{1}^{a\,c}_{d,b} = \delta^a_b \delta^c_d =$ 

trace : $T^{a\,c}_{d,b} = \delta^a_d \delta^c_b =$  .

The *characteristic equation* for T written out in the matrix, tensor and birdtrack notations is

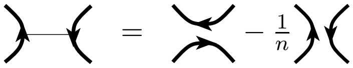
$$T^2 = nT$$

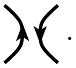
$$T^{a\,f}_{d,e}T^{e\,c}_{f,b} = \delta^a_d\delta^f_e\delta^e_f\delta^c_b = n\,T^{a\,c}_{d,b}$$

$=$ 

Here we have used $\delta^e_e = n$, the dimension of the defining vector space. The roots are $\lambda_1 = 0$, $\lambda_2 = n$, and the corresponding *projection operators* are

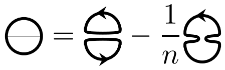
$SU(n)$ adjoint rep: $$P_1 = \frac{T-n\mathbf{1}}{0-n} = \mathbf{1} - \frac{1}{n}T$$



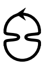
$U(n)$ singlet: $$P_2 = \frac{T-0\cdot\mathbf{1}}{n-1} = \frac{1}{n}T = \frac{1}{n}$$  (2.5)

Now we can evaluate any number associated with the $SU(n)$ adjoint representation, such as its dimension and various casimirs.

The *dimensions* of the two representations are computed by tracing the corresponding projection operators (see sect. 3.4)

$SU(n)$ adjoint: $d_1 = \mathrm{tr}\,P_1 =$  $= \delta^b_b\delta^a_a - \frac{1}{n}\delta^b_a\delta^a_b$

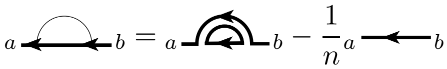
$$= n^2 - 1$$

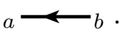
singlet: $d_2 = \mathrm{tr}\,P_2 = \frac{1}{n}$  $= 1$.

To evaluate *casimirs*, we need to fix the overall normalization of the generators of $SU(n)$. Our convention is to take

$$\delta_{ij} = \mathrm{tr}\,T_iT_j = \texttt{birdTrack}\,. \tag{2.6}$$

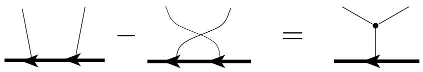
The value of the quadratic casimir for the defining representation is computed by substituting the adjoint projection operator

$SU(n):\quad C_F\delta^b_a = (T_iT_i)^b_a =$ 

$=\dfrac{n^2-1}{n}$  (2.7)

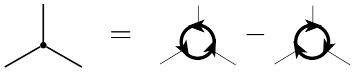
In order to evaluate the quadratic casimir for the adjoint representation, we need to replace the structure constants iC_{ijk} by their *Lie algebra* definition (see sect. 3.10)

$$T_iT_j - T_jT_i = iC_{ijk}$$



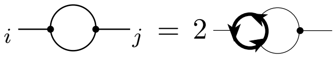
Tracing with T_k we can express C_{ijk} in terms of the defining representation traces:

$$iC_{ijk} = \mathrm{tr}\,(T_iT_jT_k) - \mathrm{tr}\,(T_jT_iT_k)$$



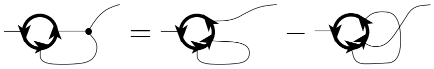
The adjoint quadratic casimir $C_{imn}C^{nmj}$ is now evaluated by first eliminating C_{ijk}'s in favor of the defining representation:

$$\delta_{ij}C_A =$$



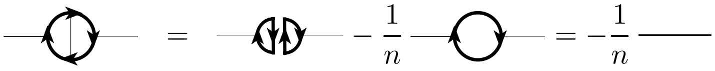
(2.8)

The remaining C_{ijk} can be unwound by the Lie algebra commutator



(2.9)

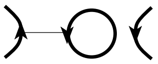
We have already evaluated the quadratic casimir (2.7) in the first term. The second term we evaluate by substituting the adjoint projection operator



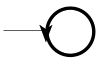
$$\mathrm{tr}\,(T_iT_kT_jT_k) = (T_i)^b_a(P_1)^a_d{}^c_b(T_j)^d_c = (T_i)^a_a(T_j)^c_c - \frac{1}{n}(T_i)^b_a(T_j)^a_b$$

The $(T_i)^a_a(T_j)^c_c$ term vanishes by the tracelessness of T_i's. This can be considered a consequence of the orthonormality of the two projection operators P_1 and P_2 in (2.5) (see (3.47)):

$$0 = P_1P_2 =$$



$$\Rightarrow \mathrm{tr}\,T_i =$$

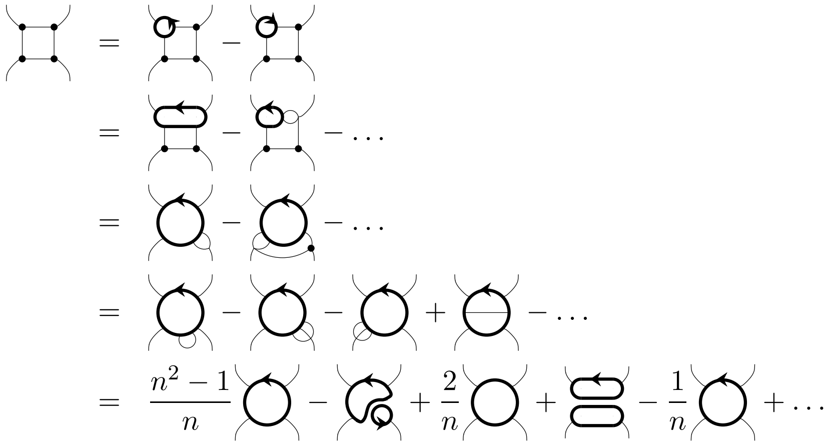


$$= 0 \qquad (2.10)$$

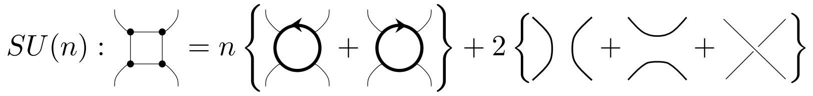
Combining the above expressions we finally obtain

$$C_A = 2\left(\frac{n^2-1}{n} + \frac{1}{n}\right) = 2n\ . \qquad (2.11)$$

The problem (1.1) that started all this is evaluated the same way. First we relate the adjoint quartic casimir to the defining casimirs:

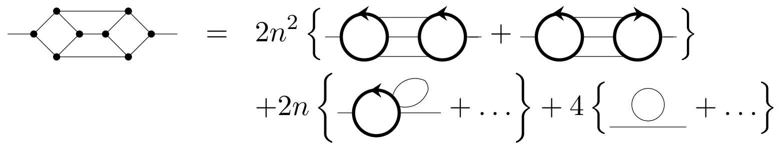


and so on. The result is

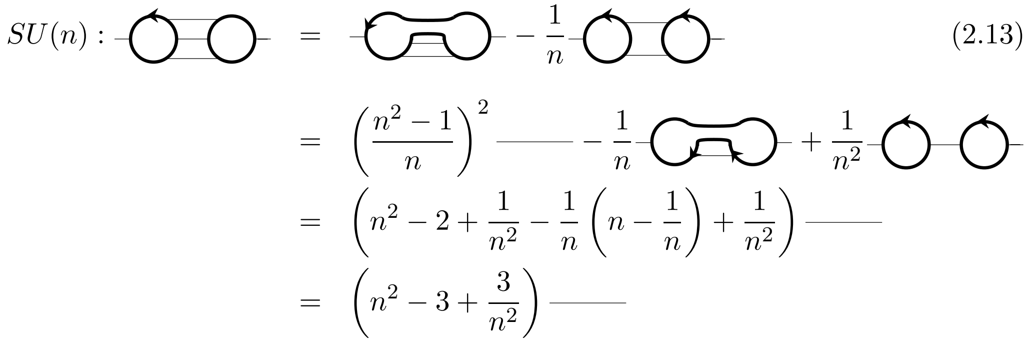


(2.12)

(1.1) is now reexpressed in terms of the defining representation casimirs:

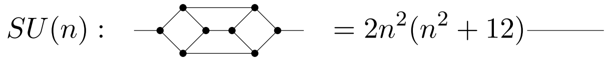


The first two terms are evaluated by inserting the gluon projection operators



(2.13)

and the remaining terms have already been evaluated. Collecting everything together, we finally obtain



(2.14)

This example was unavoidably lengthy; the main point is that the evaluation is performed by a substition algorithm and is easiliy automated. Any graph, no matter how complicated, is eventually reduced to a polynomial in traces of $\delta^a_a = n$, *ie.* the dimension of the defining representation.

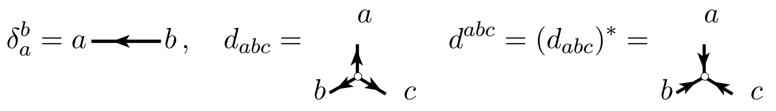
2.3 Second example: E_6 family

What invariance group preserves norms of complex vectors, as well as a symmetric cubic invariant

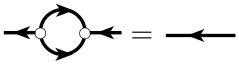
$$D(p,q,r) = d^{abc} p_a q_b r_c = D(q,p,r) = D(p,r,q)\,?$$

We analyze this case following the steps of the summary of sect. 2.1:

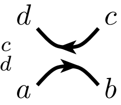
i) *primitive invariant tensors*:

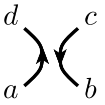


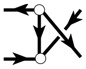
ii) *primitiveness*: $d_{aef}d^{efb}$ must be proportional to δ^a_b, the only primitive two-index tensor. We use this to fix the overall normalization of d_{abc}'s:



iii) *invariant hermitian matrices*: We shall construct here the adjoint representation projection operator on the tensor product space of the defining representation and its conjugate. All invariant matrices on this space are

$\delta^a_b\delta^c_d$ 

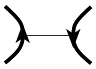
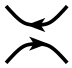
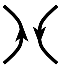
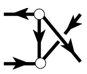
, $\delta^a_d\delta^c_b$ 

, $d^{ace}d_{ebd} =$  .

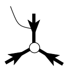
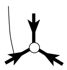
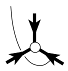
They are hermitian in the sense of being invariant under complex conjugation and transposition of indices (see (3.18)).

The adjoint projection operator must be expressible in terms of the four-index invariant tensors listed above:

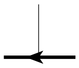
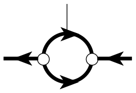
$$(T_i)^a_b(T_i)^d_c = A(\delta^a_c\delta^d_b + B\delta^a_b\delta^d_c + Cd^{ade}d_{bce})$$

 $= A\Big\{$  $+ B$  $+ C$  $\Big\}$

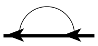
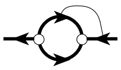
iv) *invariance*. The cubic invariant tensor satisfies (2.4)

 $+$  $+$  $= 0\,.$

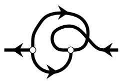
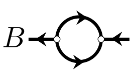
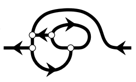
Contracting with d^{abc} we obtain

 $+\,2$  $= 0\,.$

Contracting next with $(T_i)^b_a$, we get an invariance condition on the adjoint projection operator:

 $+\,2$  $= 0\,.$

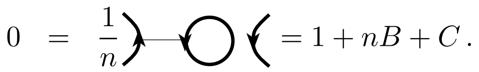
Substituting the adjoint projection operator yields the first relation between the coefficients in its expansion:

$$0 = n + B + C + 2\Big\{$$  $+ B$  $+ C$  $\Big\}$

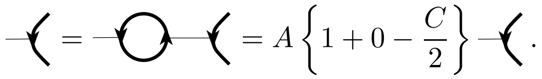
$$0 = B + C + \frac{n+2}{3}\,.$$

v) *the projection operators* should be orthonormal, $P_\mu P_\sigma = P_\mu \delta_{\mu\sigma}$. The adjoint projection operator is orthogonal to the singlet projection operator P constructed in sect. 2.2. This yields the second relation on the coefficients:

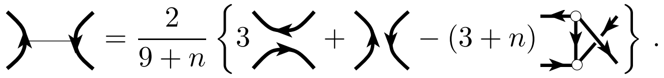
$$0 = P_A P_1$$



Finally, the overall normalization factor A is fixed by $P_A P_A = P_A$:

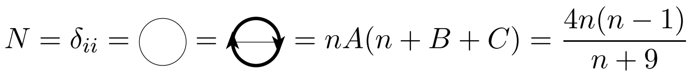


Combining the above 3 relations we obtain the adjoint projection operator for the invariance group of a symmetric cubic invariant



The corresponding *characteristic equation*, mentioned in the point iv of the summary of sect. 2.1 is given in (**??**).

The dimension of the adjoint representation is obtained by tracing the projection operator



This *Diophantine condition* is satisfied by a small family of invariance groups, discussed in chapter 17. The most interesting member of this family is the exceptional Lie group E_6, with $n = 27$ and $N = 78$.

Chapter 3

Invariants and reducibility

Basic group theoretic notions are introduced groups, invariants, tensors and the diagrammatic notation for invariant tensors.

The basic idea is simple; a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the representations of the group into direct sums of lower dimensional representations.

The key results are the construction of projection operators from invariant matrices (3.45), the Clebsch-Gordan coefficients representation of projection operators (3.73), the invariance conditions (3.91) and the Lie algebra relations (3.103).

3.1 Preliminaries

In this section we define basic building blocks of the theory to be developped here: groups, vector spaces, algebras, *etc.* This material is covered in any introduction to group theory [7, 5]. Most of sect. 3.1.2 to sect. 3.1.4 is probably known to the reader, and profitably skipped on the first reading.

3.1.1 Groups

Definition. A set of elements $g \in \mathcal{G}$ forms a group with respect to multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ if

(a) the set is *closed* with respect to multiplication; for any two elements $a, b \in \mathcal{G}$, the product $ab \in \mathcal{G}$.

(b) the multiplication is *associative*

$$(ab)c = a(bc)$$

for any three elements $a, b, c \in \mathcal{G}$.

(c) there exists an *identity* element $\mathbf{e} \in \mathcal{G}$ such that

$$\mathbf{e}g = g\mathbf{e} \qquad \text{for any } g \in \mathcal{G}$$

(d) for any $g \in \mathcal{G}$ there exists an *inverse* g^{-1} such that

$$g^{-1}g = gg^{-1} = \mathbf{e}\,.$$

If the group is finite, the number of elements is called the *order* of the group and denoted $|\mathcal{G}|$.

If the multiplication $ab = ba$ is commutative for all $a, b \in \mathcal{G}$, the group is *abelian*.

Two groups with the same multiplication table are said to be *isomorphic*.

Definition. A *subgroup* $\mathcal{H} \leq \mathcal{G}$ is a subset of $\mathcal{G}$ that forms a group under multiplication. $\mathbf{e}$ is always a subgroup; so is $\mathcal{G}$ itself.

Definition. A *cyclic group* is a group generated from one of its elements, called the generator of the cyclic group. If n is the minimum integer such that $a^n = \mathbf{e}$, the set $\mathcal{G} = \{\mathbf{e}, a, a^2, \cdots, a^{n-1}\}$ is the cyclic group. As all elements commute, cyclic groups are abelian. Every subgroup of a cyclic group is cyclic.

3.1.2 Vector spaces

Definition. A set V of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* "+" is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$.

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned}
a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x} + a\mathbf{y}\,, \qquad a, b \in \mathbb{F}\,, \quad \mathbf{x}, \mathbf{y} \in V \\
(a+b)\mathbf{x} &= a\mathbf{x} + b\mathbf{x} \\
a(b\mathbf{x}) &= (ab)\mathbf{x} \\
1\,\mathbf{x} &= \mathbf{x}\,, \qquad 0\,\mathbf{x} = \mathbf{0}\,.
\end{aligned}$$

Here the field $\mathbb{F}$ will be either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers (quaternion and octonion fields are discussed in sect. 15.5).

Definition. *n-dimensional complex vector space* V consists of all n-multiplets $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{C}$. The two elements $\mathbf{x}$, $\mathbf{y}$ are equal if $x_i = y_i$ for all $0 \leq i \leq n$. The vector addition identity element is $\mathbf{0} = (0, 0, \cdots, 0)$.

Definition. A complex vector space V is an *inner product space* if with every pair of elements $\mathbf{x}$, $\mathbf{y} \in V$ there is associated a unique *inner* (or *scalar*) product $(x, y) \in \mathbb{C}$, such that

$$\begin{aligned}
(x, y) &= (y, x)^* \\
(ax, by) &= a^* b (x, y)\,, \qquad a, b \in \mathbb{C} \\
(z, ax + by) &= a(z, x) + b(z, y)\,,
\end{aligned}$$

where * denotes complex conjugation.

Without any noteworthy loss of generality we shall here define the scalar product of two elements of V by

$$(x, y) = \sum_{j=1}^{n} x_j^* y_j \,. \tag{3.1}$$

3.1.3 Algebra

Definition. A set of elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma \in T} t_{\alpha\beta}{}^{\gamma} \mathbf{t}_\gamma \,. \tag{3.2}$$

(b) the multiplication operation is *bilinear*

$$\begin{aligned} (\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \end{aligned}$$

The set of numbers $t_{\alpha\beta}{}^{\gamma}$ are called the *structure constants* of the algebra. They form a matrix representation of the algebra

$$(t_\alpha)_\beta{}^\gamma = t_{\alpha\beta}{}^{\gamma} \tag{3.3}$$

whose dimension is the dimension of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma) \,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{3.4}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c - (t_\alpha)_c^b (t_\beta)_b^a \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{3.5}$$

which defines a *Lie algebra*.

As a plethora of vector spaces, indices and conjugations looms large in our immediate future, it pays to streamline the notation now, by singling out one vector space as “defining”, and replacing complex conjugation by raised indices.

3.1.4 Defining space, tensors, representations

Definition. Let V be the *defining* n-dimensional complex vector space. Associate with the defining n-dimensional complex vector space V a *conjugate* (or *dual*) n-dimensional vector space $\bar{V} = \{\bar{x} \mid \bar{x}^* \in V\}$ obtained by complex conjugation of elements $x \in V$. We shall denote the corresponding element of $\bar{V}$ by raising the index

$$x^a = (x_a)^* ,$$

so the components of defining space vectors, resp. conjugate vectors, are distinguished by lower, resp. upper indices

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \ , \ x \in V \\ \bar{x} &= (x^1, x^2, \ldots, x^n) \ , \ \bar{x} \in \bar{V} . \end{aligned} \tag{3.6}$$

Repeated index summation: Throughout this text the repeated indices are always summed over

$$G_a^b x_b = \sum_{b=1}^{n} G_a^b x_b \, , \tag{3.7}$$

unless explicitly stated otherwise.

Definition. Let $\mathcal{G}$ be a group of transformations acting linearly on V, with the action of a group element $g \in \mathcal{G}$ on a vector $x \in V$ given by a unitary $[n \times n]$ matrix G

$$x'_a = G_a^b x_b \qquad a, b = 1, 2, \ldots, n \, . \tag{3.8}$$

We shall refer to G_a^b as the *defining representation* of the group. The action of $g \in \mathcal{G}$ on a vector $\bar{q} \in \bar{V}$ is given by the *conjugate representation* $G^\dagger$

$$x'^a = x^b (G^\dagger)_b^a \, , \qquad (G^\dagger)_b^a \equiv (G_a^b)^* \, . \tag{3.9}$$

By defining the conjugate space $\bar{V}$ by complex conjugation and inner product (3.1), we have already chosen (without any loss of generality) δ_a^b as the invariant tensor with the bilinear form $(x, x) = x_b x^b$. From this choice it follows that in the applications considered here, the group $\mathcal{G}$ is always assumed *unitary*

$$(G^\dagger)_a^c G_c^b = \delta_a^b \, . \tag{3.10}$$

Definition. A *tensor* $x \in V^p \otimes \bar{V}^q$ is any object that transforms under the action of $g \in \mathcal{G}$ as

$$x'^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} = G^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}^{d_p \ldots d_1}_{,\, c_q \ldots c_2 c_1} \, x^{c_1 c_2 \ldots c_q}_{d_1 \ldots d_p} \, , \tag{3.11}$$

where the $V^p \otimes \bar{V}^q$ *tensor representation* of $g \in \mathcal{G}$ is defined by

$$G^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}{}^{d_q \ldots d_1}_{,\, c_p \ldots c_2 c_1} \equiv (G^\dagger)^{a_1}_{c_1} (G^\dagger)^{a_2}_{c_2} \ldots (G^\dagger)^{a_p}_{c_p} G^{d_1}_{b_1} \ldots G^{d_q}_{b_q} \, . \tag{3.12}$$

Tensors can be combined into other tensors by
(a) *addition*

$$z^{ab\dots c}_{d\dots e} = \alpha x^{ab\dots c}_{d\dots e} + \beta y^{ab\dots c}_{d\dots e}\,, \qquad \alpha, \beta \in \mathbb{C}\,, \tag{3.13}$$

(b) *product*

$$z^{abcd}_{efg} = x^{abc}_{e} y^{d}_{fg}\,, \tag{3.14}$$

(c) *contraction:* Setting an upper and lower index equal and summing over all of its values yields a tensor $z \in V^{p-1} \otimes \bar{V}^{q-1}$ without these indices:

$$z^{bc\dots d}_{e\dots f} = x^{abc\dots d}_{e\dots af}\,, \qquad z^{ad}_{e} = x^{abc}_{e} y^{d}_{cb}\,. \tag{3.15}$$

A tensor $x \in V^p \otimes \bar{V}^q$ transforms linearly under the action of g, so it can be considered a vector in the $d = n^{p+q}$ dimensional vector space $\tilde{V}$. We can replace the array of its indices by one collective index:

$$x_\alpha = x^{a_1 a_2 \dots a_q}_{b_1 \dots b_p}\,. \tag{3.16}$$

One could be more explicit and give a table like

$$x_1 = x^{11\dots 1}_{1\dots 1}\,,\; x_2 = x^{21\dots 1}_{1\dots 1}, \dots,\; x_d = x^{nn\dots n}_{n\dots n}\,, \tag{3.17}$$

but that is unnecessary, as we shall use the compact index notation only as a shorthand.

Definition. *Hermitian conjugation* is effected by complex conjugation and index transposition:

$$(h^\dagger)^{ab}_{cde} \equiv (h^{edc}_{ba})^*\,. \tag{3.18}$$

Complex conjugation interchanges upper and lower indices, as in (3.6); transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(M^\dagger)^a_b = M^a_b\,. \tag{3.19}$$

Definition. The tensor conjugate to x_α has form

$$x^\alpha = x^{b_p \dots b_1}_{a_q \dots a_2 a_1}\,. \tag{3.20}$$

Combined, the above definitions lead to the hermitian conjugation rule for collective indices: a collective index is raised or lowered by interchanging the upper and lower indices and reversing their order:

$$_\alpha = \left\{ \begin{matrix} a_1 a_2 \dots a_q \\ b_1 \dots b_p \end{matrix} \right\} \quad \leftrightarrow \quad ^\alpha = \left\{ \begin{matrix} b_p \dots b_1 \\ a_q \dots a_2 a_1 \end{matrix} \right\}. \tag{3.21}$$

This transposition convention will be motivated further by the diagrammatic rules of sect. 3.6.

The tensor representation (3.12) can be treated as a $[d \times d]$ matrix

$$G_\alpha^\beta = G^{a_1 a_2 \dots a_q}{}_{b_1 \dots b_p}{}^{d_p \dots d_1}{}_{c_q \dots c_2 c_1}\,, \tag{3.22}$$

and the tensor transformation (3.11) takes the usual matrix form

$$x'_\alpha = G_\alpha^\beta x_\beta\,. \tag{3.23}$$

3.2 Invariants

Definition. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in \mathcal{G}$

$$q = Gq\,. \tag{3.24}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} = (G^\dagger)^{a_1}_{c_1} (G^\dagger)^{a_2}_{c_2} \ldots G^{d_1}_{b_1} \ldots G^{d_q}_{b_q} x^{c_1 c_2 \ldots c_p}_{d_1 \ldots d_q}\,. \tag{3.25}$$

We can state this more compactly by using the notation of (3.22)

$$x_\alpha = G^\beta_\alpha x_\beta\,. \tag{3.26}$$

Here we treat the tensor $x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}$ as a vector in $[d \times d]$ dimensional space, $d = n^{p+q}$.

If a bilinear form $M(\bar{x}, y) = x^a M^b_a y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M^b_a = G^c_a (G^\dagger)^b_d M^d_c \tag{3.27}$$

is an *invariant matrix* . Multiplying with G^e_b and using the unitary condition (3.10), we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, M] = 0\,. \tag{3.28}$$

If we wish to treat a tensor as a matrix

$$M^\beta_\alpha = M^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}_{,}{}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1}\,, \tag{3.29}$$

then the invariance condition (3.26) will takes the commutator form (3.28).

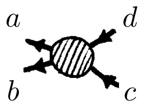
Definition. We shall refer to an invariant relation between p vectors in V and q vectors in $\bar{V}$ which can be written as a homogeneous polynomial in terms of vector components, such as

$$H(x, y, \bar{z}, \bar{r}, \bar{s}) = h^{ab}_{cde} x_b y_a s^e r^d z^c\,, \tag{3.30}$$

as an *invariant* in $V^q \otimes \bar{V}^p$ (repeated indices, as always, summed over). In this example, the coefficients h^{ab}_{cde} are components of invariant tensor $h \in V^3 \otimes \bar{V}^2$, obeying the invariance condition (3.25).

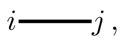
Diagrammatic represention of tensors, such as

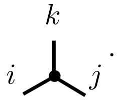
$$h^{ab}_{cd} = \tag{3.31}$$



makes it easier to distinguish different types of invariant tensors. We shall explain in great detail our conventions for drawing tensors in sect. 3.6; sketching a few simple examples should suffice for the time being.

The standard example of a defining vector space is our three-dimensional Euclidean space: $V = \bar{V}$ is the space of all three-component real vectors ($n = 3$), and examples of invariants are the length $L(x, x) = \delta_{ij} x_i x_j$ and the volume $V(x, y, z) = \epsilon_{ijk} x_i y_j z_k$. We draw the corresponding invariant tensors as

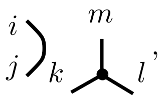
$\delta_{ij} =$ 

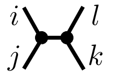
, $\epsilon_{ijk} =$ 

. (3.32)

Definition. A *composed* invariant tensor can be written as a product and/or contraction of invariant tensors.

Examples of composed invariant tensors are

$\delta_{ij}\epsilon_{klm} =$ 

, $\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} =$ 

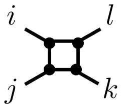
. (3.33)

The first example corresponds to a product of the two invariants $L(x, y)V(z, r, s)$. The second involves an index *contraction*; we can write this as $V\left(x, y, \frac{d}{dz}\right) V(z, r, s)$.

In order to proceed, we need to distinguish the "primitive" invariant tensors from the infinity of composed invariants. We begin by defining a finite basis for invariant tensors in $V^p \otimes \bar{V}^q$:

Definition. A *tree invariant* can be represented diagrammatically as a product of invariant tensors involving no loops of index contractions. We shall denote by $T = \{\mathbf{t}_0, \mathbf{t}_1 \ldots \mathbf{t}_r\}$ a (maximal) set of r linearly independent tree invariants $\mathbf{t}_\alpha \in V^p \otimes \bar{V}^q$. As any linear combination of $\mathbf{t}_\alpha$ can serve as a basis, we clearly have a great deal of freedom in making informed choices for the basis tensors.

Example: Tensors (3.33) are tree invariants. The tensor

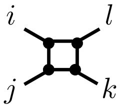
$h_{ijkl} = \epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{\ell sr} =$ 

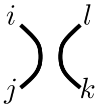
. (3.34)

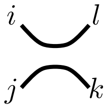
is not a tree invariant, as it involves a loop.

Definition. An invariant tensor is called a *primitive* invariant tensor if it cannot expressed as a combination of tree invariants composed from lower rank primitive invariant tensors. Let $P = \{p_1, p_2, \ldots p_k\}$ be the set of all primitives.

For example, the Kronecker delta and the Levi-Civita tensor (3.32) are the primitive invariant tensors of our three-dimensional space. The loop contraction (3.34) is not a primitive, because by the Levi-Civita completeness relation (5.32) it reduces to a sum of tree contractions:



$=$ 

$+$ 

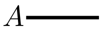
$= \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\,.$ (3.35)

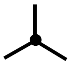
(the Levi-Civita tensor is discussed in sect. 5.4.)

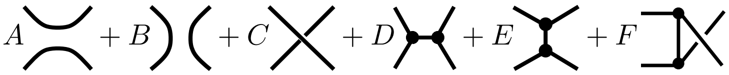
Primitiveness assumption. Any invariant tensor $h \in V^p \otimes \bar{V}^q$ can be expressed as a linear sum over the tree invariants $T \in V^q \otimes \bar{V}^p$

$$h = \sum_T h^\alpha \mathbf{t}_\alpha \,. \tag{3.36}$$

In contradistinction to arbitrary composite invariant tensors, the number of tree invariants for a fixed number of external indices is finite. For example, given the $n = 3$ dimensions primitives $P = \{\delta_{ij}, f_{ijk}\}$, any invariant tensor $h \in V^p$ (here denoted by a blob) must be expressible as

`birdTrack` $=$ A  (3.37)

`birdTrack` $=$ A 

`birdTrack` $=$ 

`birdTrack` $=$ `birdTrack` $+$ `birdTrack` $+ \ldots$ (3.38)

$\vdots$ $=$

3.2.1 Algebra of invariants

Any invariant tensor of matrix form (3.29)

$$M_\alpha^\beta = M^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}\,{}^{d_p \ldots d_1}_{,\, c_q \ldots c_2 c_1}$$

which maps $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$ can be expanded in the basis (3.36). The basis $\mathbf{t}_\alpha$ are themselves matrices in $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$, and the matrix product of two basis elements is also an element of $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$ and can be expanded in the minimal basis:

$$\mathbf{t}_\alpha \mathbf{t}_\beta = \sum_{\gamma \in T} (t_\alpha)_\beta{}^\gamma \mathbf{t}_\gamma \,. \tag{3.39}$$

As the number of tree invariants composed from the primitives is finite, under matrix multiplication the bases $\mathbf{t}_\alpha$ form a finite algebra, with the coefficients $(t_\alpha)_\beta{}^\gamma$ giving their multiplication table. The multiplication coefficients $(t_\alpha)_\beta{}^\gamma$ form a $[r \times r]$-dimensional matrix representation of $\mathbf{t}_\alpha$ acting on the vector $(\mathbf{e}, \mathbf{t}_1, \mathbf{t}_2, \cdots \mathbf{t}_{r-1})$. Given a basis, we can evaluate the matrices $\mathbf{e}_\beta{}^\gamma$, $(t_1)_\beta{}^\gamma$, $(t_2)_\beta{}^\gamma, \cdots (t_{r-1})_\beta{}^\gamma$ and their eigenvalues. For at least one of these matrices all eigenvalues will be distinct (or we have failed to chose a minimal basis). The projection operator technique of sect. 3.4 will enable us to exploit this fact to decompose the $V^q \otimes \bar{V}^p$ space into r irreducible subspaces.

This can be said in another way; the choice of basis $\{\mathbf{e}, \mathbf{t}_1, \mathbf{t}_2, \cdots \mathbf{t}_{r-1}\}$ is arbitrary, the only requirement being that the basis elements are linearly independent. Finding a $(t_\alpha)_\beta{}^\gamma$ with all eigenvalues distinct is all we need to construct an orthonormal basis $\{P_0, P_1, P_2, \cdots P_{r-1}\}$, where the basis matrices P_i are the projection operators, to be constructed below in sect. 3.4.

3.3 Invariance groups

So far we have defined invariant tensors as the tensors invariant under transformations of a given group. Now we proceed in the other direction: given a set of tensors, what is the group of transformations that leaves them invariant?

Given a full set of primitives (3.30) $P = \{p_1, p_2, \ldots, p_k\}$, meaning that *no other* primitives exist, we wish to determine all possible transformations that preserve this gvien set of invariant relations.

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations (3.25) which preserve the primitive invariant relations (and, by extension, *all* invariant relations)

$$\begin{aligned} p_1(x, \bar{y}) &= p_1(Gx, \bar{y}G^\dagger) \\ p_2(x, y, z, \ldots) &= p_2(Gx, Gy, Gz \ldots)\,, \qquad \ldots \end{aligned} \tag{3.40}$$

Unitarity (3.10) guarantees that all contractions of primitive invariant tensors, and hence all composed tensors $h \in H$ are also invariant under action of $\mathcal{G}$. As $\mathcal{G}$ we consider is unitary, it follows from (3.10) that the list of primitives must always include the Kronecker delta.

Example 1. If $p^a q_a$ is an invariant of $\mathcal{G}$

$$p'^a q'_a = p^b (G^\dagger G)_b^c q_c = p^a q_a\,, \tag{3.41}$$

then $\mathcal{G}$ is the full *unitary group* $U(n)$ (invariance group of the complex norm $|x|^2 = x^b x_a \delta_b^a$), whose elements satisfy

$$G^\dagger G = 1\,. \tag{3.42}$$

Example 2. If we wish the z-direction to be invariant in our three-dimensional space, $q = (0, 0, 1)$ is an invariant vector (3.24), and the invariance group is $O(2)$, the group of all rotations in the x-y plane.

Remark **3.1** Which representation is "defining"?.

1. The defining space V *need not* carry the lowest dimensional representation of $\mathcal{G}$; it is merely the space in terms of which we chose to define the primitive invariants.

2. We shall always assume that the Kronecker delta δ_a^b is one of the primitive invariants, *ie.* that $\mathcal{G}$ is a unitary group whose elements satisfy (3.42). This restriction to unitary transformations is not essential, but it simplifies proofs of full reducibility. The results, however, apply as well to the finite-dimensional representations of non-compact groups, such as the Lorentz group $SO(3,1)$.

3.4 Projection operators

For M a hermitian matrix, there exists a diagonalizing unitary matrix C such that:

$$CMC^{\dagger} = \begin{pmatrix} \begin{array}{cc|} \lambda_1 & 0 \\ 0 & \lambda_1 \\ \hline \end{array} & 0 & 0 \\ 0 & \begin{array}{|cccc|} \hline \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda_2 \\ \hline \end{array} & 0 \\ 0 & 0 & \begin{array}{|cc} \hline \lambda_3 & \dots \\ \vdots & \ddots \end{array} \end{pmatrix}, \quad \lambda_i \neq \lambda_j \,. \tag{3.43}$$

Here λ_i are the r distinct roots of the minimal characteristic polynomial

$$\prod_{i=1}^{r}(M - \lambda_i \mathbf{1}) = 0 \,. \tag{3.44}$$

(the characteristic equations will be discussed in sect. 5.7.) In the matrix $C(M - \lambda_2 \mathbf{1})C^{\dagger}$ the eigenvalues corresponding to λ_2 are replaced by zeroes:

$$\begin{pmatrix} \begin{array}{ccc|} \lambda_1 - \lambda_2 & & \\ & \lambda_1 - \lambda_2 & \\ & & \lambda_1 - \lambda_2 \\ \hline \end{array} & & \\ & \begin{array}{|ccc|} \hline 0 & & \\ & \ddots & \\ & & 0 \\ \hline \end{array} & \\ & & \begin{array}{|ccc} \hline \lambda_3 - \lambda_2 & & \\ & \lambda_3 - \lambda_2 & \\ & & \ddots \end{array} \end{pmatrix},$$

and so on, so the product over all factors $(M - \lambda_2 \mathbf{1})(M - \lambda_3 \mathbf{1}) \dots$ with exception of the $(M - \lambda_1 \mathbf{1})$ factor has non-zero entries only in the subspace associated with λ_1:

$$C \prod_{j \neq 1}(M - \lambda_j \mathbf{1})C^{\dagger} = \prod_{j \neq 1}(\lambda_1 - \lambda_j) \begin{pmatrix} \begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline \end{array} & 0 \\ 0 & \begin{array}{|cccc} \hline 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{array} \end{pmatrix} .$$

In this way we can associate with each distinct root λ_i a *projection operator* P_i

$$P_i = \prod_{j \neq i} \frac{M - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,, \tag{3.45}$$

which is identity on the ith subspace, and zero elsewhere. For example, the projection operator onto the λ_1 subspace is

$$P_1 = C^\dagger \left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ \hline & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{array} \right) C \,. \tag{3.46}$$

The matrices P_i are *orthonormal*

$$P_i P_j = \delta_{ij} P_j \;, \qquad (\text{no sum on } j) \,, \tag{3.47}$$

and satisfy the *completeness relation*

$$\sum_{i=1}^{r} P_i = \mathbf{1} \,. \tag{3.48}$$

As $\mathrm{tr}\,(C P_i C^+) = \mathrm{tr}\, P_i$, the dimension of the ith subspace is given by

$$d_i = \mathrm{tr}\, P_i \,. \tag{3.49}$$

It follows from the characteristic equation (3.44) and the form of the projection operator (3.45) that λ_i is the eigenvalue of M on P_i subspace:

$$M P_i = \lambda_i P_i \;, \qquad (\text{no sum on } i) \,. \tag{3.50}$$

Hence any matrix polynomial $f(M)$ takes the scalar value $f(\lambda_i)$ on the P_i subspace

$$f(M) P_i = f(\lambda_i) P_i \,. \tag{3.51}$$

This, of course, is the real reason why one wants to work with irreducible representations: they render matrices and “operators” harmless c-numbers.

3.5 Further invariants

Suppose that there exist several linearly independent invariant $[d \times d]$ hermitian matrices $M_1, M_2, \ldots$ and that we have used M_1 to decompose the d-dimensional vector space $\tilde{V} = \Sigma \oplus V_i$. Can M_2 be used to further decompose V_i? This is the standard problem of quantum mechanics (simultaneous observables), and the answer is that further decomposition is possible if, and only if, the invariant matrices commute,

$$[M_1, M_2] = 0\,, \tag{3.52}$$

or, equivalently, if all projection operators commute

$$P_i P_j = P_j P_i\,. \tag{3.53}$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators P_i constructed from M_1 can be used to project commuting pieces of M_2:

$$M_2^{(i)} = P_i M_2 P_i\ , \qquad (\text{no sum on } i)\,.$$

That $M_2^{(i)}$ commutes with M_1 follows from the orthogonality of P_i:

$$[M_2^{(i)}, M_1] = \sum_j \lambda_j [M_2^{(i)}, P_j] = 0\,. \tag{3.54}$$

Now the characteristic equation for $M_2^{(i)}$ (if nontrivial) can be used to decompose V_i subspace.

An invariant matrix M induces a decomposition only if its diagonalized form (3.43) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix, and commutes trivially with all group elements. A representation is said to be *irreducible* if all invariant matrices that can be constructed are proportional to the unit matrix.

In particular, the primitiveness relation (3.37) is a statement that the defining representation is *assumed* irreducible.

According to (3.28), an invariant matrix M commutes with group transformations $[G, M] = 0$. Projection operators (3.45) constructed from M are polynomials in M, so they also commute with all $g \in \mathcal{G}$:

$$[G, P_i] = 0\,, \tag{3.55}$$

(remember that P_i are also invariant $[d \times d]$ matrices). Hence a $[d \times d]$ matrix representation can be written as a direct sum of $[d_i \times d_i]$ matrix representations

$$G = \mathbf{1}G\mathbf{1} = \sum_{i,j} P_i G P_j = \sum_i P_i G P_i = \sum_i G_i\,. \tag{3.56}$$

In the diagonalized representation (3.46), the matrix G has a block diagonal form:

$$CGC^\dagger = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}\,, \qquad G = \sum_i C^i G_i C_i\,. \tag{3.57}$$

Representation G_i acts only on the d_i dimensional subspace V_i consisting of vectors $P_i q$, $q \in \tilde{V}$. In this way an invariant $[d \times d]$ hermitian matrix M with r distinct eigenvalues induces a decomposition of a d-dimensional vector space $\tilde{V}$ into a direct sum of d_i-dimensional vector subspaces V_i

$$\tilde{V} \overset{M}{\rightarrow} V_1 \oplus V_2 \oplus \ldots \oplus V_r \,. \tag{3.58}$$

For more detailed discussion of recursive reduction, consult appendix A.

3.6 Birdtracks

We shall often find it convenient to represent aglomerations of invariant tensors by "birdtracks", a group-theoretical version of Feynman diagrams. Tensors will be represented by "vertices", and contractions by "propagators".

Diagrammatic notation has several advantages over the tensor notation. Diagrams do not require dummy indices, so explicit labelling of such indices is unnecessary. More to the point, for a human eye it is easier to identify topologically identical diagrams than to recognize equivalence between the corresponding tensor expressions.

The main disadvantage of diagrammatic notation is lack of standardization, especially in the case of Clebsch-Gordan coefficients. Many of the diagrammatic notations [97, 98, 73] designed for atomic and nuclear spectroscopy, are complicated by various phase conventions. In our applications, explicit constructions of clebsches are superfluous, and we need no such conventions, confusing or otherwise.

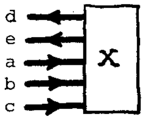
In the birdtrack notation, the Kronecker delta is a "propagator":

$$\delta^a_b = b \longleftarrow a \,. \tag{3.59}$$

For a *real* defining space there is no distinction between V and $\bar{V}$, or up and down indices, and the lines do not carry arrows.

Any invariant tensor can be drawn as a generalized vertex:

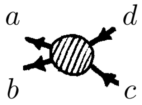
$$x_\alpha = x^{abc}_{de} = \tag{3.60}$$



Whether the vertex is drawn as a box or a circle or a dot is matter of taste. The orientation of propagators and vertices in the plane of the drawing is likewise irrelevant. The only rules are

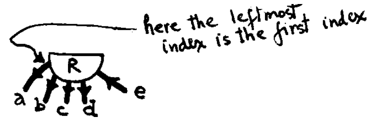
(1) Arrows point *away from the upper* indices and *toward the lower* indices; the line flow is "downward", from upper to lower indices:

$$h^{cd}_{ab} = \tag{3.61}$$



(2) Diagrammatic notation must indicate which in (out) arrow corresponds to the first upper (lower) index of the tensor (unless the tensor is cyclically symmetric);

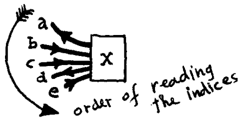
$$R^e_{abcd} =$$



. (3.62)

(3) The indices are read in the *counterclockwise* order around the vertex:

$$x^{bce}_{ad} =$$



. (3.63)

(The upper and the lower indices are read separately in the counterclockwise order; their relative ordering does not matter.)

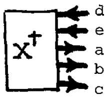
In the examples of this section we index the external lines for reader's convenience, but indices can always be omitted. An internal line implies a summation over corresponding indices, and for external lines the equivalent points on each diagram represent the same index in all terms of a diagrammatic equation.

Hermitian conjugation (3.18) does two things:

(a) it exchanges the upper and the lower indices, *ie.* it reverses the directions of the arrows

(b) it reverses the order of the indices, *ie.* it transposes a diagram into its mirror image. For example, $x^\dagger$, the tensor conjugate to (3.63), is drawn as

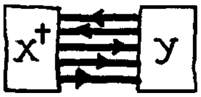
$$x^\alpha = x^{ed}_{cba} =$$



(3.64)

and a contraction of tensors $x^\dagger$ and y is drawn as

$$x^\alpha y_\alpha = x^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} y^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} =$$



. (3.65)

3.7 Clebsch-Gordan coefficients

Consider the product

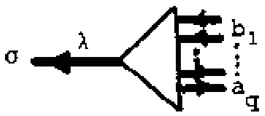
$$\left(\begin{array}{ccc}
\begin{array}{cc|}0&\\&0\\\hline\end{array} & & \\
& \boxed{\begin{array}{ccc}1&&\\&1&\\&&1\end{array}} & \\
& & \begin{array}{|cccc}\hline 0&&&\\&0&&\\&&0&\\&&&\ddots\end{array}
\end{array}\right) C \tag{3.66}$$

of the two terms in the diagonal representation of a projection operator (3.46). This matrix has non-zero entries only in the d_i rows of subspace V_i. We collect them in a $[d_i \times d]$ rectangular matrix $(C_i)^{\alpha}_{\sigma}$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_i$:

$$C_i = \underbrace{\begin{pmatrix} (C_i)^1_1 & \ldots & (C_i)^d_1 \\ \vdots & & \vdots \\ & & (C_i)^d_{d_i} \end{pmatrix}}_{d} \left.\vphantom{\begin{pmatrix} 1\\1\\1 \end{pmatrix}}\right\} d_i \, . \tag{3.67}$$

The index α in $(C_i)^{\alpha}_{\sigma}$ stands for all tensor indices associated with the $d = n^{p+q}$ dimensional tensor space $V^p \otimes \bar{V}^q$. In the birdtrack notation these indices are explicit:

$$(C_i)_{\sigma,}{}^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} = \tag{3.68}$$

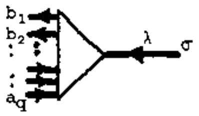


Such rectangular arrays are called *Clebsch-Gordan coefficients* (hereafter refered to as “clebsches” for short). They are explicit mappings $\tilde{V} \rightarrow V_i$. The conjugate mapping $V_i \rightarrow \tilde{V}$ is provided by the product

$$C^{\dagger} \left(\begin{array}{ccc}
\begin{array}{cc|}0&\\&0\\\hline\end{array} & & \\
& \boxed{\begin{array}{ccc}1&&\\&1&\\&&1\end{array}} & \\
& & \begin{array}{|cccc}\hline 0&&&\\&0&&\\&&0&\\&&&\ddots\end{array}
\end{array}\right) \tag{3.69}$$

which defines the $[d \times d_i]$ rectangular matrix $(C^i)^{\sigma}_{\alpha}$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_i$:

$$C^i = \underbrace{\begin{pmatrix} (C^i)^1_1 & \ldots & (C^i)^{d_i}_1 \\ \vdots & & \vdots \\ & & (C^i)^{d_i}_d \end{pmatrix}}_{d_i} \left.\vphantom{\begin{pmatrix} 1\\1\\1 \end{pmatrix}}\right\} d \, , \qquad (C^i)^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p},{}^{\sigma} \; = \; .(3.70)$$



The two rectangular Clebsch-Gordan matrices C^i and C_i are related by hermitian conjugation.

The tensors we have considered in sect. 3.6 transform as tensor products of the defining representation (3.11). In general, tensors transform as tensor products of various representations, with indices runnig over the corresponding representation dimensions:

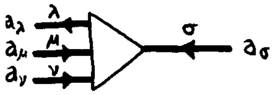
$$x^{a_{p+1}\dots a_{p+q}}_{a_1 a_2 \dots a_p} \qquad \text{where} \qquad \begin{array}{rcl} a_1 &=& 1,2,\dots,d_1 \\ a_2 &=& 1,2,\dots,d_2 \\ &\vdots& \\ a_{p+q} &=& 1,2,\dots,d_{p+q}\,. \end{array} \qquad (3.71)$$

The action of transformation g on the index a_k is given by the $[d_k \times d_k]$ matrix representation G_k.

The Clebsch-Gordan coefficients are notoriously index-overpopulated as they require a representation label and a tensor index for each representation in the tensor product. Diagrammatic notation alleviates this index plague in either of two ways:

(i) one can indicate a representation label on each line:

$$C^{a_\mu a_\nu}_{a_\lambda}{}_{,}{}^{a_\sigma} \;=\; \qquad (3.72)$$

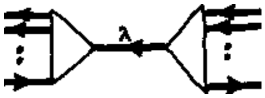


(an index, if written, is written at the end of a line; a representation label is written above the line);

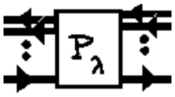
(ii) one can draw the propagators (Kronecker deltas) for different representations with different kinds of lines. For example, we shall usually draw the adjoint representation with a thin line.

By the definition of clebsches (3.46), a λ-representation projection operator can be written out in terms of Clebsch-Gordan matrices: $C^\lambda C_\lambda$:

$$\begin{array}{rcl} C^\lambda C_\lambda &=& P_\lambda\,, \qquad \text{(no sum on } \lambda) \\ (C^\lambda)^{a_1 a_2 \dots a_p}_{b_1 \dots b_q}{}_{,}{}^{\alpha}\, (C_\lambda)_{\alpha,}{}^{d_q\dots d_1}_{c_p \dots c_2 c_1} &=& (P_\lambda)^{a_1 a_2 \dots d_p}_{b_1\dots b_q}{}_{,}{}^{d_q \dots d_1}_{c_p\dots c_2 c_1} \end{array} \qquad (3.73)$$

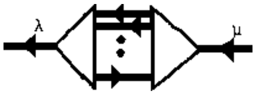


$$= \qquad . \qquad (3.74)$$



A specific choice of clebsches is quite arbitrary. All relevant properties of projection operators (orthonormality, completeness, dimensionality) are independent of the explicit form of the diagonalization transformation C. Any set of C_λ is acceptable, as long as it satisfies the orthogonality and completeness conditions. From (3.66) and (3.69) it follows that C_λ are *orthonormal*:

$$\begin{aligned} C_\lambda C^\mu &= \delta^\mu_\lambda \mathbf{1}\,. \\ (C_\lambda)_{\beta,}{}^{a_1a_2\ldots a_p}{}_{b_1\ldots b_q}(C^\mu)^{b_q\ldots b_1}{}_{a_p\ldots a_2a_1,}{}^{\alpha} &= \delta^\alpha_\beta \delta^\mu_\lambda \end{aligned}$$



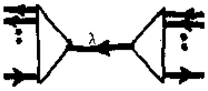
$$= \lambda \longleftarrow \mu\,. \qquad (3.75)$$

Here $\mathbf{1}$ is the $[d_\lambda \times d_\lambda]$ unit matrix, and C_λ's are multiplied as $[d_\lambda \times d]$ rectangular matrices.

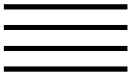
The *completeness relation* (3.48)

$$\begin{aligned} \sum_\lambda C^\lambda C_\lambda &= \mathbf{1}\,, \qquad ([d\times d] \text{ unit matrix})\,, \\ \sum_\lambda (C^\lambda)^{a_1a_2\ldots a_p}_{b_1\ldots b_q}{}_{,}{}^{\alpha}(C_\lambda)_{\alpha,}{}^{d_q\ldots d_1}_{c_p\ldots c_2c_1} &= \delta^{a_1}_{c_1}\delta^{a_2}_{c_2}\ldots\delta^{d_q}_{b_q} \end{aligned}$$

$$\sum_\lambda$$



$$=$$



$$(3.76)$$

and the orthonormality of projection operators and clebsches

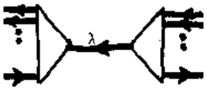
$$\begin{aligned} C^\lambda P_\mu &= \delta^\mu_\lambda C^\lambda\,, \\ P_\lambda C^\mu &= \delta^\mu_\lambda C^\mu\,, \qquad (\text{no sum on } \lambda, \mu)\,, \end{aligned} \qquad (3.77)$$

follow immediately from (3.47) and (3.75).

3.8 Zero- and one-dimensional subspaces

If a projection operator projects onto a zero-dimensional subspace, it must vanish identically

$$d_\lambda = 0 \qquad \Rightarrow \qquad P_\lambda =$$



$$= 0\,. \qquad (3.78)$$

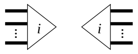
This follows from (3.46); d_λ is the number of 1's on the diagonal on the right-hand side. For $d_\lambda = 0$ the right-hand side vanishes. The general form of P_λ is

$$P_\lambda = \sum_{k=1}^{r} c_k M_k \qquad (3.79)$$

where M_k are the invariant matrices used in construction of the projector operators, and c_k are numerical coefficients. Vanishing of P_λ therefore implies a relation among invariant matrices M_k.

If a projection operator projects onto a one-dimensional subspace, its expression in terms of the Clebsch-Gordan coefficients (3.73) involves no summation, so we can omit the intermediate line

$$d_i = 1 \qquad \Rightarrow \qquad P_i =$$



$$= (C^i)^{a_1 a_2 \dots a_p}_{b_1 \dots b_q} \, (C_i)_{c_p \dots c_2 c_1}{}^{d_q \dots d_1} \, . \tag{3.80}$$

For any subgroup of $SU(n)$, the representations are unitary, with unit determinant. On the one-dimensional spaces the group acts trivially, $G = 1$. Hence if $d_i = 1$, the Clebsch-Gordan coefficient C_i in (3.80) is an invariant tensor in $V^p \otimes \bar{V}^q$.

3.9 Infinitesimal transformations

A unitary transformation G which is infinitesimally close to unity can be written as

$$G_a^b = \delta_a^b + i D_a^b \, , \tag{3.81}$$

where D is a hermitian matrix with small elements, $|D_a^b| \ll 1$. The action of $g \in \mathcal{G}$ on the conjugate space is given by

$$(G^\dagger)_b^a = \delta_b^a - i D_b^a \, . \tag{3.82}$$

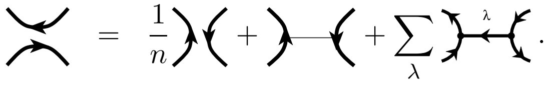
D can be parametrized by $N \leq n^2$ real parameters. N, the maximal number of independent parameters, is called the dimension of the group (also the dimension of the Lie algebra, or the dimension of the adjoint representation).

We shall consider only infinitesimal transformations, of form $G = 1 + iD$, $|D_b^a| \ll 1$. We do not study the entire group of invariances, but only the transformations (3.8) connected to the identity. For example, we shall not consider invariances under coordinate reflections.

The generators of infinitesimal transformations (3.81) are hermitian matrices and belong to the $D_b^a \in V \otimes \bar{V}$ space. However, not any element of $V \otimes \bar{V}$ generates an allowed transformation; indeed, one of the main objectives of group theory is to define the class of allowed transformations.

In sect. 3.4 we have described the general decomposition of a tensor space into (ir)reducible subspaces. As a particular case, consider the decomposition of $V \otimes \bar{V}$. The corresponding projection operators satisfy the completeness relation (3.76)

$$\begin{aligned} \mathbf{1} &= \frac{1}{n} T + P_A + \sum_{\lambda \neq A} P_\lambda \\ \delta_d^a \delta_b^c &= \frac{1}{n} \delta_b^a \delta_d^c + (P_A)_b^a{}_,{}^c_d + \sum_{\lambda \neq A} (P_\lambda)_b^a{}_,{}^c_d \end{aligned}$$



$$(3.83)$$

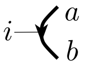
If δ_i^j is the only primitive invariant tensor, then $V \otimes \bar{V}$ decomposes into 2 subspaces, and there are no other irreducible representations. However, if there are further primitive invariant tensors, $V \otimes \bar{V}$ decomposes into more irreducible representations, and therefore the sum over λ. Examples will abound in what follows. The singlet projection operator T/n always figures in this expansion, as $\delta_b^a{}_{,}{}^c_d$ is always one of the invariant matrices (see the example worked out in sect. 2.2). Furthermore, the infinitesimal generators D_b^a must belong to at least one of the irreducible subspaces of $V \otimes \bar{V}$.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V of (3.6). We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin straight line

$$\delta_{ij} = i \text{———} j\ ,\ \ i, j = 1, 2, \ldots, N\,, \tag{3.84}$$

and the corresponding Clebsch-Gordan coefficients by

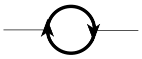
$$(C_A)_i{}_{,}{}^a_b = \frac{1}{\sqrt{a}}(T_i)^a_b =$$



$$\begin{aligned} a, b &= 1, 2, \ldots, n \\ i &= 1, 2, \ldots, N\,. \end{aligned}$$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition (3.75)

$$(T_i)^a_b (T_j)^b_a = \mathrm{tr}\,(T_i T_j) = a\,\delta_{ij}$$



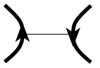
$$= a \text{———}\,. \tag{3.85}$$

The scale of T_i is not set, as any overall rescaling can be absorbed into the normalization a. For our purposes it will be most convenient to use $a = 1$ as the normalization convention. Other normalizations are commonplace. For example, $SU(2)$ Pauli matrices $T_i = \frac{1}{2}\sigma_i$ and $SU(n)$ Gell-Mann [8] matrices $T_i = \frac{1}{2}\lambda_i$ are conventionally normalized by fixing $a = 1/2$:

$$\mathrm{tr}\,(T_i T_j) = \frac{1}{2}\delta_{ij}\,. \tag{3.86}$$

The projector relation (3.73) expresses the adjoint representation projection operators in terms of the generators:

$$(P_A)^a_b{}_{,}{}^c_d = \frac{1}{a}(T_i)^a_b (T_i)^c_d = \frac{1}{a} \tag{3.87}$$



Clearly, the adjoint subspace is always included in the sum (3.83) (there must exist some allowed infinitesimal generators D_a^b, or otherwise there is no group to describe), but how do we determine the corresponding projection operator?

The adjoint projection operator is singled out by the requirement that the group transformations do not affect the invariant quantities. (Remember, the group is *defined* as the totality of all transformations that leave the invariants invariant.) For every invariant tensor q, the infinitesimal group elements $G = 1 + iD$ must satisfy the invariance condition (3.24). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = P_A H \in V \otimes \bar{V}$:

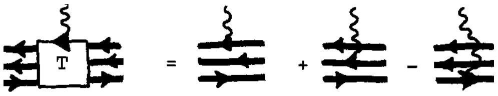
$$D^a_b = \frac{1}{a}(T_i)^a_b \epsilon_i \,, \qquad \epsilon_i = \frac{1}{a}\mathrm{tr}\,(T_i H) \tag{3.88}$$

we obtain the *invariance condition* which the *generators* must satisfy: they *annihilate* invariant tensors

$$T_i q = 0 \,. \tag{3.89}$$

To state the invariance condition for an arbitrary invariant tensor, we need to define the generators in the tensor representations. By substituting $G = 1 + i\epsilon \cdot T + O(\epsilon^2)$ into (3.12) and keeping only the terms linear in ϵ, we find that the generators of infinitesimal transformations for tensor representations act by touching one index at a time:

$$\begin{aligned}
(T_i)_{b_1 \ldots b_q}^{a_1 a_2 \ldots a_p}{}_{,\, c_p \ldots c_2 c_1}^{\;\; d_q \ldots d_1} &= (T_i)^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\
&+ \delta^{a_1}_{c_1} (T_i)^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} + \ldots + \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots (T_i)^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\
&- \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} (T_i)^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} - \ldots - \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots (T_i)^{d_q}_{b_q} \,.
\end{aligned} \tag{3.90}$$

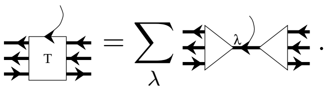


(3.91)

(with a relative minus sign between lines flowing in opposite directions). In other words, the Leibnitz rule obscured by a forest of indices.

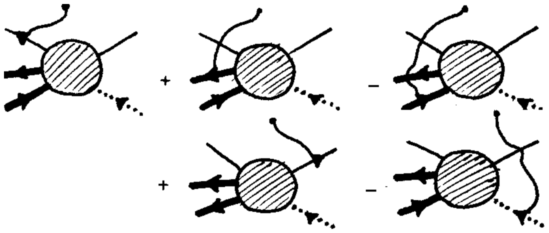
Tensor representations of the generators decompose in the same way as the group representations (3.57)

$$T_i = \sum_\lambda C^\lambda T_i^{(\lambda)} C_\lambda \,. \tag{3.92}$$



The invariance conditions take a particularly suggestive form in the diagrammatic notation. (3.89) amounts to insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:

$$0 =$$



(3.93)

The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

Clebsch-Gordan coefficients are also invariant tensors. Multiplying both sides of (3.57) with C_λ and using orthogonality (3.75), we obtain

$$C_\lambda G = G_\lambda C_\lambda \,, \qquad (\text{no sum on } \lambda)\,. \tag{3.94}$$

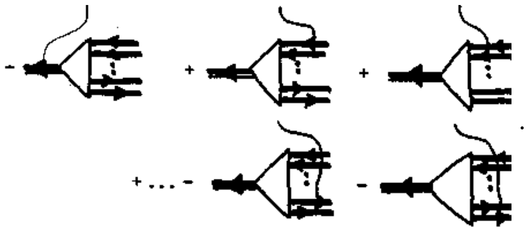
The Clebsch-Gordan matrix C_λ is a rectangular $[d_\lambda \times d]$ matrix, hence $g \in \mathcal{G}$ acts on it with a $[d_\lambda \times d_\lambda]$ representation from the left, and a $[d \times d]$ representation from the right. (3.45) is the statement of invariance for rectangular matrices, analogous to (3.27), the statement of invariance for square matrices:

$$\begin{aligned} C_\lambda &= G_\lambda^\dagger C_\lambda G \,, \\ C^\lambda &= G^\dagger C^\lambda G_\lambda \,. \end{aligned} \tag{3.95}$$

The invariance condition for the Clebsch-Gordan coefficients is a special case of (3.93), the invariance condition for any invariant tensor:

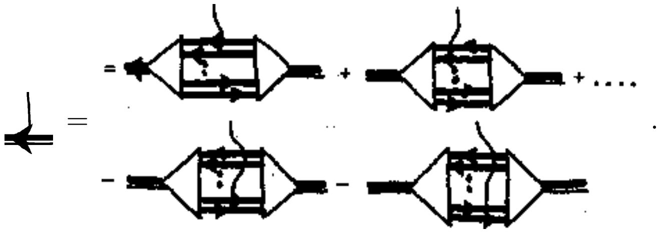
$$0 = -T_i^{(\lambda)} C_\lambda + C_\lambda T_i$$

$$0 =$$



(3.96)

The orthonormality condition (3.75) now yields the generators in λ representation in terms of the defining representation generators

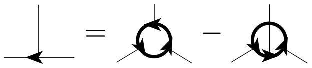


(3.97)

> Remark **3.2** The reality of the adjoint representation. For hermitian generators the adjoint representation is real, and the upper and lower indices need not be distinguished; the "propagator" needs no arrow. For non-hermitian choices of generators, the adjoint representation is complex (gluon lines carry arrows) but A and $\bar{A}$ are equivalent, as indices can be raised an lowered by the Cartan-Killing form $g_{ij} = \mathrm{Tr}\,(T_i^\dagger T_j)$. The Cartan canonical basis $D = \epsilon_i H_i + \epsilon_\alpha E_\alpha + \epsilon_\alpha^* E_{-\alpha}$ is an example of a non-hermitian choice. Here we shall always assume that T_i are chosen hermitian.

3.10 Lie algebra

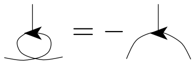
As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint representation. Using (3.97) on the $V \otimes \bar{V} \to A$ adjoint representation Clebsch-Gordan coefficients (*ie.*, generators T_i) we obtain



(3.98)

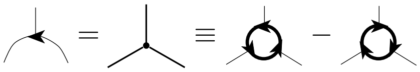
$$(T_i)_{jk} = (T_i)_a^{\ c}(T_k)_c^{\ b}(T_j)_b^{\ a} - (T_i)_a^{\ c}(T_j)_c^{\ b}(T_k)_b^{\ a} \,.$$

Our convention is to always assume that the generators T_i have been chosen hermitian. That means that ϵ_i in the expansion (3.88) are real, A is a real vector space, there is no distinction between upper and lower indices, and there is no need for arrows on the adjoint representation lines (3.84). However, the arrow on the adjoint representation generator (3.98) is necessary to define correctly the overall sign. If we interchange the two legs, the right-hand side changes sign



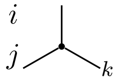
(3.99)

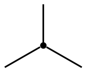
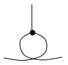
(the generators for real representations are always antisymmetric). This arrow has no absolute meaning; its direction is *defined* by (3.98). Actually, as the right-hand side of (3.98) is antisymmetric under interchange of any two legs, it is convenient to replace the arrow in the vertex by a more symmetric symbol, such as a dot:



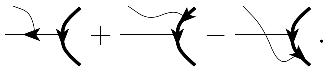
$$(T_i)_{jk} \equiv -iC_{ijk} = -\mathrm{tr}\,[T_i, T_j]T_k \,, \qquad (3.100)$$

and replace the adjoint representation generators $(T_i)_{jk}$ by the fully antisymmetric structure constants iC_{ijk}. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex

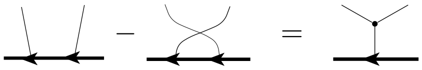
$-iC_{ijk} =$  (3.101)

 $= -$  . (3.102)

As all other clebsches, the generators must satisfy the invariance conditions (3.96):

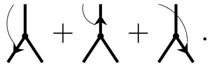
$0 = -$  .

Redrawing this a little and replacing the adjoint representation generators (3.100) by the structure constants we find that the generators obey the *Lie algebra* commutation relation

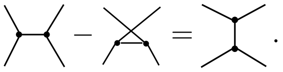


$$T_iT_j - T_jT_i = iC_{ijk}T_k \,. \tag{3.103}$$

In other words, the Lie algebra is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. The invariance condition for structure constants C_{ijk} is likewise

$0 =$  .

Rewriting this with the dot-vertex (3.100) we obtain

 . (3.104)

This is the Lie algebra commutator for the adjoint representation generators, known as the *Jacobi relation* for the structure constants

$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm} \,. \tag{3.105}$$

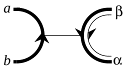
Hence the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.

> Remark **3.3** Sign convention for C_{ijk}. A word of caution about using (3.103): vertex C_{ijk} is an oriented vertex. If the arrows are reversed (matrices T_i, T_j multiplied in reverse order), the right-hand side gets an overall minus sign.

3.11 Other forms of Lie algebra commutators

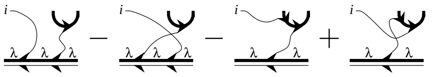
Note that in our calculations we never need explicit generators; we use instead the projection operators for the adjoint representation. For representation λ they have the form

$$(P_A)_{b}^{a}{}_{,\alpha}^{\beta} =$$



$$\begin{aligned} a,b &= 1,2,\ldots,n \\ \alpha,\beta &= 1,\ldots,d_\lambda\,. \end{aligned} \tag{3.106}$$

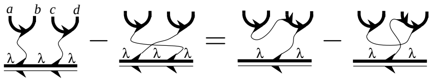
The invariance condition for a projection operator is



$$= 0\,. \tag{3.107}$$

Contracting with $(T_i)_b^a$ and defining $[d_\lambda \times d_\lambda]$ matrices $(T_b^a)_\alpha^\beta \equiv (P_A)_{b}^{a}{}_{,\alpha}^{\beta}$ we obtain

$$[T_b^a, T_d^c] = (P_A)_{b}^{a}{}_{,e}^{c} T_d^e - T_e^c (P_A)_{b}^{a}{}_{,d}^{e}$$



$$. \tag{3.108}$$

This is a common way of stating the Lie algebra conditions for the generators in an arbitrary representation λ. For example, for $U(n)$ the adjoint projection operator is simply a unit matrix (any hermitian matrix is a generator of unitary transformation, cf. chapter 8), and the right-hand side of (3.108) is given by

$$U(n), SU(n): \qquad [T_b^a, T_d^c] = \delta_b^c T_d^a - T_b^c \delta_d^a\,. \tag{3.109}$$

Another example is given by the orthogonal groups. The generators of rotations are antisymmetric matrices, and the adjoint projection operator antisymmetrizes generator indices:

$$SO(n): [T_{ab}, T_{cd}] = \frac{1}{2} \left\{ \begin{array}{c} g_{ac}T_{bd} - g_{ad}T_{bc} \\ -g_{bc}T_{ad} + g_{bd}T_{ac} \end{array} \right\}. \tag{3.110}$$

Apart from the normalization convention, these are the familiar Lorentz group commutation relations (we shal return to this in chapter 9).

3.12 Irrelevancy of clebsches

As was emphasized in sect. 3.7, an explicit choice of clebsches is highly arbitrary; it corresponds to a particular coordinatization of the d_λ-dimensional subspace V_λ. For computational purposes clebsches are largely irrelevant. Nothing that a physicist wants to compute depends on an explicit coordinatization. For example, in QCD the physically interesting objects are color singlets and all color indices are summed over: one needs only an expression for the projection operators (3.87), not for the C_λ's separately.

Again, a nice example is the Lie algebra generators T_i. Explicit matrices are often constructed (Gell-Mann λ matrices, Cartan's canonical generators); however, in any singlet they always appear summed over the adjoint representation indices, as in (3.87). The summed combination of clebsches is just the gluon projection operator, a very simple object compared with explicit T_i matrices (P_A is typically a combination of a few Kronecker deltas), and much simpler to use in explicit evaluations. As we shall show by many examples, all representation dimensions, casimirs, *etc*, are computable once the projection operators for the representations involved are known. Explicit clebsches are superfluous from the computational point of view; we use them chiefly to state general theorems, without recourse to any explicit realizations.

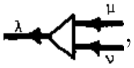
However, if one has to compute non-invariant quantities, such as subgroup embeddings, explicit clebsches might be very useful. Gell-Mann [8] invented λ_i matrices in order to embed $SU(2)$ of isospin into $SU(3)$ of the eightfold way. Cartan's canonical form for generators, summarized by Dynkin labels of a representation is a very powerful tool in the study of symmetry breaking chains [71]. The same can be achieved with decomposition by invariant matrices (a nonvanishing expectation value for a direction in the defining space defines the little group of transformations in the remaining directions), but the tensorial technology in this context is still underdeveloped compared to the canonical methods.

Chapter 4

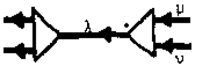
Recouplings

4.1 Couplings and recouplings

Clebsches discussed in sect. 3.7 project a tensor in $V^p \otimes \bar{V}^q$ onto a subspace λ. In practice one usually reduces a tensor step by step, decomposing a two-particle state at each step. We denote the Clebsches for $\mu \otimes \nu \rightarrow \lambda$ by



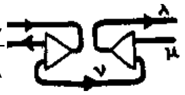
, $P_\lambda =$



. (4.1)

Here λ, μ, ν are representation labels, and the corresponding tensor indices are suppressed. Furthermore, if μ and ν are irreducible representations, the same clebsches can be used to project $\mu \otimes \bar{\lambda} \rightarrow \bar{\nu}$

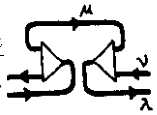
$P_\nu = \frac{d_\nu}{d_\lambda}$



, (4.2)

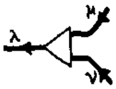
and $\nu \otimes \bar{\lambda} \rightarrow \bar{\mu}$

$P_\mu = \frac{d_\mu}{d_\lambda}$

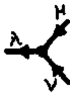


. (4.3)

Here the normalization factors come from $P^2 = P$ condition. In order to draw the projection operators in a more symmetric way, we replace clebsches by 3-vertices:

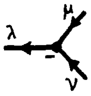


$\equiv \frac{1}{\sqrt{a_\lambda}}$

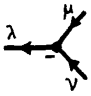


. (4.4)

In this definition one has to keep track of the ordering of the lines around the vertex. If in some context the birdtracks look better with two legs interchanged, one can use Yutsis' (1962) notation

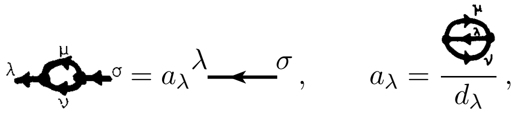


$\equiv$



. (4.5)

While all sensible clebsches are normalized by the orthonormality relation (3.75), in practice no two authors ever use the same normalization for 3-vertices (in other guises known as 3j symbols, Gell-Mann λ matrices, Cartan roots, Dirac γ matrices, *etc*, *etc*). For this reason we shall usually not fix the normalization

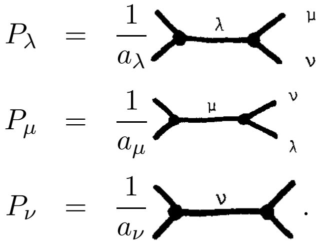

λ
μ
ν
σ
$= a_\lambda$
λ
σ
,
$a_\lambda =$
μ
λ
ν
d_λ
,

(4.6)

leaving the reader the option of substituting his favourite choice (such as $a = \frac{1}{2}$ if the 3-vertex stands for Gell-Mann $\frac{1}{2}\lambda_i$, *etc*).

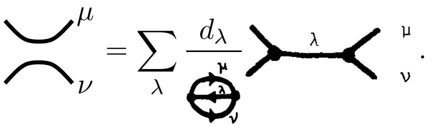
To streamline the discussion we shall drop the arrows and most of the representation labels in the remainder of this chapter - they can always easily be reinstated.

The above three projection operators now take a more symmetric form:


P_λ
$=$
$\frac{1}{a_\lambda}$
λ
μ
ν
P_μ
$=$
$\frac{1}{a_\mu}$
μ
ν
λ
P_ν
$=$
$\frac{1}{a_\nu}$
ν
.

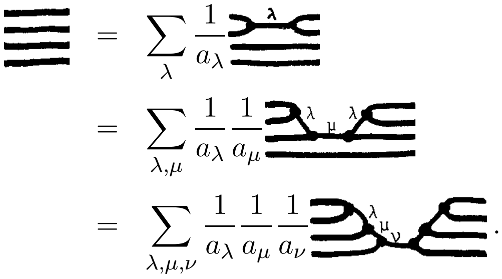
(4.7)

In terms of 3-vertices, the completeness relation (3.73) is


μ
ν
$= \sum_\lambda$
d_λ
μ
λ
ν
λ
μ
ν
.

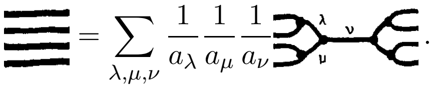
(4.8)

Any tensor can be decomposed by successive applications of the completeness relation:


$= \sum_\lambda \frac{1}{a_\lambda}$
λ
$= \sum_{\lambda,\mu} \frac{1}{a_\lambda}\frac{1}{a_\mu}$
λ
μ
λ
$= \sum_{\lambda,\mu,\nu} \frac{1}{a_\lambda}\frac{1}{a_\mu}\frac{1}{a_\nu}$
λ
μ
ν
.

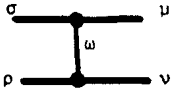
(4.9)

Hence, if we know clebsches for $\lambda \otimes \mu \to \nu$, we can also construct clebsches for $\lambda \otimes \mu \otimes \nu \otimes \ldots \to \rho$. However, there is no unique way of building up the clebsches; the above state can be equally well reduced by a different coupling scheme


$= \sum_{\lambda,\mu,\nu} \frac{1}{a_\lambda}\frac{1}{a_\mu}\frac{1}{a_\nu}$
λ
ν
μ
.

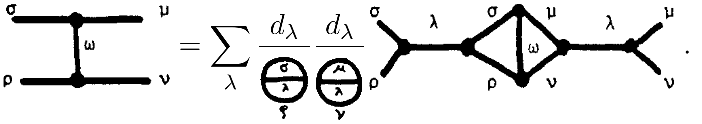
(4.10)

Consider now a process in which a particle in the representation μ interacts with a particle in the representation ν by exchanging a particle in the representation ω:



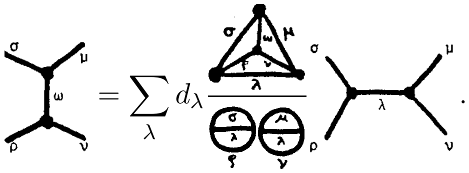
(4.11)

The final particles are in representations ρ and σ. To evaluate the contribution of this exchange to the spectroscopic levels of the $\mu - \nu$ particles system, we insert the Clebsch-Gordan series



(4.12)

By assumption λ is irreducible, so we have a recoupling relation between the exchanges in 's' and 't channels':

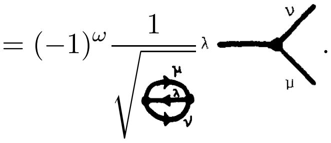


(4.13)

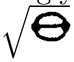
We shall refer to as $3j$ coefficients and as $6j$ coefficients, committing us to no particular normalization convention.

In atomic physics it is customary to absorb into the three-vertex and define a $3j$ symbol (Racah 1942, Wigner 1931, 1959)

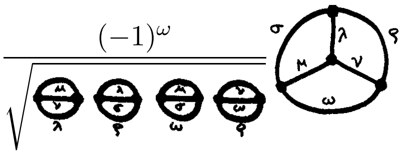
$$\begin{pmatrix} \lambda & \mu & \nu \\ \alpha & \beta & \gamma \end{pmatrix} =$$



(4.14)

Here $\alpha = 1, 2, \ldots, d_\lambda$ *etc* are indices, λ, μ, ν representation labels, and ω the phase convention. Fixing a phase convention is a waste of time, as the phases cancel in summed-over quantities. All the ugly square roots one remembers from quantum mechanics come from sticking  into $3j$ symbols. The $6j$ *symbols* (Wigner 1959) are related to our $6j$ *coefficients* by

$$\left\{ \begin{matrix} \lambda & \mu & \nu \\ \omega & \rho & \sigma \end{matrix} \right\} =$$



(4.15)

The name $3n - j$ coefficient comes from atomic physics, where a recoupling involves $3n$ angular momenta $j_1, j_2, \ldots, j_{3n}$.

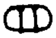
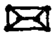
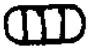
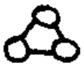
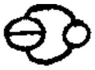
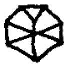
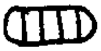
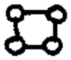
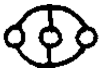
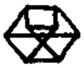
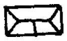
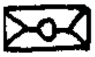
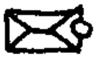
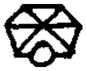
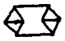
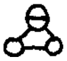
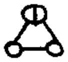
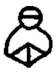
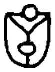
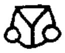
	Skeletons	Vertex insertions	Self-energy insertions	Total number
$1j$				1
$3j$				1
$6j$				2
$9j$			  	5
$12j$			  	16
			  	
			  	
			 	

Table 4.1: Topologically distinct types of Wigner $3n-j$ coefficients.

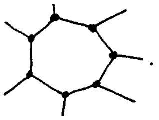
Most of the textbook symmetries of and relations between $6j$ symbols are obvious from looking at the corresponding diagrams; others follow quickly from completeness relations.

If we know the necessary $6j$'s, we can compute the level splittings due to single particle exchanges. In the next section we shall show that a far stronger claim can be made: given the $6j$ coefficients, we can compute *all* multiparticle matrix element.

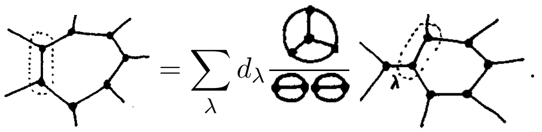
4.2 Wigner $3n-j$ coefficients

An arbitrary higher order contribution to a two-particle scattering process will give a complicated matrix element. The corresponding energy levels, cross-sections, *etc*, are expresssed in terms of scalars obtained by contracting all tensor indices; diagrammatically they look like 'vacuum bubbles', with $3n$ internal lines. The topologically distinct vacuum bubbles in low orders are given in table 4.1.

In group-theoretic literature, these diagrams are called $3n-j$ symbols, and are studied in considerable detail. Fortunately, any $3n-j$ symbol which contains as a sub-diagram a loop with, let us say, seven vertices

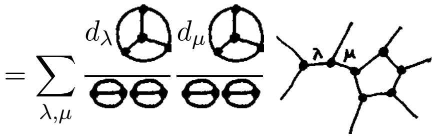


Replace the dotted pair of vertices by the cross-channel sum (4.13):


=
$\sum_\lambda d_\lambda$
λ
.

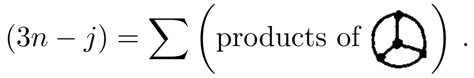
(4.16)

Now the loop has six vertices. Repeating the replacement for the next pair of vertices, we obtain a loop of length five:


=
$\sum_{\lambda,\mu}$
d_λ
d_μ
λ
μ

(4.17)

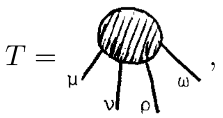
Repeating this process we can eliminate the loop altogether, producing 5-vertex-trees times bunches of $6j$ coefficients. In this way we have expressed the original $3n-j$ coefficients in terms of $3(n-1)-j$ coefficients and $6j$ coefficients. Repeating the process for the $3(n-1)-j$ coefficients, we eventually arrive at the result that


$(3n-j) = \sum$
products of
.

(4.18)

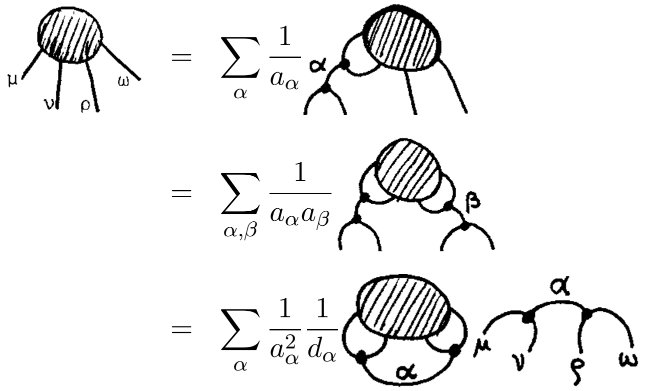
4.3 Wigner-Eckart theorem

For concreteness, consider an arbitrary invariant tensor with four indices:



(4.19)

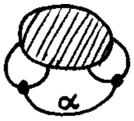
where μ, ν, ρ and ω are representation labels, and indices and line arrows are suppressed. Now insert repeatedly the completeness relation (4.8) to obtain


μ
ν
ρ
ω
=
$\sum_\alpha \frac{1}{a_\alpha}$
α
=
$\sum_{\alpha,\beta} \frac{1}{a_\alpha a_\beta}$
β
=
$\sum_\alpha \frac{1}{a_\alpha^2} \frac{1}{d_\alpha}$
α
μ
ν
ρ
ω
α

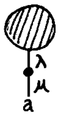
(4.20)

In the last line we have used the orthonormality of projection operators - as in (4.13) or (4.23).

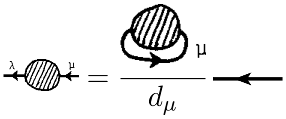
In this way any invariant tensor can be reduced to a sum over clebsches ('kinematics') weighted by 'reduced matrix elements':

$< T >_\alpha =$  (4.21)

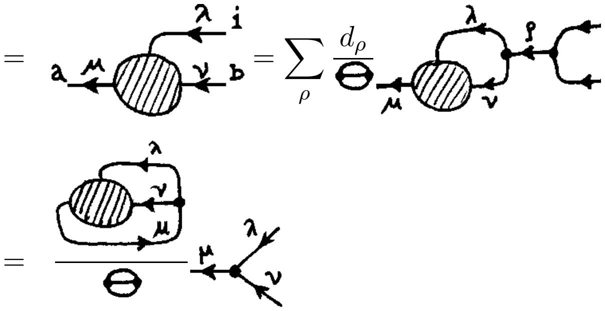
This theorem has many names, depending on how the indices are grouped. If T is a vector, then only the 1-dimensional representations (singlets) contribute

$T_a = \sum_{\lambda}^{\text{singlets}}$  (4.22)

If T is a matrix, and the representations α, μ are irreducible, the theorem is called Schur's Lemma (for an irreducible representation an invariant matrix is either zero, or proportional to the unit matrix):

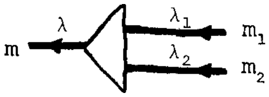
$T^{b_\mu}_{a_\lambda} =$  $\delta_{\lambda\mu}$ (4.23)

If T is an 'invariant tensor operator', then the theorem is called the Wigner (1931, 1959) - Eckart (1930) theorem:

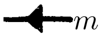
$(T_i)^b_a =$  (4.24)

(assuming that μ appears only once in $\overline{\lambda} \otimes \mu$ Kronecker product). If T has many indices, as in our original example (4.19), the theorem is ascribed to Yutsis, Levinson and Vanagas (1962). The content of all these theorems is that they reduce spectroscopic calculations to evaluation of 'vacuum bubbles' or 'reduced matrix elements' (4.21).

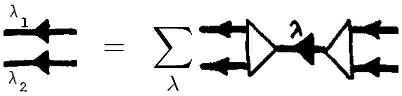
The rectangular matrices $(C_\lambda)^\alpha_\sigma$ from (3.24) do not look very much like the clebsches from the quantum mechanics textbooks; neither does the Wigner-Eckart theorem in its birdtrack version (4.22). The difference is merely a difference of notation. In the bra-ket formalism, a clebsch for $\lambda_1 \otimes \lambda_2 \to \tilde{\lambda}$ is written as

 $=< \lambda_1\lambda_2\lambda m|\lambda_1 m_1\lambda_2 m_2 >$. (4.25)

Representing the $[d_\lambda \times d_\lambda]$ representation of a group element g diagrammatically by a black triangle

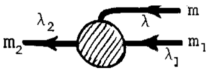
$D^{\lambda}_{m,m'}(g) =$  m' (4.26)

we can write the Clebsch-Gordan series (3.46) as



$$D^{\lambda_1}_{m_1 m_1'}(g) D^{\lambda_2}_{m_2 m_2'}(g) = \sum_{\tilde{\lambda},\tilde{m},\tilde{m}_1} < \lambda_1 m_1 \lambda_2 m_2 | \lambda_1 \lambda_2 \tilde{\lambda} \tilde{m} > D^{\tilde{\lambda}}_{\tilde{m}\tilde{m}_1}(g) < \lambda_1 \lambda_2 \tilde{\lambda} \tilde{m}_1 | \lambda_1 m_1' \lambda_2 m_2' > .$$

An 'invariant tensor operator' can be written as

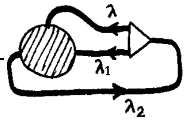
$< \lambda_2 m_2 | T^{\lambda}_{m} | \lambda_1 m_1 > =$  (4.27)

In the bra-ket formalism, the Wigner-Eckart theorem (4.24) is written as

$$< \lambda_2 m_2 | T^{\lambda}_{m} | \lambda_1 m_1 > = < \lambda \lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 > T(\lambda, \lambda_1 \lambda_2) \,, \qquad (4.28)$$

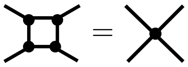
where the reduced matrix element is given by

$$T(\lambda, \lambda_1 \lambda_2) = \frac{1}{d_{\lambda_2}} \sum_{n_1,n_2,n} < \lambda n \lambda_1 n_1 | \lambda \lambda_1 \lambda_2 n_2 >< \lambda_2 n_2 | T^{\lambda}_{n} | \lambda_1 n_1 >$$

$= \frac{1}{d_{\lambda_2}}$  (4.29)

We do not find the bra-ket formalism convenient for the group-theoretic calculations that will be discussed here.

There is natural hierarchy to invariance groups that can perhaps already be grasped at this stage. Suppose that we have constructed the invariance group G_1 which preserves primitives (17.3). Adding a new primitive, let us say a quartic invariant, means that we have imposed a new constraint; only those transformations of G_1 which also preserve the additional primitive constitute G_2, the invariance group of —, ⅄, ✕. Hence G_2 is a subgroup of $G_1, G_2 \subseteq G_1$. Suppose now that you think that the primitiveness assumption is too strong, and that some quartic invariant, let us say (3.34), *cannot* be reduced to a sum of tree invariants (3.38), *ie.* it is of form

 + (rest of (3.38))

where ✕ is a new primitive, not included in the original list of primitives. By the above argument, $G_2 \subseteq G_1$. If G_1 does not exist (the invariant relations are so stringent that there is no space on which they can be realized)

Chapter 5

Permutations

The simplest example of invariant tensors are products of Kronecker deltas. On tensor spaces they represent index permutations. This is the way in which the symmetric group S_p, the group of permutations of p objects, enters into the theory of tensor representations. In this chapter, we introduce birdtracks notation for permutations, symmetrizations and antisymmetrizations and collect a few results which will be useful later on. These are the (anti)symmetrization expansion formulas (5.11) and (5.20), Levi-Civita tensor relations (5.32) and (5.34), the characteristic equations (5.54) and the invariance conditions (5.58) and (5.61).

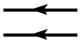
5.1 Permutations in birdtracks

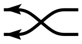
Operation of permuting tensor indices is a linear operation, and we can represent it by a $[d \times d]$ matrix:

$$\sigma_\alpha^\beta = \sigma_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}{}_{,c_q \ldots c_2 c_1}^{d_p \ldots d_1} = \delta \tag{5.1}$$

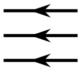
where $(\ldots)_\sigma$ stands for the desired permutation of indices. As the covariant and contravariant indices have to be permuted separately, it is sufficient to consider permutations of purely covariant tensors.

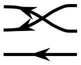
For two-index tensors, there are two permutations

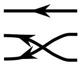
identity: $\mathbf{1}_{ab},{}^{cd} = \delta_a^b \delta_b^c =$ 

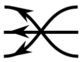
colour flip: $\sigma_{(12)ab},{}^{cd} = \delta_a^c \delta_b^d =$  (5.2)

For three-index tensors, there are six permutations

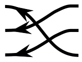
$\mathbf{1}_{a_1 a_2 a_3},{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} =$ 

$\sigma_{(12)a_1 a_2 a_3},{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} =$ 

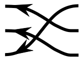
$\sigma_{(23)} =$ 

$\sigma_{(13)} =$ 

$$\sigma_{(123)} =$$



$$\sigma_{(132)} = \qquad (5.3)$$



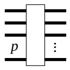
Subscripts refer to the standard cycle notation. (In the above, and for the remainder of this chapter, we shall usually omit the arrows on the Kronecker delta lines.)

5.2 Symmetrization

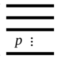
The symmetric sum of all permutations

$$S_{a_1 a_2 \dots a_p}{}^{b_p \dots b_2 b_1} = \frac{1}{p!}\left\{\delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\dots\delta^{b_p}_{a_p} + \delta^{b_1}_{a_2}\delta^{b_2}_{a_1}\dots\delta^{b_p}_{a_p} + \dots\right\}$$

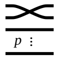
$$S =$$



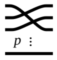
$$= \frac{1}{p!}\Bigg\{$$



$$+$$



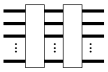
$$+$$



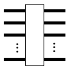
$$+ \dots\Bigg\} \qquad (5.4)$$

yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines will always denote symmetrization of the lines crossed. Factor $1/p!$ has been introduced in order that S satisfies the projection operator normalization

$$S^2 = S$$



$$=$$



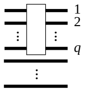
$$. \qquad (5.5)$$

A subset of indices $a_1, a_2, \dots a_q, q < p$ can be symmetrized by symmetrization matrix $S_{12\dots q}$

$$(S_{12\dots q})_{a_1 a_2 \dots a_q \dots a_p}{}^{b_p \dots b_q \dots b_2 b_1} = \qquad (5.6)$$

$$\frac{1}{q!}\left\{\delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\dots\delta^{b_q}_{a_q} + \delta^{b_1}_{a_2}\delta^{b_2}_{a_1}\dots\delta^{b_q}_{a_q} + \dots\right\}\delta^{b_q+1}_{a_q+1}\dots\delta^{b_p}_{a_p}$$

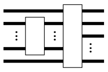
$$S_{12\dots q} =$$



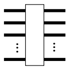
$$. \qquad (5.7)$$

Overall symmetrization also symmetrizes any subset of indices:

$$S S_{12\dots q} = S$$



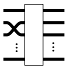
$$=$$



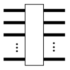
$$. \qquad (5.8)$$

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$



$$=$$



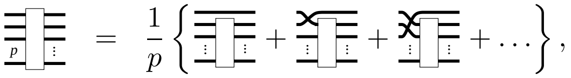
$$. \qquad (5.9)$$

Diagrammatically this means that legs can be crossed and un-crossed at will.

The definition (5.4) of the symmetrization operator as the sum of all $p!$ permutations is inconvenient for explicit calculations - a recursive definition is more useful:

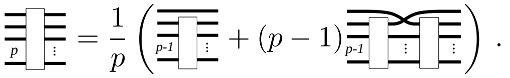
$$S_{a_1a_2\ldots a_p,}{}^{b_p\ldots b_2b_1} = \frac{1}{p}\left\{\delta_{a_1}^{b_1}S_{a_2\ldots a_p,}{}^{b_p\ldots b_2} + \delta_{a_2}^{b_1}S_{a_1a_3\ldots a_p,}{}^{b_p\ldots b_2} + \ldots\right\}$$

$$S = \frac{1}{p}\left(1+\sigma_{(21)}+\sigma_{(321)}+\ldots+\sigma_{(p\ldots321)}\right)S_{23\ldots p}$$



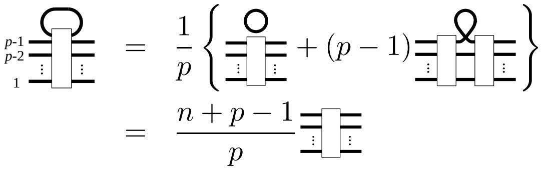
(5.10)

which involves only p terms. This equation says that if we start with the first index, we end up either with the first index, or the second index, and so on. The remaining indices are fully symmetric. Multiplying by $S_{23}\ldots p$ from the left, we obtain an even more compact recursion relation with two terms only:



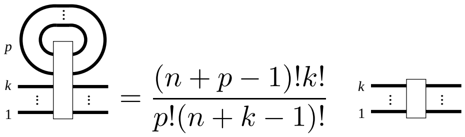
(5.11)

As a simple application, consider computation of a contraction of a single pair of indices:



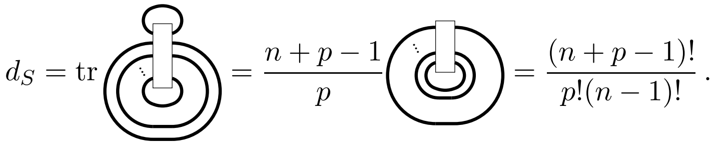
$$S_{a_pa_{p-1}\ldots a_1,}{}^{b_1\ldots b_{p-1}a_p} = \frac{n+p-1}{p}S_{a_{p-1}\ldots a_1,}{}^{b_1\ldots b_{p-1}} \,. \tag{5.12}$$

For a contraction in $(p-k)$ pairs of indices we have



(5.13)

The trace of the symmetrization operator yields the number of independent components of fully symmetric tensors:



(5.14)

For example, for two-index symmetric tensors

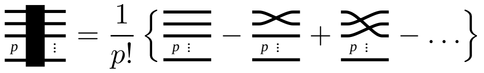
$$d_S = \frac{n(n+1)}{2} \,. \tag{5.15}$$

5.3 Antisymmetrization

The alternating sum of all permutations

$$A_{a_1a_2\ldots a_p,}{}^{b_p\ldots b_2b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p,}{}^{b_p} - \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$

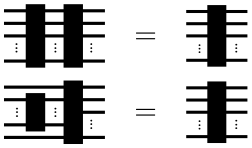
$$A =$$



(5.16)

yields the antisymmetrization projection operator A. In birdtrack notation, antisymmetrization of p lines will always be denoted by a black bar drawn across the lines. As in the previous section

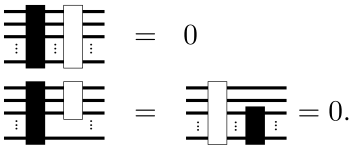
$$A^2 = A$$



(5.17)

and in addition

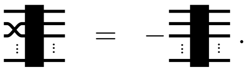
$$SA = 0$$



(5.18)

A transposition has eigenvalue -1 on the antisymmetric tensor space

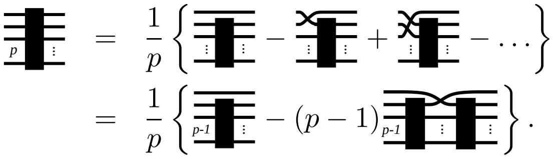
$$\sigma_{(i,i+1)}A = -A$$



(5.19)

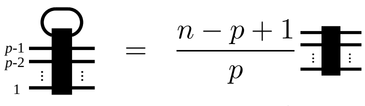
Diagrammatically this means that legs can be crossed and uncrossed at will, but with a factor of -1 for a transposition of any two neighbouring legs.

As in the case of symmetrization operators, the recursive definition is often computationally convenient



(5.20)

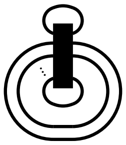
This is useful for computing contractions such as



$$A^{b_1\ldots b_{p-1}a}_{aa_{p-1}\ldots a_1} = \frac{n-p+1}{p}A^{b_1\ldots b_{p-1}}_{a_{p-1}\ldots a_1}\,. \tag{5.21}$$

The number of independent components of fully antisymmetric tensors is given by

$$d_A = \mathrm{tr}\, A =$$

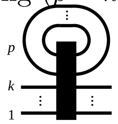


$$= \frac{n-p+1}{p}\frac{n-p+2}{p-1}\cdots\frac{n}{1}$$
$$= \frac{n!}{p!(n-p)!} n \geq p$$
$$= 0 \qquad n \leq p\,. \qquad (5.22)$$

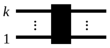
For example, for rank-two antisymmetric tensors the number of independent components is

$$d_A = \frac{n(n-1)}{2}\,. \qquad (5.23)$$

Tracing $(p-k)$ pairs of indices yields

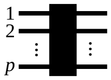


$$= \frac{k!(n-k)a5.23.b8}{p!(n-p)!}$$



$$(5.24)$$

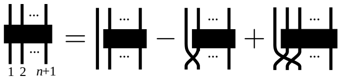
The antisymmetrization tensor $A_{a_1 a_2 \ldots}^{b_p \ldots b_2 b_1}$ has non-vanishing components only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no non-vanishing components



$$= 0 \qquad \text{if } p > n\,. \qquad (5.25)$$

This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n+1$ case

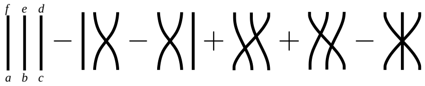
$$0 =$$



$$- \ldots\,. \qquad (5.26)$$

for example, for $n = 2$ we have

$$n = 2; 0 =$$

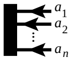


$$0 = \delta_a^f \delta_b^e \delta_c^d - \delta_a^f \delta_c^e \delta_b^d - \delta_b^f \delta_a^e \delta_c^d + \delta_b^f \delta_c^e \delta_a^d + \delta_c^f \delta_a^e \delta_b^d - \delta_c^f \delta_b^e \delta_a^d\,. \qquad (5.27)$$

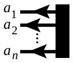
5.4 Levi-Civita tensor

An antisymmetric tensor with n indices in defining dimension n has only one independent component (by (5.22) $d_n = 1$). The clebsches (3.72) are in this case proportional to the Levi-Civita tensor

$$(C_A)_1^{a_n \ldots a_2 a_1} = C\epsilon^{a_n \ldots a_2 a_1} =$$



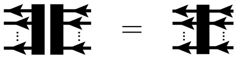
$$(C_A)_{a_1 a_2 \ldots a_n}{}^{,1} = C\epsilon_{a_1 a_2 \ldots a_n} =$$



$$(5.28)$$

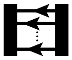
with $\epsilon^{12\ldots n} = \epsilon_{12\ldots n} = 1$. This diagrammatic notation for the Levi-Civita tensor was introduced by R. Penrose [1]. The normalization factors C are physically irrelevant[1] they adjust the phase and the overall normalization in order that the Levi-Civita tensors satisfy the projection operator (3.73) and orthonormality (3.75) conditions:

$$\frac{1}{N!}\varepsilon_{b_1 b_2 \ldots b_n}\varepsilon^{a_1 a_2 \ldots a_n} = A_{b_1 b_2 \ldots b_n},{}^{a_n \ldots a_2 a_1}$$



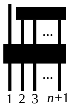
(5.32)

$$\frac{1}{N!}\varepsilon_{a_1 a_2 \ldots a_n}\varepsilon^{a_1 a_2 \ldots a_n} = \delta_{1,1} = 1$$



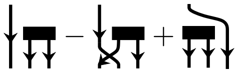
$= 1$ (5.33)

Given n dimensions we cannot label more than n indices, so Levi-Civita tensors satisfy

$0 =$ 

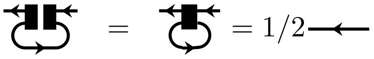
(5.34)

For example, for two colours

$n = 2 : 0 =$ 

$$0 = \delta_a^d \varepsilon_{bc} - \delta_b^d \varepsilon_{ac} + \delta_c^d \varepsilon_{ab}\,. \tag{5.35}$$

This is actually the same as the completeness relation (5.32), as can be seen by contracting (5.35) with ε_{cd} and using

$n = 2 :$ 

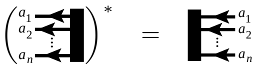
$$\varepsilon_{ac}\varepsilon^{bc} = \delta_a^b\,. \tag{5.36}$$

---

[1] With our conventions

$$C = \frac{in(n-1)/2}{\sqrt{n}!}. \tag{5.29}$$

The phase factor arises from the hermiticity condition (**??**) for clebsches (remember that indices are always read in the counterclockwise order around a diagram)



$$i^{-\phi}\varepsilon_{a_1 a_2 \ldots a_n} = i^{-\phi}\varepsilon_{a_n \ldots a_2 a_1}\,. \tag{5.30}$$

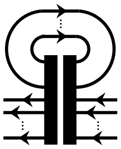
Transposing the indices

$$\varepsilon_{a_1 a_2 \ldots a_n} = -\varepsilon_{a_2 a_1 \ldots a_n} = \ldots = (-1)^{n(n-1)/2}\varepsilon_{a_n \ldots a_2 a_1} \tag{5.31}$$

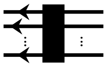
yields $\phi = n(n-1)/2$. The factor $(n!)^{1/2}$ is needed for the projection operator normalization (**??**)

This relation is one of a series of relations obtained by contracting indices in the completeness relation (5.32) and substituting (5.24):

$$\varepsilon_{a_n\ldots a_{k+1}b_k\ldots b_1}\varepsilon^{a_n\ldots A_{k+1}a_k\ldots a_1} = (n-k)!k!A_{b_k\ldots b_1,}{}^{a_1\ldots a_k}$$



$$= \frac{k!(n-l)!}{n!}$$



(5.37)

Such identities are familiar from relativistic calculations ($n = 4$):

$$\begin{aligned}
\varepsilon_{abcd}\varepsilon^{agfe} &= \delta^{gfe}_{bcd} \\
\varepsilon_{abcd}\varepsilon^{abfe} &= 2\delta^{fe}_{cd} \\
\varepsilon_{abcd}\varepsilon^{abce} &= 6\delta^{e}_{d} \\
\varepsilon_{abcd}\varepsilon^{abcd} &= 24
\end{aligned} \tag{5.38}$$

where the generalized Kronecker delta is defined by

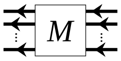
$$\frac{1}{p!}\delta^{b_1b_2\ldots b_p}_{a_1a_2\ldots a_p} = A_{a_1a_2\ldots a_p,}{}^{b_p\ldots b_2b_1}\,. \tag{5.39}$$

5.5 Determinants

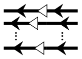
Consider a $[n^p\times n^p]$ matrix M_α^β defined by a direct product of $[n\times n]$ matrices M_a^b

$$M_\alpha^\beta = M_{a_1a_2\ldots a_p,}{}^{b_p\ldots a_2a_1} = M_{a_1}^{b_1}M_{a_2}^{b_2}\ldots M_{a_p}^{b_p}$$

$$M =$$



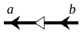
$$=$$



, (5.40)

where

$$M_a^b =$$

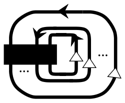


. (5.41)

The trace of the antisymmetric projection of M_α^β is given by

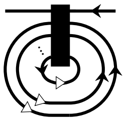
$$\mathrm{tr}_{\,p}AM = A_{abc\ldots d,}{}^{d'\ldots c'b'a'}\,M_{a'}^{a}M_{b_1}^{b}\ldots M_{d'}^{d}$$

$$=$$

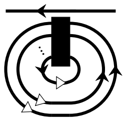


. (5.42)

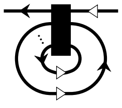
The subscript p on $\mathrm{tr}_{\,p}(\ldots)$ distinguishes the traces on $[n^p\times n^p]$ matrices M_α^β from the $[n\times n]$ matrix trace $\mathrm{tr}\,M$. To derive a recursive evaluation rule for $\mathrm{tr}_{\,p}AM$ use (5.20) to obtain



$$= \frac{1}{p}\Bigg\{$$

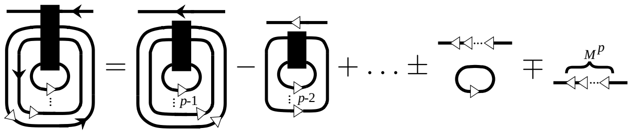


$$-(p-1)$$



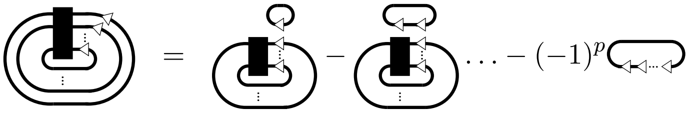
$$\Bigg\}$$ (5.43)

Iteration yields



(5.44)

Contracting with M_a^b we obtain

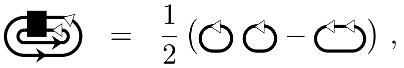


$$^{p}_{k=1}(-1)^{k-1}\,(\mathrm{tr}_{\,p-k}AM)\,\mathrm{tr}\,M^k\,. \tag{5.45}$$

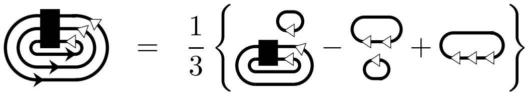
This formula enables us to compute recursively all $\mathrm{tr}_{\,p}AM$ as polynomials in traces of powers of M:

$$\mathrm{tr}_{\,0}AM = 1\,, \tag{5.46}$$

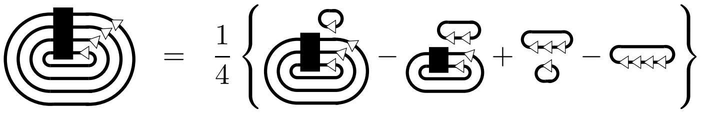
$\mathrm{tr}_{\,1}AM =$ $= \mathrm{tr}\,M$



$$\mathrm{tr}_{\,2}AM = \frac{1}{2}\left\{(\mathrm{tr}\,M)^2 - \mathrm{tr}\,M^2\right\}\,, \tag{5.47}$$



$$\mathrm{tr}_{\,3}AM = \frac{1}{3!}\left\{(\mathrm{tr}\,M)^3 - 3(\mathrm{tr}\,M)(\mathrm{tr}\,M^2) + 2\mathrm{tr}\,M^3\right\}\,, \tag{5.48}$$



$$\begin{aligned}\mathrm{tr}_{\,4}AM &= \frac{1}{4}\left\{(\mathrm{tr}\,M)^4 - 6(\mathrm{tr}\,M)^2\mathrm{tr}\,M^2\right.\\ &\left.+3(\mathrm{tr}\,M^2)^2 + 8\mathrm{tr}\,M^3\mathrm{tr}\,M - 6\mathrm{tr}\,M^4\right\}\,.\end{aligned} \tag{5.49}$$

For $p = n$ (M_a^b are $[n \times n]$ matrices) the antisymmetrized trace is the determinant

$$\mathrm{det}M = \mathrm{tr}_{\,n}AM = A_{a_1a_2\ldots a_n},^{b_n\ldots b_2b_1}\,M_{b_1}^{a_1}M_{b_2}^{a_2}\ldots M_{b_n}^{a_n}. \tag{5.50}$$

The coefficients in the above expansions are simple combinatoric numbers. A general term for $(\mathrm{tr}\,M^{\ell_1})^{\alpha_1}(\mathrm{tr}\,M^{\ell_2})^{\alpha_s}$, with α_1 loops of length ℓ_1, α_2 loops of length ℓ_2 and so on, is divided by the number of ways in which this pattern may be obtained [2]:

$$\ell_1^{\alpha_1}\ell_2^{\alpha_2}\ldots\ell_s^{\alpha_s}\alpha_1!\alpha_2!\ldots\alpha_s!\,. \tag{5.51}$$

---

[2] A.D. Kennedy, unpublished

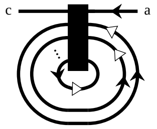
5.6 Characteristic equations

We have noted that the dimension of the antisymmetric tensor space is zero in $n < p$. This is rather obvious; antisymmetrization allows each label to be used at most once, and it is impossible to label more legs than there are labels. In terms of the antisymmetrization operator this is given by the identity

$$A = 0 \qquad if\ p > n\,. \tag{5.52}$$

This trivial identity has an important consequence: it guarantees that any $[n \times n]$ matrix satisfies a characteristic (or Hamilton-Cayley) equation. Take $p = n + 1$ and contract with $M^b_1 n$ index pairs of A:

$$A_{ca_1a_2\ldots a_n},{}^{b_n\ldots b_2b_1d}\, M^{a_1}_{b_1} M^{a_2}_{b_2} \ldots M^{a_n}_{b_n} \;=\; 0$$



$$= \; 0\,. \tag{5.53}$$

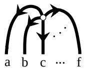
We have already expanded this in (5.44). For $p = n + 1$ we obtain the *characteristic equation*

$$\begin{aligned} 0 &= \sum_{k=0}^{n} (-1)^k (\mathrm{tr}_{\,n-k} AM) M^k\,, \\ 0 &= M^n - (\mathrm{tr}\, M) M^{n-1} + \big(\mathrm{tr}_{\,2} M^{n-2} - \ldots + (-1)^n \mathrm{det} M\big)\, \mathbf{1}\,. \end{aligned} \tag{5.54}$$

5.7 Fully (anti)symmetric tensors

As we shall often use fully symmetric and antisymmetric tensors, it is convenient to introduce special birdtrack symbols for them. We shall denote the fully *symmetric* tensors by small circles

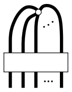
$$d_{abc\ldots f} =$$



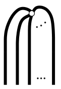
$$. \tag{5.55}$$

A symmetric tensor $d_{abc\ldots d} = d_{bac\ldots d} = d_{acb\ldots d} = \ldots$ satisfies

$$Sd \;=\; d$$



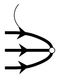
$$=$$



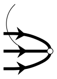
$$. \tag{5.56}$$

If this tensor is also an invariant tensor, the invariance condition (**??**) can be written as

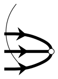
$$0 \;=$$



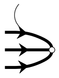
$$+$$



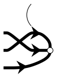
$$+$$



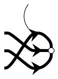
$$=$$



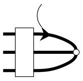
$$+$$



$$+$$

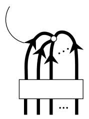


$$= p$$

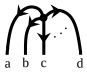


$$\text{(p = number of indices)}\,. \tag{5.57}$$

Hence the invariance condition for symmetric tensors is

$0 =$  . (5.58)

The fully *antisymmetric* tensors with *odd* numbers of legs will be denoted by black dots

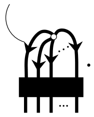
$f_{abc\ldots d} =$  . (5.59)

If the number of legs is *even*, an antisymmetric tensor is *anticyclic*

$$f_{abc\ldots d} = -f_{bc\ldots da} \,, \tag{5.60}$$

and the birdtrack notation must distinguish the first leg. A black dot is inadequate for the purpose. A bar, as for the Levi-Civita tensor (5.28), or a semicircle or similar notation fixes the problem.

For antisymmetric tensors, the invariance condition can be stated compactly as

$0 =$  . (5.61)

5.8 Young tableaux, Dynkin labels

It is standard to

Chapter 6

Casimir operators

The construction of invariance groups developed elsewhere in this monograph is self-contained, and none of the material covered in this chapter is necessary for understanding the remainder of the report. We have argued in sect. 4.2 that all relevant group-theoretic numbers are given by vacuum bubbles (reduced matrix elements, $3n-j$ coefficients, *etc*), and we have described the algorithms for their evaluation. That is all that is really needed in applications.

However, one often wants to cross-check one's calculation against the existing literature. In this chapter we discuss why and how one introduces casimirs (or Dynkin indices), we construct independent Casimir operators for the classical groups, and finally we compile values of a few frequently used casimirs.

Our approach emphasizes the role of primitive invariants in constructing representations of Lie groups. Given a list of primitives, we present a systematic algorithm for constructing invariant matrices M_i and the associated projection operators (3.45).

In the canonical, Cartan-Killing approach one faces a somewhat different problem. Instead of the primitives, one is given the generators T_i explicitly, and no other invariants. Hence the invariant matrices M_i can be constructed only from contractions of generators; typical examples are matrices

$$M_2 = \texttt{birdTrack},\ , M_4 = \texttt{birdTrack}, \ldots \tag{6.1}$$

where λ, μ could be any representations, reducible or irreducible. Such invariants are called *Casimir operators.*

What is a minimal set of Casimir operators, sufficient to reduce any representation to its irreducible subspaces? (Such bases can be useful, as the corresponding r Casimir operators label uniquely each irreducible representation by their eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_r$).

The invariance condition for any invariant matrix (3.28) is

$$0 = [T_i, M] = \texttt{birdTrack} - \texttt{birdTrack} \tag{6.2}$$

so all Casimir operators commute

$$M_2 M_4 = \texttt{birdTrack} = \texttt{birdTrack} = M_4 M_2 \tag{6.3}$$

and, according to sect. 3.5, can be used to simultaneously decompose the representation μ. If $M_1, M_2 \ldots$ have been used in the construction of projection operators (3.45), any matrix polynomial $f(M_1, M_2 \ldots)$ takes value $f(\lambda_1, \lambda_2, \ldots)$ on the irreducible subspace projected by P_i, so polynomials induce no further decompositions. Hence it is sufficient to determine the finite number of M_i's which form a polynomial basis for all Casimir operators (6.1). Furthermore, as we show in the next section, it is sufficient to restrict the consideration to the symmetrized casimirs. This observation enables us to explicitly construct in sect. 6.2 a set of independent casimirs for each classical group in sect. 6.2 . Exceptional groups are harder.

6.1 Casimirs and Lie algebra

6.2 Independent casimirs

6.3 Casimir operators

Most physicists would not refer to $\operatorname{tr} X^k$ as a casimir. Casimir's (1931) quadratic operator and its generalizations (Racah 1950) are $[d_\mu \times d_\mu]$ matrices

6.4 Dynkin indices

As we have seen so far, there are many ways of defining casimirs; in practice it is usually quicker to directly evaluate a given birdtrack diagram than to relate it to standard casimirs. Still, it is good to have an established convention, if for no other reason than to be able to cross-check one's calculation against the tabulations available in the literature.

6.5 Quadratic, cubic casimirs

As the low-order Casimir operators appear so often in physics, it is useful to list them and their relations.

6.6 Quartic casimirs

6.7 Sundry relations between quartic casimirs

6.8 Identically vanishing tensors

6.9 Dynkin labels

It is standard to identify a representation of a simple group of rank r by its Dynkin labels, a set of r integers $(a_1 a_2 \ldots a_r)$ assigned to the simple roots of the group by the Dynkin diagrams. The Dynkin diagrams, table 2.1, are the most concise summary of the Cartan-Killing construction of the semi-simple Lie Algebras. We list them here only to facilitate the identification of the representations, and do not attempt to derive or explain them. Dynkin's canonical construction is described in Slansky's (1981) review. In this report we develop only the tensor techniques for constructing representations. However, in order to help the reader connect the two approaches, we will state the correspondence between the tensor representations (identified by the Young tableaux) and the canonical representations (identified by the Dynkin labels) for each group separately, in the appropriate chapters.

Chapter 7

Group integrals

In this chapter we discuss evaluation of group-theoretic integrals of form

$$\int dg G_a^b G_c^d \ldots {G^e}_f G_h^g \tag{7.1}$$

where G_a^b is the $[n \times n]$ defining matrix representation of $g \in G_c$ and the integration is over the entire range of g. As always, we assume that G_c is a compact Lie group, and G_a^b is unitary.

The integral (7.1) is defined by two rules:

1. As the overall normalization is rarely of interest in calculation of physically interesting group-averaged quantities, we normalize the group volume to 1:

$$\int dg = 1 \tag{7.2}$$

2. How do we define $\int dg G_a^b$? The action of $g \in G_c$ is to rotate a vector x_a into $x_a^1 = G_a^{ab} x_b$

The averaging smears x in all directions, hence the second integration rule

$$\int dg G_a^b = 0, \, Gisanon - trivialrepresentationofg \tag{7.3}$$

simply states that the average of a vector is zero.

A representation is trivial if $G = 1$ for all colour rotations g. In this case no averaging is taking place, and the first integration rule (7.2) applies.

What happens if we average a pair of vectors x, y? There is no reason why a pair should average to zero; for example, we know that $|x|^2 = \sum_a x_a x_a^* = x_a x^a$ is invariant (we are considering only unitary representations), so it cannot have a vanishing average. Therefore, in general

$$\int dg G_a^b G_d^c \neq 0 \, . \tag{7.4}$$

To get a feeling for what the right-hand side looks like, let us work out a few examples

7.1 Group integrals for arbitrary representations

Let G_a^b be the defining $[nxn]$ matrix representation of $SU(n)$. The defining representation is non-trivial, so it averages to zero by (7.3). The first non-vanishing average is the integral over $G^\dagger$. $G^\dagger$ is the matrix representation of the action of g on the conjugate vector space, which we write as (3.4)

$$G_b^a = (G^\dagger)_b^a\,. \tag{7.5}$$

7.2 Characters

Physics calculations (such as lattice gauge theories) often involve group invariant quantities formed by contracting G with invariant tensors. Such invariants are of the form $\mathrm{tr}\,(hG) = H_b^a G_a^b$, where h stands for any invariant tensor. The trace of an irreducible $[d \times d]$ matrix representation λ of g is called the *character* of the representation:

$$\chi_\lambda(g) = \mathrm{tr}_{\,\lambda} G = G_a^a. \tag{7.6}$$

The character of the conjugate representation is

$$\chi^\lambda(g) = \chi_\lambda(g)* = \mathrm{tr}\, G^\dagger = (G^\dagger)_a^a. \tag{7.7}$$

Contracting (**??**) with two arbitrary invariant $[d \times d]$ matrices h_d^a and $(f^\dagger)_b^c$ we obtain the *character orthonormality relation*

$$\begin{aligned}\int dg \chi_\lambda(hg)\chi^\mu(gf) &= \delta_\lambda^\mu \frac{1}{d_\lambda}\chi_\lambda(hg^\dagger) \\ \int dg\texttt{birdTrack} &= \frac{1}{d_\lambda}\texttt{birdTrack}\left(\begin{array}{c}\lambda,\mu\text{irreducible}\\ \text{representations}\end{array}\right)\end{aligned} \tag{7.8}$$

The character orthonormality tells us that if two group variant quantities share a $GG^\dagger$ pair, the group averaging sews them into a single group invariant quantity. The replacement of G_a^b by the trace $\chi_\lambda(h^\dagger g)$ does not mean that any of the tensor index structure is lost; G_a^b can be recovered by differentiating

$$G_a^b = \frac{d}{dh_b}\chi_\lambda(h^\dagger g)\,. \tag{7.9}$$

The birtracks and the characters are two equivalent notations for evaluating group integrals.

7.3 Examples of group integrals

Chapter 8

Unitary groups

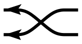
(P. Cvitanović, H. Elvang, and A. D. Kennedy)

$U(n)$ is the group of all transformations which leave the norm of a complex vector $\overline{q}q = \delta_b^a q^b q_a$ invariant. For $U(n)$ there are no other invariant tensors beyond those constructed of products of Kronecker deltas. They can be used to decompose the tensor representations of $U(n)$. For purely covariant or contravariant tensors, the symmetric group can be used to construct the Young projection operators. In sects. 8.1–8.2 we show how to do this for 2- and 3-index tensors by constructing the appropriate characteristic equations. For tensors with more indices it is easier to construct the Young projection operators directly from the Young tableaux. We use the projection operators so constructed to evaluate characters and 3-j coefficients of $U(n)$.

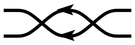
For mixed tensors reduction also involves index contractions and the symmetric group methods alone do not suffice. In sects. 8.8–8.10 the mixed $U(n)$ tensors are decomposed by the projection operator techniques introduced in chapter 3.

8.1 Two-index tensors

Consider 2-index tensors $q^{(1)} \otimes q^{(2)} \in V^2$. According to (5.1), all permutations are represented by invariant matrices. Here there are only two permutations, the identity and the color flip (5.2)

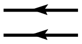
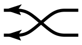
$\sigma =$  . (8.1)

The color flip satisfies

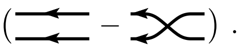
$\sigma^2 =$  $= \mathbf{1}\,,$

$$(\sigma + \mathbf{1})(\sigma - \mathbf{1}) = 0\,. \tag{8.2}$$

Hence the roots are $\lambda_1 = 1, \lambda_2 = -1$, and the corresponding projection operators (**??**) are

$P_1 = \frac{\sigma - (-1)\mathbf{1}}{1-(-1)} = \frac{1}{2}(\mathbf{1} + \sigma) = \frac{1}{2}\Big($  $+$  $\Big)\,,$ (8.3)

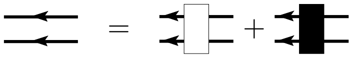
$$P_2 = \frac{\sigma - \mathbf{1}}{-1-1} = \frac{1}{2}(\mathbf{1} - \sigma) = \frac{1}{2}$$



(8.4)

We recognize the symmetrization, antisymmetrization operators (5.4), (5.16); $P_1 = S, P_2 = A$, with subspace dimensions $d = n(n+1)/2, d_2 = n(n-1)/2$. In other words, under general linear transformations the symmetric and the antisymmetric parts of a tensor x_{ab} transform separately:

$$\begin{aligned} x &= Sx + Ax\,, \\ x_{ab} &= \frac{1}{2}(x_{ab}+x_{ba}) + \frac{1}{2}(x_{ab}-x_{ba}) \end{aligned}$$



(8.5)

The Dynkin indices for the two representations follow by (**??**) from $6j's$:

$$\begin{aligned} \texttt{birdTrack} &= \frac{1}{2}\texttt{birdTrack} = \frac{N}{2} \\ \ell_1 &= \frac{2\ell}{n}\cdot d_1 + \frac{2\ell}{N}\cdot\frac{N}{2} \\ &= \ell(n+2) \end{aligned} \qquad (8.6)$$

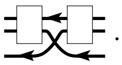
(The defining representation Dynkin index $\ell^{-1} = C_A = 2n$ was computed in sect. **??**). The result is

$$\ell = \frac{n+2}{2n}\,, \qquad \ell_2 = \frac{n-2}{2n} \qquad (8.7)$$

8.2 Three-index tensors

Three-index tensors can be reduced to irreducible subspaces by adding the third index to each of the 2-index subspaces, the symmetric and the antisymmetric. The results of this section are summarized in table 8.2. We mix the third index into the symmetric 2-index subspace using the invariant matrix

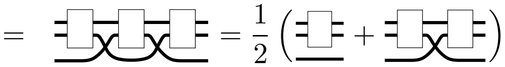
$$Q = S_{12}\sigma_{(23)}S_{12} =$$



(8.8)

Here projection operators S_{12} ensure the restriction to the 2-index symmetric subspace, and the transposition $\sigma_{(23)}$ mixes in the third index. To find the characteristic equation for Q, we compute Q^2:

$$Q^2 = S_{12}\sigma_{(23)}S_{12}\sigma_{(23)}S_{12} = \frac{1}{2}\left(S_{12} + S_{12}\sigma_{(23)}S_{12}\right)$$



$$= \frac{1}{2}S_{12} + \frac{1}{2}Q\,. \qquad (8.9)$$

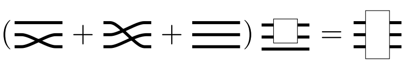
Hence Q satisfies

$$(Q-1)(Q+1/2)S_{12} = 0\,, \qquad (8.10)$$

and the corresponding Young projection operators are

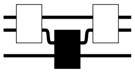
$$P_1 = \frac{Q+\frac{1}{2}\mathbf{1}}{1+\frac{1}{2}}S_{12} = \frac{1}{3}\left(\sigma_{(23)}+\sigma_{(123)}+1\right)S_{12} = S$$

$$= \frac{1}{3}$$



(8.11)

$$P_2 = \frac{Q-\mathbf{1}}{-\frac{1}{2}-1}S_{12} = \frac{4}{3}S_{12}A_{23}S_{12} = \frac{4}{3}$$

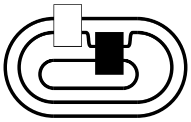


. (8.12)

Hence the symmetric 2-index subspace combines with the third index into a symmetric 3-index subspace (5.14) and a mixed symmetry subspace with dimensions

$$d_1 = \operatorname{tr} P_1 = \frac{n(n+1)(n+2)}{3!} \quad (8.13)$$

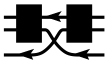
$$d_2 = \operatorname{tr} P_2 = \frac{4}{3}$$



$$= \frac{n(n^2-1)}{3}\,. \quad (8.14)$$

The antisymmetric 2-index subspace can be treated in the same way using invariant matrix

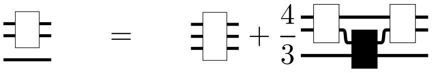
$$Q = A_{12}\sigma_{(23)}A_{12} =$$



. (8.15)

The resulting projection operators for the antisymmetric and mixed symmetry 3-index tensors are given in table 8.2. Symmetries of the subspace are indicated by the corresponding Young tableaux, table 8.2. For example, we have just constructed

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$



$$\frac{n^2(n+1)}{2} = \frac{n(n+1)(n+2)}{3!} + \frac{n(n^2-1)}{3}\,. \quad (8.16)$$

Reduction stuff

Table 8.1: Table summarizing the reduction procedure.

Old table 6.4

Table 8.2: Table summarizing the reduction procedure.

Old table 6.5

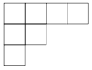
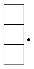
Table 8.3: Table summarizing the reduction procedure.

8.3 Young tableaux

As we have seen in the above examples, the projection operators for two-index and three-index tensors can be constructed using the characteristic equations. This, however, becomes cumbersome when applied to tensors with more than 3 indices. We now show how to construct Young projection operators for the irreducible representations of $U(n)$ directly from the Young tableaux.

8.3.1 Definitions

Partition k boxes into D subsets, so that the mth subset contains λ_i boxes. Order the partition so the set $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ fulfils $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D \geq 1$ and $\sum_{i=1}^{D} \lambda_i = k$. The diagram obtained by drawing the D rows of boxes on top of each other, left alligned, starting with λ_1, is called a *Young diagram* Y.

Examples: For $k = 4$ the ordered partitions for $k = 4$ are $[4], [3,1], [2,2], [2,1,1]$ and $[1,1,1,1]$. For the $k = 7$ partition $[4,2,1]$ the Young diagram is  and for the $k = 3$ partition $[1,1,1]$ it is .

A box in a Young diagram can be assigned a coordinate (i,j) such that $\mathrm{Y} = \{(i,j) \in \mathbb{Z}^2 | 1 \leq j \leq \lambda_i\}$. Here i label the rows and j the columns.

Inserting a number from the set $\{1, \ldots, n\}$ into every box of a Young diagram Y_λ in such a way that numbers increase when reading a column from top to bottom and numbers do not decrease when reading a row from left to right yields a *Young tableaux* Y_a. The subscript a labels different tableaux derived from a given Young diagram, that is different admissible ways of inserting the numbers into the boxes. Denoting the number in the (i,j)th box by $\tau_a(i,j)$ we have

$$\begin{array}{lll} \mathrm{Y}_a = & \{(\tau_a(i,j)) \in \{1, \ldots, n\}^k \mid & (i,j) \in \mathrm{Y}, \\ & & \tau_a(i,j+1) \geq \tau_a(i,j), \\ & & \tau_a(i+1,j) > \tau_a(i,j)\} \end{array}$$

A Young tableaux with numbers inserted as above is called a *standard arrangement.* The monotonically ordered arrangement

$$\begin{array}{lll} \mathrm{Y}_a = & \{(\tau_a(i,j)) \in \{1, \ldots, k\} \mid & (i,j) \in \mathrm{Y}, \\ & & \tau_a(i,j+1) > \tau_a(i,j), \\ & & \tau_a(i+1,j) > \tau_a(i,j)\} \end{array}$$

is called a *k-standard arrangement*.

In the following we denote by Young diagram Y a box diagram without numbers, and by Young tableaux Y_a a diagram filled with a standard arrangement. Often we simplify the notation by using Y, Z, ... to denote *both* Young diagrams and Young tableaux.

The *transpose* diagram Y^t is obtained from Y by interchanging rows and columns. For example, the transpose of $[3, 1]$ is $[2, 1, 1]$, or in the alternative labelling ()

An alternative labelling of a Young diagram is to list the number b_m of columns with m boxes as $(b_1 b_2 \ldots)$. Having k boxes we must have $\sum_{m=1}^{k} m b_m = k$. As an example we see that $[4, 2, 1]$ and $(21100\ldots)$ label the same Young diagram. Similarly for $[2, 2]$ and $(020\ldots)$. This notation is handy when considering Dynkin labels.

8.3.2 $SU(n)$ Young tableaux

We now show that a Young tableau with no more than n rows corresponds to an irreducible representation of $SU(n)$.

A k-index tensor is represented by a Young diagram with k boxes — one may think of this as a k particle state. For $SU(n)$ there are n one-particle states available and the irreducible k-particle states correspond to a Young tableaux obtained by inserting the numbers $1, \ldots, n$ into the k boxes of the Young diagrams. Boxes in a row correspond to indices that are symmetric under interchanges (symmetric multiparticle states), and boxes in a column correspond to indices antisymmetric under interchanges (antisymmetric multiparticle states).

Consider the reduction of a two-particle state, that is a two-index tensor, into a symmetric and an antisymmetric state (8.5). Using Young diagrams we would write this as

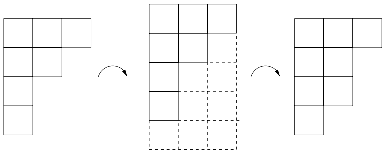
$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} = \begin{array}{|c|c|}\hline & \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \tag{8.17}$$

For the $n = 2$ case the Young tableaux of SU(2) are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \tag{8.18}$$

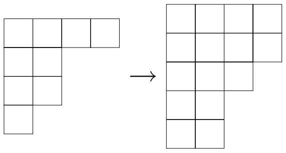
The dimension of an irreducible representation of $SU(n)$ is found by counting the number of standard arrangements. Thus for SU(2) the symmetric state is 3 dimensional whereas the antisymmetric state is 1 dimensional, in agreement with the formulas (5.4) and (5.16) for the dimensions of the symmetry operators. In section sect. 8.4.1 we shall state and prove the dimension formula for a general irreducible $U(n)$ representation.

For $SU(n)$ columns cannot contain more than n boxes, as it is impossible to antisymmetrize more than n labels. Columns of n boxes can be contracted away by means of the Levi-Civita tensor sect. **??**. Hence the highest column is of height $n - 1$, which is also the rank of $SU(n)$. Furthermore, for $SU(n)$ a column with k boxes (antisymmetrization of covariant k indices) can be converted by contraction with the Levi-Civita tensor into a column of $(n-k)$ boxes (corresponding to $(n-k)$ contravariant indices). This operation associates with each tableau a conjugate representation. Thus the *conjugate* of a $SU(n)$ Young diagram Y is constructed from the missing pieces needed to complete the rectangle of n rows:



(8.19)

That is, add squares below the diagram of Y such that the resulting figure is rectangle with height n and width of the top row in Y. Remove the squares corresponding to Y and rotate the rest by 180 degrees. The result is the conjugate diagram of Y. For example, for $SU(6)$, representation (20110)



(8.20)

has as its conjugate representation (01102). In general, the $SU(n)$ representations $(b_1 b_2 \dots b_{n-1})$ and $(b_{n-1} \dots b_2 b_1)$ are conjugate. For example, if $(10\dots0)$ stand for the defining representation, then its conjugate is represented by $(00\dots01)$, *ie.* a column of $n-1$ boxes.

We prefer to keep the conjugate representations conjugate, rather than replacing them by columns of $(n-1)$ defining representations, as this will give us $SU(n)$ expressions valid for any n.

8.3.3 Reduction of direct products

We now state the rules for reduction of direct products such as (8.17) in terms of Young diagrams:

Draw the two diagrams next to one another and place in each box of the second diagram an a_i, $i = 1, \dots, k$, such that the boxes in the first row all have a_1 in them, second row boxes have a_2 in them *etc.* The boxes of the second diagram are now added to the first diagram to create new diagrams in accordance to the rules

1. Each diagram must be a Young diagram.
2. The number of boxes in the new diagram must be equal to the sum of the number of boxes in the original two diagrams.
3. For SU(n) no diagram has more than n rows.
4. Making a journey through the diagram starting with the top row and entering each row from the right, at any point the number of a_i's encountered in any of the attached boxes must not exceed the number of previously encountered a_{i-1}'s.

5. The numbers must not increase when reading across a row from left to right.

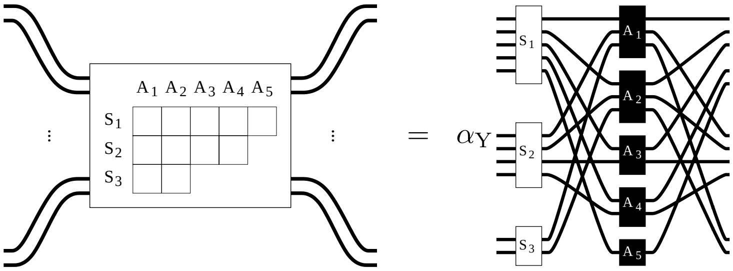
6. The numbers must decrease when reading a column from top to bottom.

The rules 4-6 ensure that states which were previously symmetrized are not antisymmetrized in the product and vice versa, and to avoid counting the same state twice.

8.4 Young projection operators

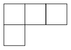
Given a Young tableau Y of $U(n)$ with an k-standard arrangement we construct the corresponding Young projection operator $P_{\rm Y}$ in birdtrack notation by identifying each box in the diagram with a directed line. The operator $P_{\rm Y}$ is a block of symmetrizers to the left of a block of antisymmetrizers, all imposed on the n lines. The blocks of symmetry operators are dictated by the Young *diagram* whereas the attachment of lines to these operators follows from the k-standard arrangement.

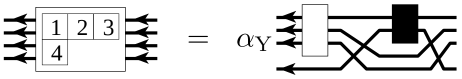
For a Young diagram Y with s rows and t columns we refer to the rows as $\rm S_1$, $\rm S_2$, ...,$\rm S_s$ and to the columns as $\rm A_1$, $\rm A_2$, ...,$\rm A_t$. Each symmetry operator in $P_{\rm Y}$ is associated to a row/column in Y, hence we label a symmetry operator after the corresponding row/column,



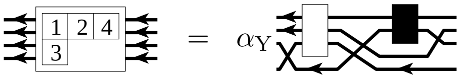
(8.21)

We denote by $|\rm S_i|$ or $|\rm A_i|$ the *length* of a row or column, respectively, that is the number of boxes it contains. Thus $|\rm A_i|$ also denotes the number of lines entering the antisymmetrizer $\rm A_i$. In the above example we have $|\rm S_1| = 5$, and $|\rm A_2| = 3$, *etc.*

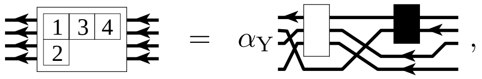
An example of the construction of the Young projection operators: The Young diagram  tells us to use one symmetrizer of length three, one of length one, one antisymmetrizer of length two, and two of length one. There are three distinct k-standard arrangements, each corresponding to a projection operator



(8.22)



(8.23)



(8.24)

where α_{Y} is a normalization constant. We use the convention that if the lines pass straight through the symmetry operators they appear in the same order as they entered. More examples of Young projection operators are given in sect. 8.5. The normalization is given by

$$\alpha_{\mathrm{Y}} = \frac{\prod_{i=1}^{s} |\mathrm{S}_i|! \prod_{j=1}^{t} |\mathrm{A}_j|!}{|\mathrm{Y}|} \tag{8.25}$$

where $|\mathrm{Y}|$ is a combinatoric number calculated by the following hook rule. For each box of the Young diagram Y write the number of boxes below and to the left of the box (including the box itself — once). Then $|\mathrm{Y}|$ is the product of the numbers in all the boxes. For instance,

$$\mathrm{Y} = \begin{array}{|c|c|c|c|} \hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 & \\ \cline{1-3} 2 & 1 & & \\ \cline{1-2} \end{array} \tag{8.26}$$

has $|\mathrm{Y}| = 6! \cdot 3$. We prove that this is the correct normalization in appendix B. The normalization only depends on the Young diagram, not the particular tableau.

For multidimensional irreducible representations the Young projection operators constructed as above will generally be different from the ones constructed from characteristic equations, see sects. 8.1–8.2, but the difference amounts to a choice of basis, so they are equivalent.

We prove in appendix B that the above construction indeed yields well-defined projection operators. Some of the properties of the Young projection operators:

- The Young projection operators are indeed *projections*, $P_{\mathrm{Y}}^2 = P_{\mathrm{Y}}$.
- The Young projection operators are *orthogonal*: If Y and Z are two different k-standard arrangement, then $P_{\mathrm{Y}} P_{\mathrm{Z}} = 0 = P_{\mathrm{Z}} P_{\mathrm{Y}}$.
- For a given k the Young projection operators constitute a complete set such that $\mathbf{1} = \sum P_{\mathrm{Y}}$, where the sum is over all k-standard arrangements Y with k boxes and $\mathbf{1}$ is the $[k \times k]$ unit matrix.

The dimension $d_{\mathrm{Y}} = \operatorname{tr} P_{\mathrm{Y}}$ of a Young projection operator P_{Y} can be calculated directly by tracing P_{Y} and expanding it using (5.11) and (5.20). In practice, this is unnecessarily laborious. Instead, we offer two simple ways of computing the dimension of an irreducible representation from its Young diagram.

8.4.1 A dimension formula

Let $f_{\mathrm{Y}}(n)$ be the polynomial in n obtained from the Young diagram Y by multiplying the numbers written in the boxes of Y, according to the following rules:

1. The upper left box contains an n.

2. The numbers in a row increases by one when reading from left to right.
3. The numbers in a column decrease by one when reading from top to bottom.

Hence, if k is the number of boxes in Y, $f_{\mathrm{Y}}(n)$ is a polynomial in n of degree k.

For U(n) the dimension of the irreducible representation labelled by the Young diagram Y is

$$d_{\mathrm{Y}} = \frac{f_{\mathrm{Y}}(n)}{|Y|} \tag{8.27}$$

EXAMPLE: With Y = [4,2,1] we have

$$f_{\mathrm{Y}}(n) = \begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n \\ \cline{1-2} n-2 \\ \cline{1-1}\end{array} = n^2(n^2-1)^2(n^2-4)(n+3),$$

and

$$|\mathrm{Y}| = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} = 144.$$

hence

$$d_{\mathrm{Y}} = \frac{n^2(n^2-1)^2(n^2-4)(n+3)}{144} \tag{8.28}$$

This dimension formula is derived in appendix B. Next we give an intuitive interpretation of what this formula means.

8.4.2 Dimension as the number of strand colorings

The dimension of a Young projection operator P_{Y} of $SU(n)$ can be calculated by counting the number of distinct ways in which the trace diagram of a Young projection operator can be colored.

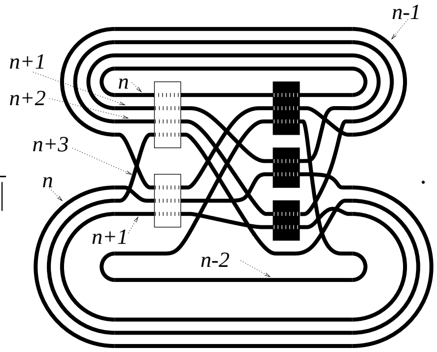
Draw the trace of the Young projection operator. Each line is strand, a closed path which we draw as passing straight through the symmetry operators. Order the paths in accordance to the k-standard arrangement (see example). The lines are colored in this order. Having n colors we can color the first line in n different ways.

Rule 1: If a path, which could be colored in k ways, enters an antisymmetrizer, the lines below it can be colored in $k-1$, $k-2$, ... ways.

Rule 2: If a path, which could be colored in k ways, enters a symmetrizer, the lines below it can be colored in $k+1$, $k+2$, ... ways.

Label each path with the number of ways it can be colored. The number of ways to color the trace diagram is the product of all the factors obtained above; but this is simply $f_{\mathrm{Y}}(n)$ defined in sect. 8.4.1. An example:

$$d_{\mathrm{Y}} = \frac{f_{\mathrm{Y}}(n)}{|\mathrm{Y}|} = \mathrm{tr}\,\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&7\\\cline{1-3} 8\\\cline{1-1}\end{array} = \frac{1}{|\mathrm{Y}|}$$



. (8.29)

8.5 Reduction of tensor products

We now apply the rules for decomposition of direct products of Young diagrams/tableaux to several explicit examples. We use the tableaux to compute the dimensions and construct the Young projection operators. We have already treated the decomposition of the two-index tensor into the symmetric and the anti-symmetric tensors, but we shall reconsider the three-index tensor, since the projection operators will be different from those derived from the characteristic equations in sect. 8.2.

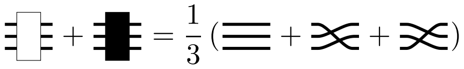
8.5.1 Three- and four-index tensors

According to the rules in sect. 17.10, the three-index tensor reduces to

$$\boxed{1}\otimes\boxed{2}\otimes\boxed{3} = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right)\otimes\boxed{3} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}. \qquad (8.30)$$

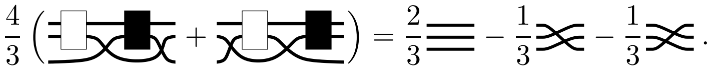
The corresponding dimensions and Young projection operators are given in table 8.5.1. For simplicity, we neglect the arrows on the lines where this leads to no confusion.

Let us check the completeness by an explicit computation. In the sum of the fully symmetric and the fully antisymmetric tensors all the odd permutions cancel, and we are left with



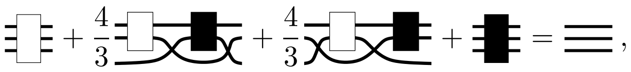
(8.31)

Expanding the two tensors of mixed symmetry, we obtain



(8.32)

Adding (8.31) and (8.32) we get



(8.33)

varifying the completeness relation.

For four-index tensors the decomposition is performed as in the three-index case, resulting in table 8.5.1.

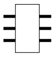
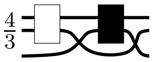
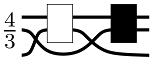
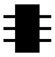
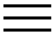
Y_a	d_{Y_a}	P_{Y_a}
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline\end{array}$	$\frac{n(n+1)(n+2)}{6}$	
$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$	$\frac{n(n^2-1)}{3}$	
$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$	$\frac{n(n^2-1)}{3}$	
$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$	$\frac{(n-2)(n-1)n}{6}$	
$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$	n^3	

Table 8.4: Reduction of three-index tensor. The bottom row is the direct sum of the Young tableaux, the sum of the dimensions, and the sum of the projection operators (completeness).

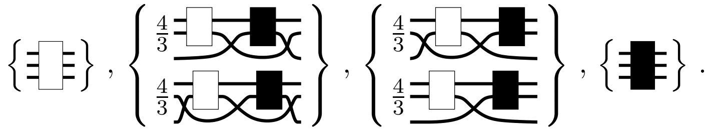
8.5.2 Basis vectors

The Young projection operators as constructed above are also projection operators of the symmetric group S_n. If we let Y be a Young tableau labelling an irreducible representation of S_n, the dimension of the representation is

$$d_{\mathrm{Y}} = \frac{n!}{|\mathrm{Y}|}. \tag{8.34}$$

For the two-index tensors we see that application of the projection operators project any group element to the subspace of the projection in question.

For the three-index tensors the result is not as simple as that, because the S_n representation $\begin{array}{|c|c|}\hline \ &\ \\\hline \ &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$ is two-dimensional. Instead, when the three-index projection operators are applied from the right, the group elements of S_n are projected to the set



(8.35)

of basis vectors. For higher-index tensors there are similar sets of basis vectors. The number of components in each basis vector is the dimension of the projection operator in S_n.

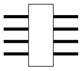
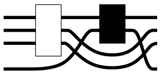
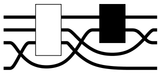
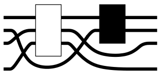
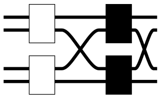
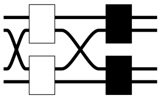
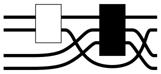
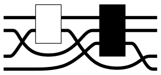
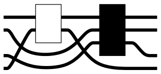
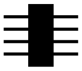
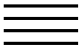
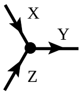
Y_a	d_{Y_a}	P_{Y_a}
$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}$	$\frac{n(n+1)(n+2)(n+3)}{24}$	
$\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$ 
$\begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$ 
$\begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$ 
$\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$	$\frac{n^2(n^2-1)}{12}$	$\frac{4}{3}$ 
$\begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}$	$\frac{n^2(n^2-1)}{12}$	$\frac{4}{3}$ 
$\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}$	$\frac{(n-1)n(n+1)(n+2)}{8}$	$\frac{3}{2}$ 
$\begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}$	$\frac{(n-2)(n-1)n(n+1)}{8}$	$\frac{3}{2}$ 
$\begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}$	$\frac{(n-2)(n-1)n(n+1)}{8}$	$\frac{3}{2}$ 
$\begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$	$\frac{(n-2)(n-1)n(n+1)}{8}$	
$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{4}$	n^4	

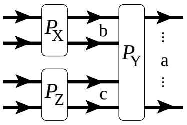
Table 8.5: Reduction of four-index tensor.

8.6 3-j symbols

The $SU(n)$ three-vertex is written



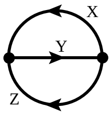
$$= \sqrt{\alpha_{\mathrm{X}}\alpha_{\mathrm{Y}}\alpha_{\mathrm{Z}}}$$



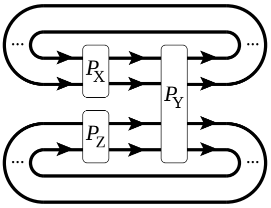
(8.36)

in terms of the Young projection operators P_{X}, P_{Y}, and P_{Z}. If $b+c \neq a$ the vertex vanishes; if $a = b+c$ the vertex might be non-vanishing. The overall normalization is arbitrary, but $\sqrt{\alpha_{\mathrm{X}}\alpha_{\mathrm{Y}}\alpha_{\mathrm{Z}}}$ is a natural choice, see (8.25).

A 3-j consists of two fully contracted three-vertices. We therefore have



$$= \alpha_{\mathrm{X}}\alpha_{\mathrm{Y}}\alpha_{\mathrm{Z}}$$

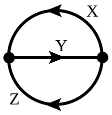


(8.37)

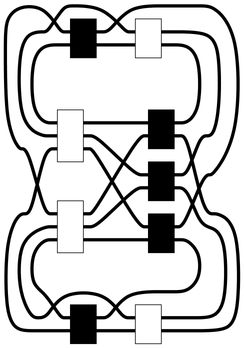
which we write $\mathrm{tr}\,(\mathrm{X} \oplus \mathrm{Z}) \otimes \mathrm{Y}$. As an example, take

$$\mathrm{X} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\,, \quad \mathrm{Y} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\,, \qquad \text{and} \qquad \mathrm{Z} = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 6 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\,.$$

Then



$$= \frac{4}{3} \cdot 2 \cdot \frac{4}{3}$$



For economy of notation we omit the arrows on the Kronecker delta lines.

8.6.1 Evaluation by direct expansion

The simplest 3-j's to evaluate are $\mathrm{tr}\,(\square \oplus \square) \otimes \square\square$ and $\mathrm{tr}\,(\square \oplus \square) \otimes \begin{array}{c}\square\\[-4pt]\square\end{array}$.

Any $SU(n)$ 3-j may be evaluated by direct expansion of the symmetry operators, but the resulting number of terms grows combinatorially with the total number of boxes in the Young diagram Y, making brute force expansion an unattractive method.

There is a slightly less brutal expansion method. Expanding one symmetry operator may lead to simplifications of the diagram, for instance by using rules such as (5.8), (5.9), (5.18), and (5.19). An example of the application of this method is given in (Elvang).

If Y is a Young diagram with a single row or a single column it is easily seen that the 3-j $\mathrm{X} \otimes \mathrm{Y} \otimes \mathrm{Z}$ is either 0 or d_{Y}.

8.6.2 Application of the negative dimension theorem

An $SU(n)$ invariant scalar is a fully contracted object (vacuum bubble) consisting of Kronecker deltas and Levi-Civita symbols. Since there are no external legs, the Levi-Civitas appear only in pairs, making it possible to combine them into antisymmetrizers. In birdtrack notation, an $SU(n)$ invariant scalar is therefore a vacuum bubble graph built only from symmetrizers and antisymmetrizers.

The negative dimensionality theorem for $SU(n)$ states that for any $SU(n)$ invariant scalar exchanging symmetrizers and antisymmetrizers is equivalent to replacing n by $-n$:

$$\mathrm{SU}(n) = \overline{\mathrm{SU}}(-n)\ . \tag{8.38}$$

where the bar on $\overline{\mathrm{SU}}$ indicates transposition, *ie.* exchange of symmetrizations and antisymmetrizations. The theorem also applies to $U(n)$ invariant scalars, since the only difference between $U(n)$ and $SU(n)$ is the invariance of the Levi-Civita tensor in $SU(n)$. The proof of this theorem is given in chapter 12.

For the dimensions of the Young projection operators we have $d_{\mathrm{Y}^t}(n) = d_{\mathrm{Y}}(-n)$ by the negative dimensionality theorem, where Y^t is the transpose of the k-standard arrangement Y; hence it suffices to compute the dimension once, either for Y or Y^t.

Now for k-standard arrangements X, Y, and Z, compare the diagram of $\mathrm{X}^t \otimes \mathrm{Y}^t \otimes \mathrm{Z}^t$ to that of $\mathrm{X} \otimes \mathrm{Y} \otimes \mathrm{Z}$. The diagrams are related by a reflection in a vertical line, reversal of the arrows on the lines, and interchange of symmetrizers and antisymmetrizers. The first two operations do not change the value of the diagram, hence the value of $\mathrm{X}^t \otimes \mathrm{Y}^t \otimes \mathrm{Z}^t$ is the value of $\mathrm{X} \otimes \mathrm{Y} \otimes \mathrm{Z}$ with $n \leftrightarrow -n$ (and possibly an overall sign). Hence it is sufficient to calculate approximately half of all 3-j's.

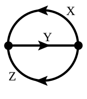
Challenge

We have seen that there is a coloring algorithm for the dimensionality of the Young projection operators. Find a coloring algorithm for the 3-j's of $SU(n)$ — *open question.*

8.6.3 A sum rule for 3-j's

Let Y be a k-standard arrangement with k boxes and let Λ be the set of all k-standard arrangements and Λ_p the set of k-standard arrangements with p boxes. Then

$$\sum_{(X,Z)\in\Lambda}$$

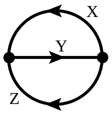


$$= \quad (k-1)d_{\mathrm{Y}}. \tag{8.39}$$

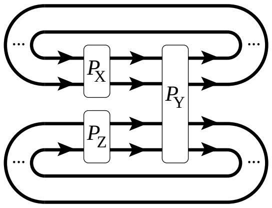
First of all, the sum is well-defined, *ie.* finite, because the 3-j is non-vanishing only if the number of boxes in X and Z add up to k, and this only happens for finitely many tableaux.

To prove this, recall that the Young projection operators constitute a complete set, $\sum_{X\in\Lambda_p} P_{\mathrm{X}} = \mathbf{1}$, where $\mathbf{1}$ is the $[p \times p]$ unit matrix. Hence,

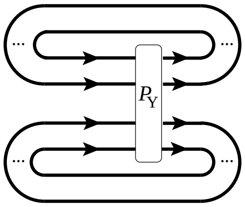
$$\sum_{X,Z\in\Lambda}$$



$$= \sum_{m=1}^{k-1} \sum_{\substack{X\in\Lambda_m \\ Z\in\Lambda_{k-m}}}$$



$$= \sum_{m=1}^{k-1}$$



$$= \sum_{m=1}^{k-1} d_{\mathrm{Y}} = (k-1) d_{\mathrm{Y}}\,. \tag{8.40}$$

This sum rule offers a cross-check on the individual 3-j calculations.

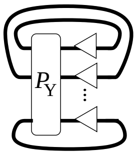
8.7 Characters

Now that we have explicit Young projection operators we should be able to compute any $SU(n)$ invariant scalar. As an example we will consider calculations of characters of $SU(n)$.

Given an irreducible representation we have the corresponding Young tableau k-standard arrangement Y, which enable us to calculate the character $\chi_{\mathrm{Y}}(M) = \mathrm{tr}_{\mathrm{Y}} M$, where M is a unitary $[n \times n]$ matrix.

Diagrammatically we shall denote M as $M_{ij} = j$ —◁— i. Then

$$\chi_{\mathrm{Y}}(M) = \tag{8.41}$$



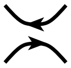
Expanding the symmetry operators and collecting terms we find

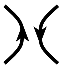
$$\chi_{\mathrm{Y}}(M) = \sum_{m=0}^{k} c_m (\mathrm{tr}\, M)^m \mathrm{tr}\, M^{k-m}\,, \tag{8.42}$$

where k is the number of boxes in Y and the c_m's are coefficients of the expansion.

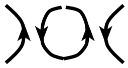
8.8 Mixed two-index tensors

As the next trivial example consider mixed tensors $q^{(1)} \otimes \overline{q}^{(2)} \in V \otimes \overline{V}$. The Kronecker delta invariants are the same as in sect. 8.1, but now they are drawn differently (we are looking at a "cross channel"):

identity: $\mathbf{1} = 1^{b\,c}_{a,d} = \delta^c_a \delta^b_d =$  ,

trace: $T = T^{b\,c}_{a,d} = \delta^b_a \delta^c_d =$  . (8.43)

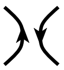
The matrix has a trivial characteristic equation

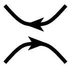
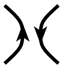
$T^2 =$  $= nT\,,$ (8.44)

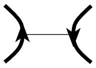
with roots $\lambda_1 = 0, \lambda_2 = n$;

$$T(T-n) = 0\,. \tag{8.45}$$

The corresponding projection operators (**??**) are

$P_1 = \frac{1}{n}T = \frac{1}{n}$  , (8.46)

$P_2 = 1 - \frac{1}{n}T =$  $- \frac{1}{n}$  (8.47)

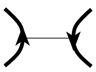
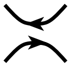
$=$  (8.48)

with dimensions $d_1 = \operatorname{tr} P_1 = 1, d_2 = \operatorname{tr} P_2 = n^2 - 1$. (9.14) is the projection operator for the adjoint representation of $SU(n)$. In this way the invariant matrix T has resolved the space of tensors $x^a_b \in V \otimes \overline{V}$ into

$$\text{singlet:}\quad P_1 x = \frac{1}{n} x^c_c \delta^b_a\,, \tag{8.49}$$

$$\text{traceless part:}\quad P_2 x = x^b_a - \left(\frac{1}{n} x^c_c\right) \delta^b_a\,. \tag{8.50}$$

Both projection operators obviously leave δ^a_b invariant, so the generators of the unitary transformations are given by their sum

$U(n) : \frac{1}{a}$  $=$  , (8.51)

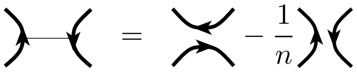
and the dimension of $U(n)$ is $N \operatorname{tr} P_A = \delta^a_a \delta^b_b = n^2$. If we extend the list of primitive invariants from the Kronecker delta to the Kronecker delta and the Levi-Civita tensor (5.28), the singlet subspace does not satisfy the invariance condition (5.61)

$$\texttt{birdTrack} \neq 0\,. \tag{8.52}$$

For the traceless subspace (8.48), the invariance condition is

$$\texttt{birdTrack} - \frac{1}{n}\texttt{birdTrack} = 0\,. \tag{8.53}$$

This is the same relation as (5.26). (Expand the antisymmetrization operator using (5.20) so the invariance condition is satisfied.) The adjoint representation is given by

$SU(n) : \frac{1}{a}$ 

$$\frac{1}{a}\,(T_i)_b^{\,a}\,(T_i)_c^{\,d} = \delta_c^a\delta_b^d - \frac{1}{n}\delta_b^a\delta_c^d\,. \tag{8.54}$$

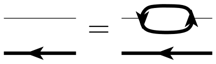
$SU(n)$ is, by definition, the invariance group of the Levi-Civita tensor (hence 'special') and the Kronecker delta (hence 'unitary'), and its dimension is $N = n^2 - 1$. The defining representation Dynkin index follows from (**??**) and (**??**)

$$\ell^{-1} = 2n \tag{8.55}$$

(This was evaluated in the example of sect. **??**). The Dynkin index for the singlet representation (8.49) vanishes identically (as it does for any singlet representation).

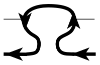
8.9 Mixed defining × adjoint tensors

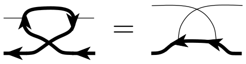
In this and the following section we generalize the reduction by invariant matrices to spaces other than the defining representation. Such techniques will be very useful later on, in our construction of the exceptional Lie groups. We consider the defining × adjoint tensor space as a projection from $V \otimes \overline{V}$ space:



(8.56)

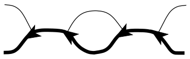
The following two invariant matrices acting on $V^2 \otimes \overline{V}$ space contract or interchange defining representation indices:

R =  (8.57)

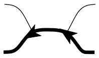
Q = 

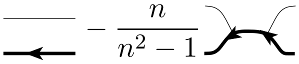
(8.58)

R projects onto the defining space, and satisfies characteristic equation

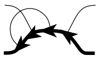
$R^2 =$  $= \frac{n^2-1}{n} R\,.$ (8.59)

The corresponding projection operators are

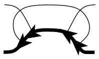
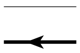
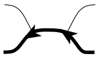
$P_1 = \frac{n}{n^2-1}$  ,

$P_4 =$  . (8.60)

Q takes a single eigenvalue on the P_1 subspace

$$QR =$$  $$= -\frac{1}{n}R . \qquad (8.61)$$

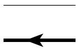
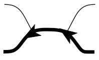
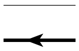
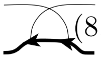
Q^2 is computed by inserting the adjoint representation projection operator (8.54):

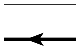
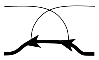
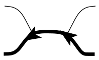
$$Q^2 =$$  $$=$$  $$- \frac{1}{n}$$  $$. \qquad (8.62)$$

The projection on the P_4 subspace yields the characteristic equation

$$P_4(Q^2 - 1) = 0 , \qquad (8.63)$$

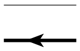
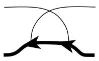
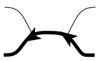
with the associated projection operators

$$P_2 = \frac{1}{2}P_4(1+Q) = \frac{1}{2}\Bigg($$  $$- \frac{n}{n^2-1}$$  $$\Bigg)\Bigg($$  $$+$$  $$\Bigg) \qquad (8.64)$$

$$= \frac{1}{2}\Bigg($$  $$+$$  $$- \frac{1}{n+1}$$  $$\Bigg) , \qquad (8.65)$$

$$(8.66)$$

$$P_3 = \frac{1}{2}P_4(1-Q)$$

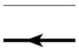
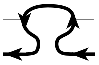
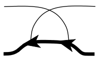
$$= \frac{1}{2}\Bigg($$  $$-$$  $$- \frac{1}{n-1}$$  $$\Bigg) .$$

The dimensions of the two subspaces are computed by taking traces of their projection operators:

$$\begin{aligned} d_2 &= \mathrm{tr}\, P_2 = \texttt{birdTrack} = \frac{1}{2}\left(\texttt{birdTrack} + \texttt{birdTrack} - \frac{1}{n+1}\texttt{birdTrack}\right) \\ &= \frac{1}{2}\left(nN + N - \frac{1}{n+1}N\right)\left(n+1-\frac{1}{n+1}\right) \\ &= \frac{(n-1)n(n+2)}{2} , \qquad (8.67) \end{aligned}$$

and similarly for d_3. This is tabulated in table 8.5.

Mostly for illustration purposes, let us now perform the same calculation by utilizing the algebra of invariants method outlined in sect. 3.3. A possible basis set picked from the $V \otimes A \to V \otimes A$ linearly independent tree invariants consists of

$$(\mathbf{e}, R, Q) = \Bigg($$  $$,$$  $$,$$  $$\Bigg) . \qquad (8.68)$$

The multiplication table (3.39) has been worked out in (8.59), (8.61) and (8.62). For example, the $(t_\alpha)_\beta{}^\gamma$ matrix representation for $Q\mathbf{t}$ is

$$\sum_{\gamma\in\mathcal{T}}(Q)_\beta{}^\gamma\mathbf{t}_\gamma = Q\begin{pmatrix}\mathbf{e}\\ R\\ Q\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & -1/n & 0\\ 1 & -1/n & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ R\\ Q\end{pmatrix} \qquad (8.69)$$

and similarly for R. In this way we obtain the [3×3] matrix representation of the algebra of invariants

$$\{\mathbf{e}, R, Q\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & n-\frac{1}{n} & 0 \\ 0 & -1/n & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/n & 0 \\ 1 & -1/n & 0 \end{pmatrix} \right\} \tag{8.70}$$

From (8.59) we already know that the eigenvalues of R are $\{0, 0, n-\frac{1}{n}\}$. The last eigenvalue yields the projection operator $P_1 = \frac{n}{n^2-1}$, but the projection operator P_4 yields a 2-dimensional degenerate representation. Q has 3 distinct eigenvalues $\{-\frac{1}{n}, 1, -1\}$ and is thus more interesting; the corresponding projection operators fully decompose the $V \otimes A$ space. $-\frac{1}{n}$ eigenspace projection operators is again P_1, but P_4 is split into 2 subspaces, verifying (8.66) and (8.64):

$$\begin{aligned} P_2 &= \frac{(Q+\mathbf{1})(Q+\frac{1}{n}\mathbf{1})}{(1+1)(1+\frac{1}{n})} = \frac{1}{2}\left(\mathbf{1} + Q - \frac{1}{n+1}R\right) \\ P_3 &= \frac{(Q-\mathbf{1})(Q+\frac{1}{n}\mathbf{1})}{(-1-1)(-1+\frac{1}{n})} = \frac{1}{2}\left(\mathbf{1} - Q - \frac{1}{n-1}R\right) . \end{aligned} \tag{8.71}$$

We see that the matrix representation of the algebra of invariants is a fine tool for implementing the full reduction, and perhaps easies to implement as computation than out ans out birtracks manipulation.

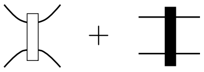
To summarize the invariant matrix R projects out the one-particle subspace P_1. The particle exchange matrix Q splits the remainder into the irreducible particle-adjoint subspaces P_2 and P_3.

8.10 Two-index adjoint tensors

Consider the Kronecker product of two adjoint representations. We want to reduce the space of tensors $x_{ij} \in A \otimes A$, with $i = 1, 2, \ldots d_A$. The first decomposition is the obvious decomposition (9.14) into the symmetric and antisymmetric subspaces

$$1 = S + A$$

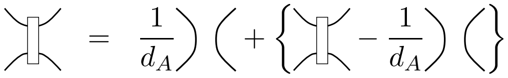
`birdTrack` =



. (8.72)

As the adjoint representation is real, the symmetric part can be split into the trace and the traceless part, as in (**??**)

$$S = \frac{1}{d_A}T + P_S$$



. (8.73)

8.11 Casimirs for the fully symmetric representations of $SU(n)$

In this section we carry out a few explicit birdtrack Casimir evaluations.

8.12 $SU(n)$, $U(n)$ equivalence in adjoint representation

8.13 Dynkin labels for $SU(n)$ representations

Chapter 9

Orthogonal groups

Orthogonal group $SO(n)$ is the group of transformations which leave invariant a symmetric quadratic form $(q, q) = g_{\mu\nu} q^\mu q^\nu$:

$$g_{\mu\nu} = g_{\nu\mu} = \texttt{birdTrack} \tag{9.1}$$

If (q, q) is an invariant, so is its complex conjugate $(q, q)* = g^{\mu\nu} q^\mu q^\nu$, and

$$g^{\mu\nu} = g^{\nu\mu} = \mu\texttt{birdTrack}\nu \tag{9.2}$$

is also an invariant tensor. Matrix $A_\mu^\nu \equiv g_{\mu\sigma} g^{\sigma\nu}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional representation. A convenient normalization is

$$\begin{aligned} g_{\mu\sigma} g^{\sigma\nu} &= \delta_\mu^\nu \\ \texttt{birdTrack} &= \texttt{birdTrack} \end{aligned} \tag{9.3}$$

As the indices can be raised and lowered at will, nothing is gained by keeping arrows. Our convention will be to perform all contractions with metric tensors with upper indices, and omit the arrows and the open dots:

$$g^{\mu\nu} \equiv \texttt{birdTrack}\,. \tag{9.4}$$

All other tensors will have lower indices. For example, Lie group generators $(T_i)_\mu^\nu$ from (**??**) will be replaced by

$$\begin{aligned} (T_i)_{\mu\nu} &= \texttt{birdTrack} \\ &\equiv \texttt{birdTrack}\,. \end{aligned} \tag{9.5}$$

The invariance condition (**??**) for the metric tensor

$$\begin{aligned} \texttt{birdTrack} + \texttt{birdTrack} &= 0 \\ (T_i)_\mu^\sigma g_{\sigma\nu} + (T_i)_\nu^\sigma g_{\mu\sigma} &= 0 \end{aligned} \tag{9.6}$$

becomes in this notation a statement that the $SO(n)$ generators are antisymmetric:

$$\begin{aligned} \texttt{birdTrack} + \texttt{birdTrack} &= 0 \\ (T_i)_{\mu\nu} &= -(T_i)_{\nu\mu} \end{aligned} \tag{9.7}$$

Our analysis of the representations of $SO(n)$ will depend only on the existence of a symmetric metric tensor and its invertability, and not on its eigenvalues. The resulting Clebsch-Gordan series applies both to the compact $SO(n)$, and non-compact orthogonal groups such as the Minkowski group $SO(1,3)$. In this chapter, we outline the construction of $SO(n)$ tensor representations. Spinor representations will be taken up in chapter **??**

9.1 Two-index tensors

9.2 Three-index tensors

9.3 Mixed defining × adjoint tensors

9.4 Two-index adjoint tensors

9.5 Gravity tensors

In a different application of the birdtracks, we now change the language, and construct 'irreducible rank-four gravity curvature tensors". (Birdtracks for Young projection operators had originally been invented by Penrose (1971) in this context.) The Riemann-Christoffel curvature tensor has the following symmetries (Weinberg 1972):

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta'} \tag{9.8}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\gamma\delta\alpha\beta'} \tag{9.9}$$

$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\beta} + R_{\gamma\alpha\beta\delta} = 0 \, . \tag{9.10}$$

9.6 Dynkin labels of $SO(n)$ representations

In general, one has to distinguish between the odd and the even dimensional orthogonal groups, as well as their spinor and ono-spinor representations.

For $SO(2r+1)$ representations there are r Dynkin labels $(a_1 a_2 \ldots a_{r-1} Z)$. If Z is odd, the representation is spinor; if Z is even, it is tensor.

In this chapter, we study only the tensor representations. For the tensor representations, the corresponding Young tableau (9.14) is given by

$$(a_1 a_2 \ldots a_{r-1} Z) \rightarrow (a_1 a_2 \ldots a_{r-1} \frac{Z}{2} 00 \ldots) \, . \tag{9.11}$$

For example, for $SO(7)$ representation (102) have

$$(102) \to (1010\ldots) = \texttt{birdTrack} \tag{9.12}$$

For orthogonal groups the Levi-Civita tensor can be used to convert a long column of k boxes into a short column of $(2r+1-k)$ boxes. The highest column which cannot be shortened by this procedure has r boxes. r is the rank of $SO(2r+1)$. For $SO(2r)$ representations, the last two Dynkin labels are spinor roots $(a_1a_2\ldots a_{r-2}YZ)$. Tensor representations have $Y+Z$ =even. However, as spinors are complex, tensor representations can also be complex, conjugate representations being related by

$$(a_1a_2\ldots YZ) = (a_1a_2\ldots ZY)^* \,. \tag{9.13}$$

For $Z \geq Y, Z+Y$ even, the corresponding Young tableau is given by

$$(a_1a_2\ldots a_{r-2}YZ) \to (a_1a_2\ldots a_{r-2}, \frac{Z-Y}{2}, 0, 0, \ldots) \tag{9.14}$$

The Levi-Civita tensor can be used to convert long columns into short columns. For columns of r boxes, the Levi-Civita tensor splits $0(2r)$ representations into conjugate pairs of $SO(n)$ representations. Of the various expressions for the dimensions of $SO(n)$ tensor representations, we find the formula of King (1972) and Murtaza and Rashid (1973) the most convenient.

Chapter 10

Spinors

In chapter 9 we have discussed only the tensor representation of orthogonal groups. However, the spinor representations of $SO(n)$ play a fundamental role in particle physics, both as representations of space-time symmetries (Pauli spin matrices, Dirac gamma matrices, fermions in D-dimensional supergravities), and as representations of internal symmetries ($SO(10)$ grand unified theory, for example). In calculations of radiative corrections, the QED 'colour' weights (*ie.*, spin traces) can easily run up to traces of some twelve gamma matrices (Kinoshita 1981), and efficient evaluation algorithms are of great practical importance. A most straightforward algorithm would evaluate such a trace in some $11!! = 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \simeq 10\ 000$ steps. Even computers shirk such tedium. A good algorithm will do the job in some $6^2 \simeq 100$ steps.

Spinors came to Cartan (1864) as an unexpected fruit of his labours on the complete classification of representations of the simple Lie groups. Dirac (1928) rediscovered them while looking for a linear version of the relativistic Klein-Gordon equation. He introduced matrices γ_μ which were required to satisfy

$$(p_0\gamma_0 + p_1\gamma_1 + \ldots)^2 = (p_0^2 - p_1^2 - p_2^2 - \ldots)\ . \tag{10.1}$$

For $n = 4$ he constructed gammas as $[4 \times 4]$ complex matrices. For $SO(2r)$ and $SO(2r+1)$ gamma matrices were constructed explicitly as $[2^r \times 2^r]$ complex matrices by Weyl and Brauer (1935). In the early days, such matrices were taken as a literal truth, and Klein and Nishina (1929) are reputed to have computed their cross-section by multiplying by hand explicit $[4 \times 4]$ matrices. Nevertheless, all information that is actually needed for spin traces evaluation is contained in the Dirac algebraic condition (10.2), *ie.* the Clifford algebra of γ matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\ . \tag{10.2}$$

Iterative application of this condition immediately yields a spin traces evaluation algorithm in which the only residue of gamma matrices is the normalization factor $\mathrm{tr}\,\mathbf{1}$. However, this simple algorithm is inefficient in the sense that it requires a combinatorially large number of evaluation steps. The most efficient algorithm on the market (for arbitrary n) appears to be the one given by Kennedy

(1982) [Cvitanović and Kennedy (1982)]. In Kennedy's algorithm, one views the spin trace to be evaluated as a $3n-j$ coefficient. Fierz (1934) identities are used to express this $3n-j$ coefficient in terms of $6j$ coefficients (cf. sect. 9.14). Gamma matrices are $[2^{n/2} \times 2^{n/2}]$ in even dimensions, $[2^{(n-1)/2} \times 2^{(n-1)/2}]$ in odd dimensions, and at first sight it is not obvious that a smooth analytic continuation in dimension should be possible for spin-traces. The reason why the Kennedy algorithm succeeds is that spinors are really not there at all. Their only role is to restrict the $SO(n)$ Clebsch-Gordan series to fully antisymmetric representations. The corresponding $3j$ and $6j$ coefficients are relatively simple combinatoric numbers, with analytic continuations in terms of gamma functions. The case of four spacetime dimensions is special because of the reducibility of $SO(4)$ to $SU(2) \times SU(2)$. Farrar and Neri (1983), who have (as of 18th April 1983) computed in excess of 58 149 Feynman diagrams, have used this structure to develop a very efficient method for evaluating $SO(4)$ spinor expressions. An older technique is the Kahane (1968) algorithm, which implements diagrammatically the Chisholm (1963) identities. (REDUCE (Hearn 1970) uses the Kahane algorithm). Thrnblad (1967) has used $SO(4) \subset SO(5)$ embedding to speed-up evaluation of traces for massive fermions.

10.1 Spinograpy

Kennedy (1982) introduced diagrammatic notation for gamma matrices

10.2 Fierzing around

10.3 Fierz coefficients

10.4 $6j$ coefficients

10.5 Exemplary evaluations

10.6 Invariance of γ-matrices

10.7 Handedness

10.8 Kahane algorithm

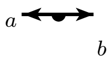
For the case of four dimensions, there is a fast algorithm for trace evaluation, due to Kahane (1968).

Chapter 11

Symplectic groups

Symplectic group $SP(n)$ is the group of all transformations which leave invariant a skew symmetric quadratic form $(p,q) = g_{ab}p^a q^b$:

$$g_{ab} \;=\; -\, g_{ba} \qquad\qquad a,b = 1,2,\dots n$$



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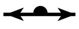
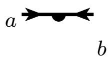

n even (11.1)

The birdtrack notation is motivated by the need to distinguish the first and the second index: it is a special case of the birdtracks for antisymmetric tensors of even rank (**??**). If (p,q) is an invariant, so is its complex conjugate $(p,q)^* = g^{ba}p_a q_b$, and

$$g^{ab} \;=\; -\, g^{ba}$$



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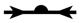
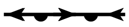

(11.2)

is also an invariant tensor. Matrix $A_a^b = g_{ac}g^{cb}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional representation. A convenient normalization is

$$g_{ac}g^{cb} \;=\; -\, \delta_a^b \tag{11.3}$$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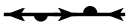
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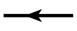
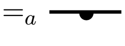
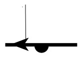

(11.4)

Indices can be raised and lowered at will, so the arrows on lines can be dropped. However, omitting symplectic warts (the black half-circles) appears perilous, as without them it is hard to keep track of signs. Our convention will be to perform all contractions with g^{ab}, and omit the arrows but not the warts:

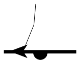
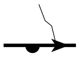
$g^{ab} =$


(11.5)

All other tensors will have lower indices. The Lie group generators $(T_i)_a{}^b$ will be replaced by

$(T_i)_{ab} =$  . (11.6)

The invariance condition (**??**) for the symplectic invariant tensor is

 $+$  $= 0$

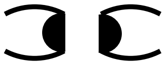
$$(T_i)_a{}^c g_{cb} + (T_i)_b{}^c g_{ac} = 0\,. \qquad (11.7)$$

A skew symmetric matrix g_{ab} has the inverse in (8.3) only if $Det|g| \neq 0$. That is possible only in even dimensions, so $Sp(n)$ can be realized only for even $\underline{n}$.

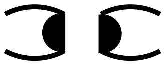
In this chapter we shall outline the construction of $Sp(n)$ tensor representations. They are obtained by contracting the irreducible tensors of $SU(n)$ with the symplectic metric g^{ab} and decomposing them into traces and traceless parts. The representation theory for $Sp(n)$ is analogous in step-by-step fashion to the representation theory for $SO(n)$. In chapter ??????? we shall show that this arises because the two groups are related by supersymmetry, and in chapter ??????? we shall exploit this connection by showing that all group-theoretic weights for the two groups are related by analytic continuation into negative dimensions.

11.1 Two-index tensors

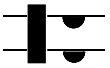
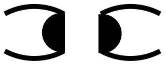
The decomposition goes the same way as for $SO(n)$, sect. 9.14. The matrix (**??**) is given by:

$T =$  (11.8)

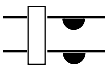
and satisfies the same characteristic equation (11.11) as before. Now T is antisymmetric, $AT = T$, and only the antisymmetric subspace gets decomposed. The final decomposition of $Sp(n)$ two-index tensors is

singlet: $(P_1)_{ab,cd} = \frac{1}{n} g_{ab} g_{cd} = \frac{1}{n}$ 

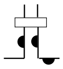
traceless antisymmetric: $(P_2)_{ab,cd} = \frac{1}{2}(g_{ad}g_{bc} - g_{ac}g_{bd}) - \frac{1}{n} g_{ab} g_{cd}$

$=$  - $\frac{1}{n}$  (11.9)

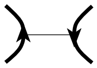
symmetric: $(P_3)_{ab,cd} = \frac{1}{2}(g_{ad}g_{bc} + g_{ac}g_{bd})$

$=$ 

The $SU(n)$ adjoint representation (**??**) is now split into traceless symmetric and antisymmetric parts. The adjoint representation of $Sp(n)$ is given by the symmetric subspace, as only P_3 satisfies the invariance condition (11.7):

 $+$ `birdTrack` $= 0$ (11.10)

Hence the projection operator for $Sp(n)$ is given by

$\frac{1}{a}$  $=$ `birdTrack` (11.11)

The dimension of $Sp(n)$ is

$$N = \mathrm{tr}\, P_A = \texttt{birdTrack} = \frac{n(n+1)}{2} \tag{11.12}$$

Remember that all contractions are carried out by g^{ab} - hence the extra warts in the trace expression. Dimensions of other representation and the Dynkin indices are listed in ???? .

11.2 Mixed defining × adjoint tensors

11.3 Dynkin labels of $Sp(n)$ representations

Chapter 12

Negative dimensions

A cursory examination of the expressions for the dimensions and the Dynkin indices listed in the tables of chapter **??**, chapter **??** and chapter **??** reveal intriguing symmetries under substitution $n \to -n$. This kind of symmetry is best illustrated by the representations of $SU(n)$; if λ stands for a Young tableau with p boxes, and $\overline{\lambda}$ for the transposed tableau obtained by flipping λ across the diagonal, (*ie.*, exchanging symmetrizations and antisymmetrizations), then the dimensions of the two tableaux are related by

$$SU(n) : d_\lambda(n) = (-1)^p d_{\overline{\lambda}}(-n) \ . \tag{12.1}$$

This is evident from the standard recipe (**??**) for computing the $SU(n)$ representation dimensions, as well as from the expressions listed in the tables of chapter **??**. In all cases, exchanging symmetrizations and antisymmetrizations amounts to replacing n by $-n$.

Such relations have been noticed before, Parisi and Soulas (1979), have suggested that a Grassmann vector space of dimension n can be interpreted as an ordinary vector space of dimension $-n$. Penrose (1971) has introduced the term 'negative dimensions' in his construction of $SU(2) \simeq Sp(2)$ representations as $SO(-2)$. King (1972) has proved that the dimension of any irreducible representation of $Sp(n)$ is equal to that of $SO(n)$ with symmetrizations exchanged with antisymmetrizations (*ie.* corresponding to the transposed Young tableau), and n replaced by $-n$. Mkrtchyan (1981) has observed this relation for the QCD loop equations. With the advent of supersymmetries, $n \to -n$ relations have become commonplace, as they are built into the structure of groups such as the orthosymplectic group $OSp(b, f)$. Some highly nontrivial examples of $n \to -n$ symmetries for the exceptional groups (Cvitanović 1981) will be discussed elsewhere in these notes.

Here we shall prove the following:

Theorem 1. For any $SU(n)$ invariant scalar exchanging symmetrizations and antisymmetrizations is equivalent to replacing n by $-n$:

$$SU(n) = \overline{SU}(-n) \ . \tag{12.2}$$

Theorem 2. For any $SO(n)$ invariant scalar there exists the corresponding $Sp(n)$ invariant scalar (and vice versa) obtained by exchanging symmetrizations and antisymmetrizations, replacing the $SO(n)$ symmetric bilinear invariant d_{ab} by the $Sp(n)$ antisymmetric bilinear invariant f_{ab}, and replacing n by $-n$:

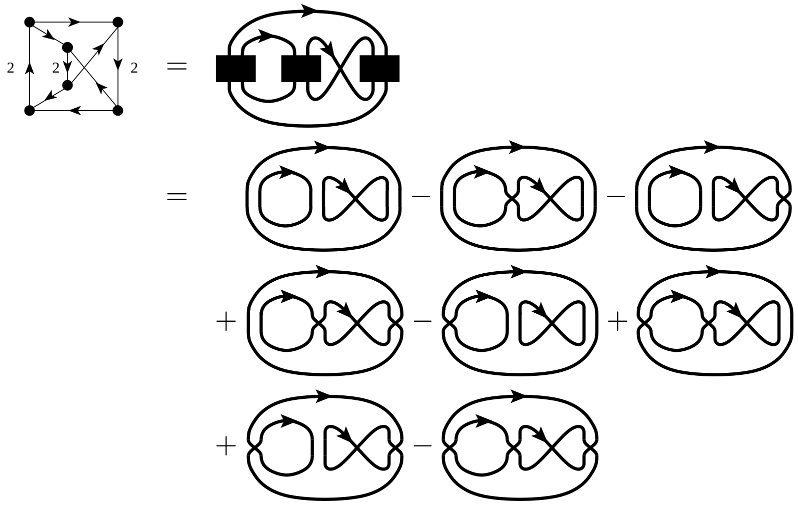
$$SO(n) = \overline{Sp}(-n) \, , \tag{12.3}$$

$$Sp(n) = \overline{SO}(-n) \, , \tag{12.4}$$

(the bars on $\overline{SU}$, $\overline{Sp}$, $\overline{SO}$ indicate transposition, *ie.* exchange of symmetrizations and antisymmetrizations).

All previous $n \to -n$ relations are special cases of these general theorems, the general proof is much simpler than the published proofs for the special cases.

As we have argued in sect. 9.14, all physical consequences of a symmetry (representation dimensions, level splittings, *etc*) can be expressed in terms of invariant scalars ($3n-j$ coefficients, casimirs, *etc*), so it suffices to prove the equivalence (12.2), (12.4) for arbitrary scalar invariants. The idea of the proof is illustrated by the following typical computation: evaluate, for example, the following $SU(n)$ $9j$ coefficient for recoupling of three antisymmetric rank-two-representations:



$$\begin{aligned} &= n^3 - n^2 - n^2 + n - n^2 + n + n - n^2 \\ &= n(n-1)(n-3) \, . \end{aligned} \tag{12.5}$$

Notice that in the expansion of the symmetry operators the graphs with an odd number of crossings give an even power of n, and vice versa. If we change the three symmetrizers into antisymmetrizers, the terms which change the sign are exactly those with an even number of crossings. The crossing in the original graph, which had nothing to do with any symmetry operator, appears in every term of the expansion, and this does not affect our conclusion; an exchange of symmetrizations and antisymmetrizations amounts to substitution $n \to -n$.

(The overall sign is only a matter of convention; it depends on how we define vertices in $3n-j$'s.) The proof for the general $SU(n)$ case is even simpler than the above example:

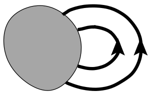
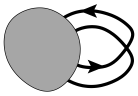
12.1 $SU(n) = SU(-n)$

The primitive invariant tensors of $SU(n)$ are the Kronecker tensor δ^a_b and the Levi-Civita tensor $\varepsilon_{a_1 \cdots a_n}$. All other invariants of $SU(n)$ are built from these two objects.

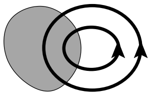
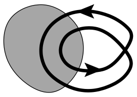
A scalar ($3n-j$ coefficient, vacuum bubble) is a number which, in birdtrack notation, corresponds to a graph with no external legs.

As the directed lines must end somewhere, the Levi-Civita tensors can be present only in pairs, and can always be eliminated by the identity (5.32). An $SU(n)$ $3n-j$ coefficient, therefore, corresponds to a diagram made solely of closed loops of directed lines and symmetry projection operators, like the example (12.5).

Consider the graph corresponding to an arbitrary $SU(n)$ scalar and expand all its symmetry operators as in (12.5). The expansion can be arranged (in any of many possible ways) as a sum of pairs of form

$\ldots +$  $\pm$  $+ \ldots$ (12.6)

with a plus sign if the crossing arises from a symmetrization, and a minus sign if it arises from an antisymmetrization. Each graph consists only of closed loops, *ie.* a definite power of n, and thus uncrossing two lines can have one of two consequences. If the two crossed line segments come from the same loop, then uncrossing splits this into two loops, whereas if they come from two loops, it joins them into one loop. The power of n is changed by the uncrossing:

 $= n$  (12.7)

Hence the pairs in the expansion (12.6) always differ by $n^{\pm 1}$, and exchanging symmetrizations and antisymmetrizations has the same effect as substituting $n \to -n$ (up to an irrelevant overall sign). This completes the proof of (12.2).

Some examples of $n \to -n$ relations for $SU(n)$ representations:

(i) Dimensions of the fully symmetric representations (5.14) and the fully antisymmetric representations (5.22) are related by the gamma-function analytic continuation formula

$$\frac{n!}{(n-p)!} = (-1)^p \frac{(-n+p-1)!}{(-n-1)!} \,. \tag{12.8}$$

(ii) The representations (**??**), (**??**) correspond to the 2−??? symmetric, antisymmetric, respectively. Therefore, their dimensions in table 6.1 are related by $n \to -n$.

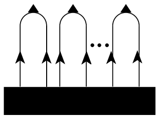
(iii) The representations (**??**) and (**??**) (see table **??** are related by $n \to -n$ for the same reason.

(iv) $n \to -n$ symmetries in table **??**.

(v) Dimension formula (12.1).

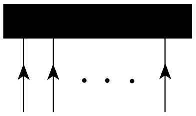
12.2 $SO(n) = Sp(-n)$

In addition to δ^a_b and $\varepsilon_{ab\ldots d}$, $SO(n)$ preserves a symmetric bilinear invariant d_{ab} for which we have introduced birdtrack open circle notation in (7.1). Such open circles can occur in $SO(n)$ $3n-j$ graphs, flipping the line directions. The Levi-Civita tensor still cannot occur, as directed lines starting on a ε tensor would have to end on a d tensor, which gives zero by symmetry. $Sp(n)$ differs from $SO(n)$ by having a skew symmetric bilinear tensor f_{ab}, for which we have introduced birdtrack wart notation in (**??**). A Levi-Civita tensor can appear in an $Sp(n)$ $3n-j$ graph, but as

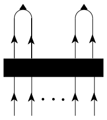


$$= \sqrt{det\ f}\ , \qquad (12.9)$$

(an exercise for the reader), a Levi-Civita can always be replaced by an antisymmetrization

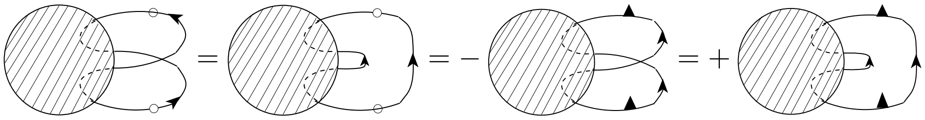


$$= (det\ f)^{-\frac{1}{2}}$$



$$(12.10)$$

For any $SO(n)$ scalar there exists a correponding $Sp(n)$ scalar, obtained by exchanging the symmetrizations and antisymmetrizations *and* the d_{ab}'s and f_{ab}'s in the corresponding graphs. The proof that the two scalars are transformed into each other by replacing n by $-n$ is the same as for $SU(n)$, except that the two line segments at a crossing could come from a new kind of loop containing d_{ab}'s or f_{ab}'s. In that case, equation (12.7) is replaced by



$$(12.11)$$

While now uncrossing the lines does not change the number of loops, changing d_{ab}'s to f_{ab}'s does provide the necessary minus sign. This completes the proof of

(12.4) for the tensor representations of $SO(n)$ and $Sp(n)$. To extend the proof to the spinor representations, we will first have to invent the $Sp(n)$ analog of spinor representations. We postpone this until the next chapter.

Some examples of $SO(n) = \overline{Sp}(-n)$ relations:

(i) The $SO(n)$ antisymmetric adjoint representation (**??**) corresponds to the $Sp(n)$ symmetric adjoint representation (**??**)

(ii) Compare table **??** and table **??**.

(iii) Penrose (1971) binors: $SU(2) = \overline{SO}(-2)$.

Chapter 13

Spinsters

This chapter is based on P. Cvitanović and A.D. Kennedy, *Phys. Scripta* **26**, 5 (1982).

Chapter 14

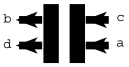
$SU(n)$ family of invariance groups

$SU(n)$ preserves the Levi-Civita tensor, in addition to the Kronecker δ sect. **??**. This additional invariant induces non-trivial decompositions of $U(n)$ representations. In this chapter we show how the theory of $SU(2)$ representations (the quantum mechanics textbooks' theory of angular momentum) is developed by birdtracking; that $SU(3)$ is the unique group with the Kronecker delta and a rank-three antisymmetric primitive invariant; that $SU(4)$ is isomorphic to $SO(6)$; and that for $n \geq 4$, only $SU(n)$ has the Kronecker δ and rank-n antisymmetric tensor primitive invariants.

14.1 Representations of $SU(2)$

For $SU(2)$ we can construct an additional invariant matrix which would appear to induce a decomposition of $n \otimes \overline{n}$ representations

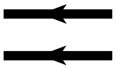
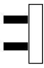
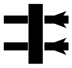
$$E^{a\ c}_{b,\ d} = \frac{1}{2}\varepsilon^{ac}\varepsilon_{bd} \tag{14.1}$$

= 

. (14.2)

However, by (5.32) this can be written as a sum over Kronecker deltas, and is not an independent invariant. So what does ε^{ac} do? It does two things; it removes the distinction between quark and antiquark; (if q_a transforms as a quark, then $\varepsilon^{ab}q_b$ transforms as an antiquark), and it reduces the representations of $SU(2)$ to the fully symmetric ones. Consider $n \otimes n$ decomposition (**??**)

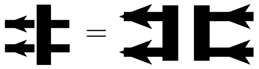
$$\boxed{1} \otimes \boxed{2} = \texttt{birdTrack} + \bullet \tag{14.3}$$

 =  + 

(14.4)

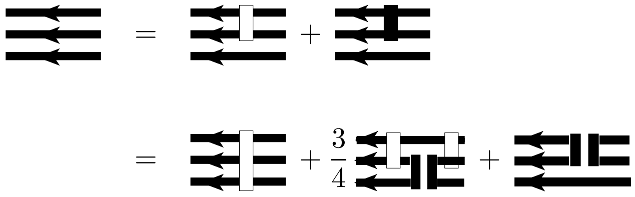
$$2^2 = \frac{2\cdot 3}{2}+\frac{2\cdot 1}{2}\,. \tag{14.5}$$

The antisymmetric representation is a singlet



(14.6)

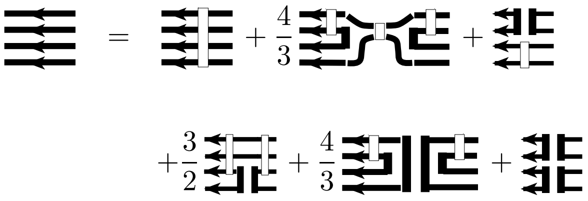
Now consider the $\otimes V^3$ and $\otimes V^4$ space decompositions, obtained by adding successive quarks one at a time:



(14.7)

$$\boxed{1}\times\boxed{2}\times\boxed{3} = \boxed{1}\boxed{2}\boxed{3}+\boxed{1}+\boxed{3} \tag{14.8}$$

(14.9)



(14.10)

(14.11)

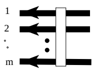
$$\boxed{1}\times\boxed{2}\times\boxed{3}\times\boxed{4} = \boxed{1}\boxed{2}\boxed{3}\boxed{4}+\boxed{1}\boxed{4}+\boxed{3}\boxed{4} \tag{14.12}$$

$$+\boxed{1}\boxed{2}+\bullet+\bullet+\bullet \tag{14.13}$$

This is clearly leading us into the theory of $SO(3)$ angular momentum addition, described in any quantum mechanics textbook. We shall anyway persist a little while longer, just to illustrate how birdtracks can be used to recover some familiar results.

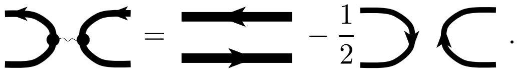
The projector for m-quark representation is

$P_m =$

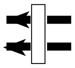


(14.14)

The dimension is $trP_m = \frac{2(2+1)(2+2)...(2+m-1)}{m!} = m+1$ (usually $m = 2j$, where j is the spin of the representation). The projection operator (**??**) for the adjoint representation (spin 1) is



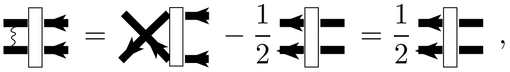
(14.15)

(This can be rewritten as  using (14.6)).

The quadratic casimir for the defining representation is

$$\texttt{birdTrack} = \frac{3}{2} \longleftarrow . \tag{14.16}$$

Using



(14.17)

we can compute the quadratic casimir for any representation

14.2 $SU(3)$ as invariance group of a cubic invariant

We have proven that the only group that satisfies the conditions (i) - (iii) at the beginning of this section is $SU(3)$. Of course, it is well-known that the colour group of physical hadrons is $SU(3)$, and this result might appear rather trivial. That it is not so will become clear from the further examples of invariance groups, such as the G_2 family of the next chapter.

14.3 Levi-Civita tensors and $SU(n)$

In chapter ????? we have shown that the invariance group for a skew-symmetric invariant f^{ab} is $Sp(n)$. In particular, for $f^{ab} = \varepsilon^{ab}$, the Levi-Civita tensor, the invariance group is $SU(2) = Sp(2)$. In the preceding section, we have proven that the invariance group of a skew-symmetric invariant f^{abc} is $SU(3)$, and that f^{abc} must be proportional to the Levi-Civita tensor. Now we shall show that for $f^{abc\ldots d}$ with r indices, the invariance group is $SU(r)$, and f is always proportional to the Levi-Civita tensor. $r = 2$ and $r = 3$ cases had to be treated separately because it was possible to construct from f^{ab} and f^{abc} tree invariants on the $\overline{q} \times q \to \overline{q} \times q$ space which could reduce the colour group $SU(n)$ to a subgroup. For $f^{ab}, n \geq 4$ this is indeed what happens: $SU(n) \to Sp(n)$, for n even.

14.4 $SU(4)$ - $SO(6)$ isomorphism

Chapter 15

G_2 family of invariance groups

[1]

In this chapter we begin the construction of all invariance groups which posses a symmetric quadratic and an antisymmetric cubic invariant in the defining representation. The resulting classification is summarized in fig. 15.1. I find that the cubic invariant must satisfy either the Jacobi relation (15.7), or the alternativity relation (15.11). In the former case the invariance group can be any semi-simple Lie group in its adjoint representation; we pursue this possibility in the next chapter. The latter case is developed in this chapter; we find that the invariance group is either $SO(3)$ or the exceptional Lie group G_2. The problem of evaluation of $3n-j$ coefficients for G_2 is solved completely by the reduction identity (15.14). As a byproduct of the construction I give a proof of the Hurwitz's theorem, sect. 15.5. I also demonstrate that the independent casimirs for G_2 are of order 2 and 6, by explicitely reducing the 4-th order casimir in sect. 15.4.

Consider the following list of primitive invariants:

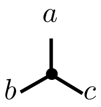
(i) δ^a_b, so the invariance group is a subgroup of $SU(n)$.

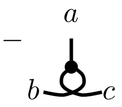
(ii) symmetric $g^{ab}=g^{ba}$, $g_{ab}=g_{ba}$, so the invariance group is a subgroup of $SO(n)$. We take this invariant in its diagonal, Kronecker delta form δ_{ab}.

(iii) a cubic antisymmetric invariant f_{abc}.

Primitivness assumption requires that all other invariants can be expressed in terms of the tree contractions of δ_{ab} , f_{abc}.

In the diagrammatic notation one keeps track of the antisymmetry of the cubic invariant by reading the indices off the vertex in a fixed order:

$f_{abc} =$ 

$= -$ 

$= -f_{acb}\,.$ (15.1)

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[1] superceeds chapter 19 of ref. [13]

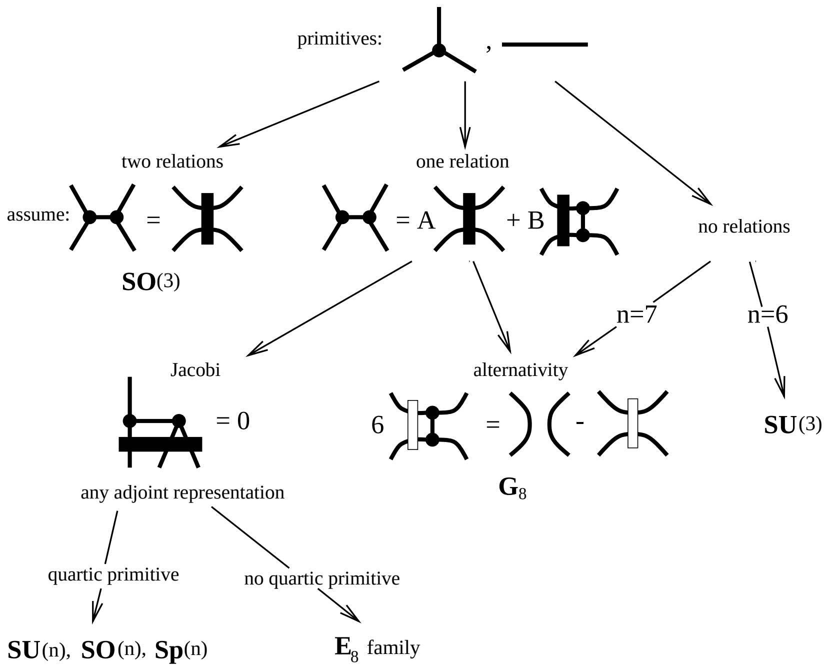
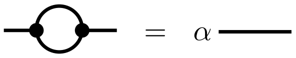

primitives:
two relations
one relation
no relations
assume:
=
= A
+ B
SO(3)
n=7
n=6
Jacobi
= 0
alternativity
6
=
-
SU(3)
G8
any adjoint representation
quartic primitive
no quartic primitive
SU(n), SO(n), Sp(n)
E8 family

Figure 15.1: Logical organization of chapter 15 and chapter 16. The invariance groups $SO(3)$ and G_2 are derived in this chapter, while the E_8 family is derived in chapter 16.

The primitiveness assumption implies that the double contraction of a pair of f's is proportional to the Kronecker delta. We can use this relation to fix the overall normalization of f's:

$$f_{abc}f_{cbd} = \alpha\,\delta_{ad}$$



(15.2)

For convenience we shall often set $\alpha = 1$ in what follows.

The next step in our construction is to identify all invariant matrices on $\bigotimes V^2$, and construct the Clebsch-Gordan series for decomposition of two-index tensors. There are six such invariants, the three distinct permutations of indices of $\delta_{ab}\delta_{cd}$, and the three distinct permutations of free indices of $f_{abe}f_{ecd}$. For reasons of clarity, we shall break up the discussion into two steps. In the first step, sect. 15.1, we assume that a linear relation between these six invariants exists. Pure symmetry considerations, together with the invariance condition, completely fix the algebra of invariants and restrict the dimension of the defining space to either 3 or 7. In the second step, sect. 15.3, we show that a relation *assumed* in the first step must exist because of the invariance condition.

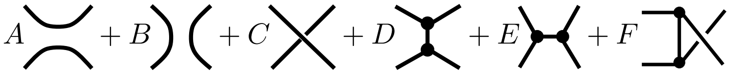
Remark **15.1** Quarks and hadrons. An example of a theory with above invariants would be QCD with the hadronic spectrum consisting of following singlets:

(i) quark-antiquark mesons
(ii) mesons built of two quarks (or antiquarks) in a symmetric color combination
(iii) baryons built of three quarks (or antiquarks) in a fully antisymmetric color combination
(iv) no exotics, *ie.* no hadrons built from other combinations of quarks and antiquarks

As we shall now demonstrate, for this hadronic spectrum the color group is either $SO(3)$, with quarks of three colors, or the exceptional Lie group G_2, with quarks of seven colors.

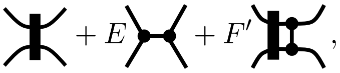
15.1 Jacobi relation

If the six invariant tensors mentioned above are not independent, they satisfy a relation of form



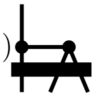
$$0 = A \; + B \; + C \; + D \; + E \; + F \qquad (15.3)$$

Antisymmetrizing a pair of indices yields



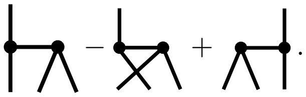
$$0 = A' \; + E \; + F' \; , \qquad (15.4)$$

and antisymmetrizing any three indices yields



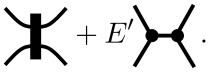
$$0 = (E + F') \; . \qquad (15.5)$$

If the tensor itself vanishes, f's satisfy the *Jacobi relation* (**??**)



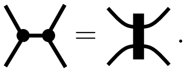
$$0 = \; - \; + \; . \qquad (15.6)$$

If $A' \neq 0$ in (15.4), the Jacobi relation relates the second and the third term



$$0 = \; + E' \; . \qquad (15.7)$$

The normalization condition (15.2) fixes $E' = -1$.



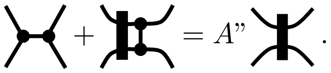
$$= \; . \qquad (15.8)$$

Contracting with δ_{ab} we obtain $1 = (n-1)/2$, so $n = 3$. We conclude that if pair contraction of f's is expressible in terms of δ's, the invariance group is $SO(3)$ and f_{abc} is proportional to the 3-index Levi-Civita tensor. To spell it out; in 3 dimensions an antisymmetric rank-3 tensor can take only one value, $f_{abc} = \pm f_{123}$ which can be set equal to ± 1 by appropriate normalization convention (15.2).

If $A' = 0$ in (15.4), the Jacobi relation is the only relation we have, and the adjoint representation of any simple Lie group is a possible solution. I return to this case in chapter 16.

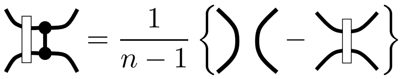
15.2 Alternativity and reduction of f-contractions

If the Jacobi relation does not hold we must have $E = -F'$ in (15.5), and (15.4) takes form



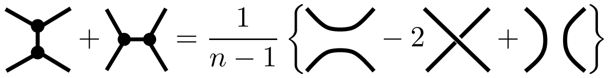
(15.9)

Contracting with δ_{ab} fixes $A” = 3/(n-1)$. Symmetrizing the top two lines and rotating the diagrams by 90^0 we obtain the *alternativity relation*:



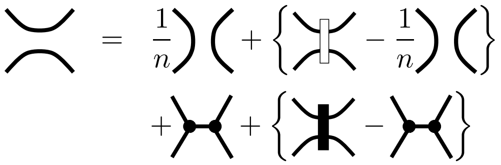
(15.10)

The name comes from the octonian interpretation given in sect. 15.4. Adding the two equations we obtain

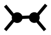


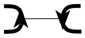
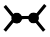
(15.11)

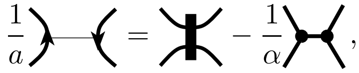
The Clebsch-Gordon decomposition of $\bigotimes V^2$ follows:



$$n^2 = 1 + \frac{(n-1)(n+2)}{2} + n + \frac{n(n-3)}{2} \qquad (15.12)$$

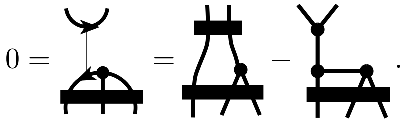
By (15.9) the invariant is reducible on the antisymmetric subspace. By (15.10) it is also reducible on the symmetric subspace. The only independent $f \cdot f$ invariant is  which, by the normalization (15.2), is already the projection operator which projects the antisymmetric two-index tensors onto the n-dimensional defining space. The dimensions of the representations are obtained by tracing the corresponding projection operators.

The adjoint representation  of $SO(n)$ is now split into two representations. Which one is the new adjoint representation? That we determine by considering (5.61), the invariance condition for f_{abc}. If we take  to be the projection operator for the adjoint representation, we again get the Jacobi condition with $SO(3)$ as the only solution. However, if we assume that the last term in (15.12) is the adjoint projection operator



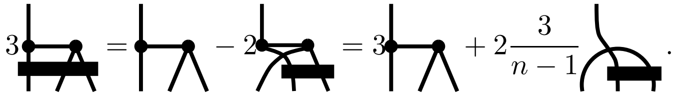
(15.13)

the invariance condition becomes a non-trivial condition:

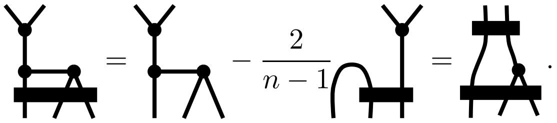


(15.14)

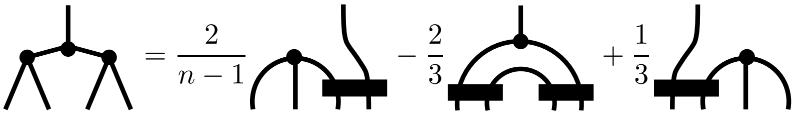
The last term can be simplified by (15.9) and (5.20)



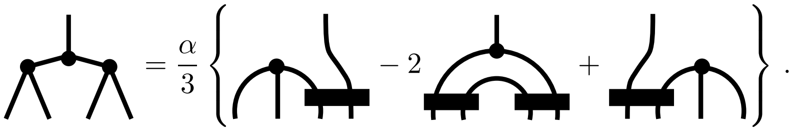
Substituting back into (15.14) yields



Expanding the last term and redrawing the equation slightly we have



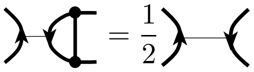
This equation is antisymmetric under interchange of the left and the right index pairs. Hence $2/(n-1) = 1/3$, and the invariance condition is satisfied *only* for $n = 7$. Furthermore, the above relation gives us the G_2 *reduction identity*



(15.15)

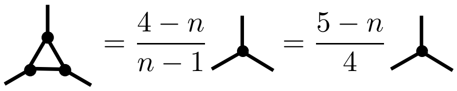
This identity is the key result of this chapter: it enables us to recursively reduce *all* contractions of products of δ-functions and pairwise contractions $f_{abc}f_{cde}$, and thus completely solves the problem of evaluating any casimir or $3n-j$ coefficient of G_2.

The invariance condition (15.14) for f_{abc} implies that



(15.16)

The "triangle graph" for the defining representation can be computed in two ways, either by contracting (15.10) with f_{abc}, or by contracting the invariance condition (15.14) with δ_{ab}:



(15.17)

So the alternativity and the invariance conditions are consistent iff $(n-3)(n-7) = 0$, *ie. only* for 3 or 7 dimensions. In the latter case, the invariance group is the exceptional Lie group G_2, and the above derivation is also a proof of Hurwitz's theorem, see sect. 15.4.

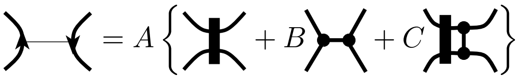
In this way symmetry considerations together with the invariance conditions, suffice to determine the algebra satisfied by the cubic invariant. The invariance condition fixes the defining dimension to $n = 3$ or 7. Having assumed only that *a* cubic antisymmetric invariant exists, we find that if the cubic invariant is not a structure constant, it can be realized only in 7 dimensions, and its algebra is *completely* determined. The identity (15.14) plays the role analogous to that the Dirac relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{I}$ plays for evaluation of traces of products of Dirac gamma-matrices. [2] Just as the Dirac relation obviates need for explicit representations of γ's, (15.14) reduces any $f \cdot f \cdot f$ contraction to a sum of terms linear in f, and obviates any need for explicit construction of f's.

The above results now enable us to compute any group-theoretic weight for G_2 in two steps. First we replace all adjoint representation lines by the projection operators P_A (15.13). The resulting expression contains Kronecker deltas and chains of contractions of f_{abc}, which can then be reduced by systematic application of the reduction identity (15.15).

15.3 Primitivity implies alternativity

The only detail which remains to be proven is the assertion that the alternativity relation (15.10) follows from the primitiveness assumption. I complete the proof in this section. The proof is rather inellegant and can probably be easily streamlined.

If no relation (15.3) between the three $f \cdot f$ contraction is assumed, then by the primitiveness assumption the adjoint representation projection operator P_A is of the form



(15.18)

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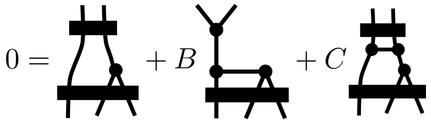
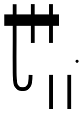
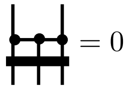
[2] The Klein-Nishina formula of quantum electrodynamics was computed by Klein and Nishina by explicitely multiplying [4×4] matrix representations of γ_μ's and then summing over μ's. Day after day they would multiply away whole morning, and then meet in the Niels Bohr Institute's cafeteria to compare their results. Today the Klein-Nishina trace over Dirac γ's is a texbook exercise, reducible by several applications of $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{I}$.

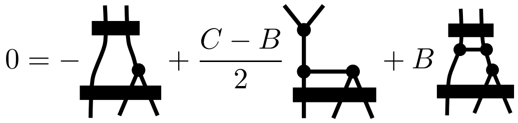
Assume that the Jacobi relation does not hold; otherwise this immediately reduces to $SO(3)$. The generators must be antisymmetric, as the group is a subgroup of $SO(n)$. Substitute the adjoint projection operator into the invariance condition (5.61) (or (15.14)) for f_{abc}:


(15.19)

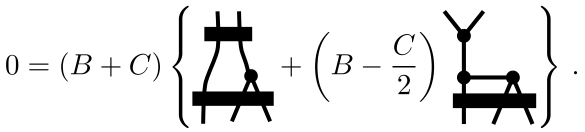
Resymmetrize this equation by contracting with . This is evaluated expanding with (5.20) and using a relation due to the antisymmetry of f_{abc}:


(15.20)

The result is:

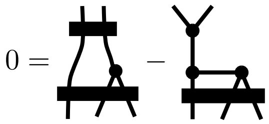

(15.21)

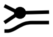
Multiplying (15.19) by B, (15.21) by C and subtracting, we obtain

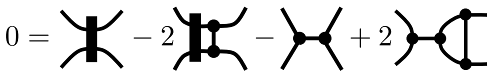

(15.22)

I return to the case $B + C = 0$ below, in (15.26).

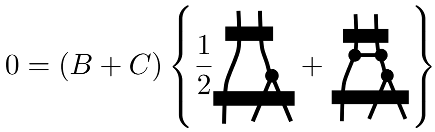
If $B + C \neq 0$, by contracting with f_{abc} we get $B - C/2 = -1$, and

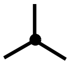

(15.23)

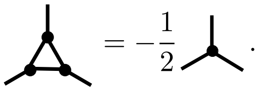
To prove that this is equivalent to the alternativity relation, we contract with , expand the 3-leg antisymmetrization, and obtain


(15.24)

The triangle subdiagram can be computed by adding (15.19) and (15.21)



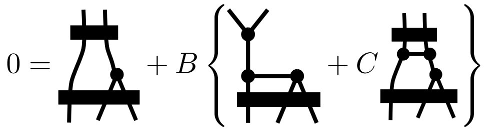
and contracting with . The result is



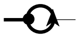
(15.25)

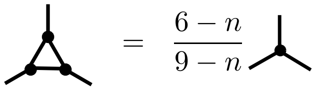
Substituting into (15.24) we recover the alternativity relation (15.10). Hence we have proven that the primitivity assumption implies the alternativity relation for the case $B + C \neq 0$ in (15.22).

If $B + C = 0$, (15.19) becomes

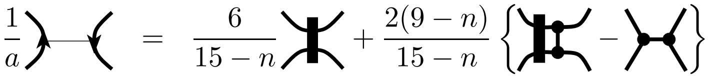


(15.26)

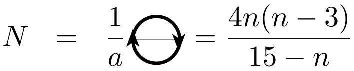
Using the normalization (**??**) and orthonormality  conditions we obtain



(15.27)

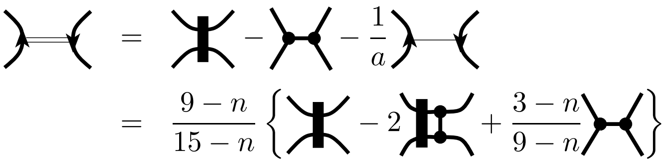


(15.28)



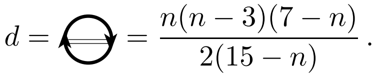
(15.29)

The remaining antisymmetric representation



(15.30)

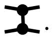
has dimension

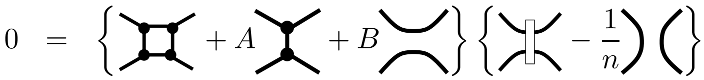


(15.31)

The dimension cannot be negative, so $d \leq 7$. For $n = 7$ the projection operator (15.30) vanishes identically, and we recover the alternativity relation (15.10).

The Diophantine condition (15.31) has two further solutions: $n = 5$ and $n = 6$.

The $n = 5$ is eliminated by examining the decomposition of the traceless symmetric subspace in (15.12) induced by the invariant $\mathbf{Q} =$ . By the primitiveness assumption $\mathbf{Q}^2$ is reducible on the symmetric subspace



$$0 = (\mathbf{Q}^2 + A\mathbf{Q} + B)P_2 \,.$$

Contracting the top two indices with δ_{ab} and $(T_i)_{ab}$ we obtain

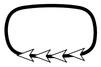
$$\left(\mathbf{Q}^2 - \frac{1}{2}\frac{3-n}{9-n}\mathbf{Q} - \frac{5}{2}\frac{6-n}{(2+n)(9-n)}\mathbf{I}\right) P_2 = 0 \tag{15.32}$$

For $n = 5$ the roots of this equation are rational and the dimensions of the two representations induced by decomposition with respect to $\mathbf{Q}$ are not integers. Hence $n = 5$ is not a solution. We turn to the case $n = 6$ in appendix **??**.

15.4 Casimirs for G_2

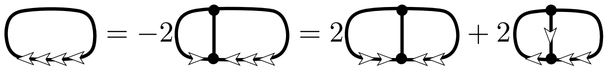
In this section we prove that the independent casimirs for G_2 are of order 2 and 6, as indicated in the table **??**. As G_2 is a subgroup of $SO(7)$, its generators are antisymmetric and only even order casmirs are nonvanishing.

The quartic casimir (in the notation of (**??**))

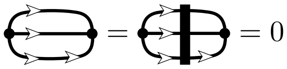


$$= \mathrm{tr}\, X^4 = \sum_{ijkl} x_i x_j x_k x_l \mathrm{tr}\ (T_i T_j T_k T_l)$$

can be reduced by manipulating it with the invariance condition (5.61)

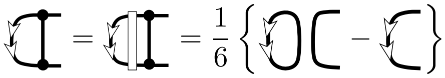


The last term vanishes by further manipulation with the invariance condition

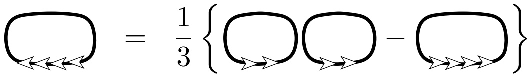


(15.33)

The remaining term is reduced by the alternativity relation (15.10)



This yields the explicit expression for the reduction of quartic casimirs in the defining representation of G_2



$$\mathrm{tr}\, X^4 = \frac{1}{4}\left(\mathrm{tr}\, X^2\right)^2 \tag{15.34}$$

As the defining representation is 7-dimensional, the characteristic equation (**??**) reduces the 8th and all higher order casimirs. Hence the independent casimirs for G_2 are of order 2 and 6.

15.5 Hurwitz's theorem

Definition (Curtis 1963): A normed algebra A is an $n+1$ dimensional vector space over a field F with a product xy such that

$$\begin{array}{rrcll} (i) & x(cy) = (cx)y & = & c(xy)\,, & c \in F \\ (ii) & x(y+z) & = & xy + xz\,, & x,y,z \in A \\ & (x+y)z & = & xz + yz, & \end{array}$$

and a non-degenerate quadratic norm which permits composition

$$(iii) \qquad N(xy) = N(x)N(y)\,, \qquad N(x) \in F. \tag{15.35}$$

Here F will be the field of real numbers. Let $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of A over F:

$$x = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n\,, \qquad x_a \in F\,, \quad \mathbf{e}_a \in A\,. \tag{15.36}$$

It is always possible to choose $\mathbf{e}_o = \mathbf{I}$ (see Curtis 1963). The product of remaining bases must close the algebra

$$\mathbf{e}_a\mathbf{e}_b = -d_{ab}\mathbf{I} + f_{abc}\mathbf{e}_c\,, \qquad d_{ab}, f_{abc} \in F \qquad a,\ldots,c = 1,2,\ldots,n \tag{15.37}$$

The norm in this basis is

$$N(x) = x_0^2 + d_{ab}x_a x_b. \tag{15.38}$$

From the symmetry of the associated inner product (Tits 1966)

$$(x,y) = (y,x) = -\frac{N(x+y) - N(x) - N(y)}{2}, \tag{15.39}$$

it follows that $-d_{ab} = (\mathbf{e}_a, \mathbf{e}_b) = (\mathbf{e}_b, \mathbf{e}_a)$ is symmetric, and it is always possible to choose bases $\mathbf{e}_a$ such that

$$\mathbf{e}_a\mathbf{e}_b = -\delta_{ab} + f_{abc}\mathbf{e}_c. \tag{15.40}$$

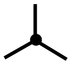
Furthermore, from

$$\begin{aligned} -(xy,x) &= \frac{N(xy+x) - N(x)N(y)}{2} = N(x)\frac{N(y+1) - N(y) - 1}{2} \\ &= N(x)(y,1), \end{aligned} \tag{15.41}$$

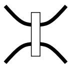
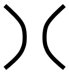
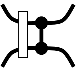
it follows that $f_{abc} = (\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$ is fully antisymmetric. [In Tit's (1966) notation, the multiplication tensor f_{abc} is replaced by a cubic antisymmetric form (a, a', a''), his equation (14)]. The composition requirement (15.35) expressed in terms of bases (15.36) is

$$\begin{aligned} 0 &= N(xy) - N(x)N(y) \\ &= x_a x_b y_c y_d \left(\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} + f_{ace}f_{cbd}\right). \end{aligned} \tag{15.42}$$

To make a contact with sect. 15.2 we introduce diagrammatic notation (factor $i\sqrt{6/\alpha}$ adjusts the normalization to (15.2))

$f_{abc} = i\sqrt{\frac{6}{\alpha}}$  . (15.43)

Diagrammatically, (15.42) is given by

$0 =$  $-$  $+ \frac{6}{\alpha}$  . (15.44)

This is precisely the relation (15.10) which we have proven to be nontrivially realizable only in 3 and 7 dimensions. The trivial realizations are $n = 0$ and $n = 1$, $f_{abc} = 0$. So we have inadvertently proven

Hurwitz's theorem (Curtis (1963): *$n + 1$ dimensional normed algebras over reals exist only for $n = 0, 1, 3, 7$ (real, complex, quaternion, octonion).*

I call (15.10) the *alternativity* relation because it can also be obtained by substituting (15.40) into the alternativity condition for octonions (Schafer 1966)

$$\begin{aligned} [xyz] &\equiv (xy)z - x(yz)\,, \\ [xyz] &= [zxy] = [yzx] = -\,[yxz]\;. \end{aligned} \tag{15.45}$$

Cartan (1894, 1952) was first to note that $G_2(7)$ is the isomorphism group of octonions, *ie.* the group of transformations of octonion bases (written here in the infinitesimal form)

$$\mathbf{e}'_a = (\delta_{ab} + iD_{ab})\mathbf{e}_b$$

which preserve the octonionic multiplication rule (15.40). The reduction identity (15.15) was first derived by Behrends *et al.* (1966) [in very different notation, their equation (16)]. Tits also constructed the adjoint representation projection operator for $G_2(7)$ by defining the derivation on an octonion algebra as

$$Dz =< x, y > z = -\frac{1}{2}((x \cdot y) \cdot z) + \frac{3}{2}[(y, z)x - (x, z)y],$$

[Tits 1966, equation (23)] where

$$\mathbf{e}_a \cdot \mathbf{e}_b \equiv f_{abc}\mathbf{e}_c, \tag{15.46}$$

$$(\mathbf{e}_a, \mathbf{e}_b) \equiv -\delta_{ab}. \tag{15.47}$$

Substituting $x = x_a\mathbf{e}_a$, we find

$$(Dz)_d = -3x_a y_b \left(\frac{1}{2}\delta_{ab}\delta_{bd} + \frac{1}{6}f_{abe}f_{ecd}\right) z_c \tag{15.48}$$

The term in the brackets is just the $G_2(7)$ adjoint representation projection operator P_A in (15.13) with normalization $\alpha = -3$.

15.6 Representations of G_2

G_2 is characterized by the fully antisymmetric cubic primitive invariant f_{abc}. Contracting with f_{abc}, we are able to reduce any column higher than two boxes. Hence, representations of G_2 are specified by Young tableaux (**??**) of form ($qp00...$). Patera and Sankoff (1973) have chosen to label the simple roots in such a way that the correspondence is

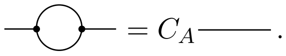
Chapter 16

E_8 family of invariance groups

In this chapter we continue the construction of invariance groups characterized by a symmetric quadratic and an antisymmetric cubic primitive invariant. In the preceeding chapter we proved that the cubic invariant must either satisfy the alternativity relation (15.11), or the Jacobi relation (15.7). The first case has $SO(3)$ and G_2 as the only solutions. Here we pursue the second possibility, restricted to the case of *no quartic primitive* invariant (see fig. 15.1). The main results will be the Diophantine conditions (16.10)-(16.15) and the projection operators for E_8 family, given in table 16.1. [1]

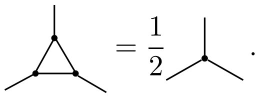
As, by assumption, the defining representation satisfies the Jacobi relation (15.7), it is also the adjoint representation of some Lie group. Hence, in this chapter, we denote the dimension of the defining representation by N, the cubic invariant by the Lie algebra structure constants $-iC_{ijk}$, and draw the invariants with the thin (adjoint) lines, as in (3.84).

The assumption that the defining representation is irreducible means in this case that the Lie group is simple, and the quadratic casimir (Cartan-Killing tensor) is proportional to the identity



(16.1)

In this chapter we shall usually chose normalization $C_A = 1$. The Jacobi relation (??) reduces a loop with three structure constants



(16.2)

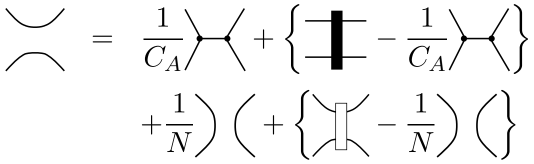
In order to reduce loops with four structure constants we turn to the reduction of the $A \otimes A$ space.

---

[1] The inspiration for construction of sect. 15.1 came from Okubo (1979).

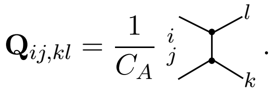
16.1 Two-index tensors

Consider the decomposition of $A \bigotimes A$ tensors. As in sect. **??** they immediately decompose into four subspaces:


=
$\frac{1}{C_A}$
+
−
$\frac{1}{C_A}$
+$\frac{1}{N}$
+
−
$\frac{1}{N}$

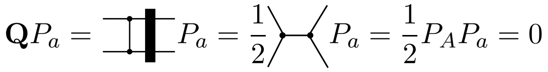
$$\mathbf{1} = P_{|??|} + P_{\substack{|??|\\|??|}} + P_{\bullet} + P_s\,. \qquad (16.3)$$

Consider $A \bigotimes A \to A \bigotimes A$ invariant matrix


$\mathbf{Q}_{ij,kl} = \frac{1}{C_A}$
i
j
l
k
.

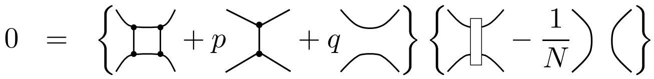
(16.4)

By the Jacobi relation (**??**) or (15.6), $\mathbf{Q}$ has zero eigenvalue on the antisymmetric subspace


$\mathbf{Q}P_a =$
$P_a = \frac{1}{2}$
$P_a = \frac{1}{2}P_A P_a = 0$

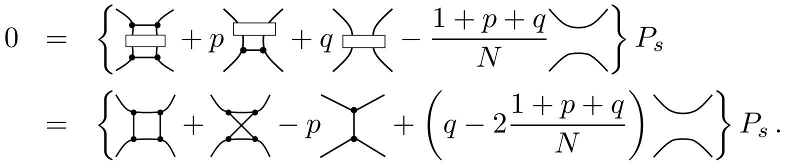
so $\mathbf{Q}$ can decompose only the symmetric subspace.

The assumption that there is no primitive quartic invariant is the *defining relation* for the E_8 family. By this assumption $\mathbf{Q}^2$ is not linearly independent of $\mathbf{Q}_{ij,k\ell}$, $f_{ijm}f_{mk\ell}$ and δ_{ij}'s. On the traceless symmetric subspace this implies that $\mathbf{Q}^2 P_s$ satisfies a relationship of form

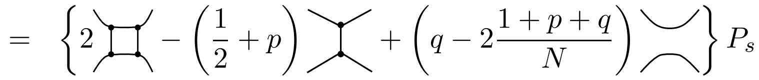


$$0 = (\mathbf{Q}^2 + p\mathbf{Q} + q)P_s\,. \qquad (16.5)$$

The coefficients p, q follow from symmetry considerations and the Jacobi relation. Rotate each term in the above equation by 90^o and the project onto the traceless symmetric subspace;


0 =
+ p
+ q
−
$\frac{1+p+q}{N}$
P_s
=
+
− p
+
$\left(q - 2\frac{1+p+q}{N}\right)$
P_s .

Jacobi relation (**??**) relates the second term to the first:


=
2
$-\left(\frac{1}{2} + p\right)$
+
$\left(q - 2\frac{1+p+q}{N}\right)$
P_s

$$0 = \left(\mathbf{Q}^2 - \frac{1+2p}{4}\mathbf{Q} + \frac{q}{2} - \frac{1+p+q}{N}\right)P_s\,. \qquad (16.6)$$

Comparing the coefficients in (16.5) and (16.6) we obtain the characteristic equation for $\mathbf{Q}$

$$\left(\mathbf{Q}^2-\frac{1}{6}\mathbf{Q}-\frac{5}{3(N+2)}\right)P_s=0\,. \tag{16.7}$$

We shall use this equation to obtain a Diophantine condition on admissible dimensions of the adjoint representation. Either eigenvalue of $\mathbf{Q}$ satisfies the characteristic equation

$$\lambda^2-\frac{1}{6}\lambda-\frac{5}{3(N+2)}=0\,,$$

hence N can be expressed in terms of the eigenvalue

$$N+2=\frac{5}{3\lambda(\lambda-1/6)}=10\left\{(6-\lambda^{-1})-12+\frac{6^2}{6-\lambda^{-1}}\right\}. \tag{16.8}$$

It is convenient to reparametrize the two eigenvalues as

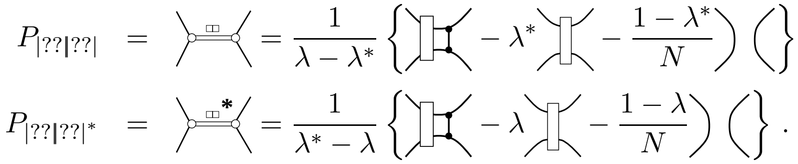
$$\lambda=-\frac{1}{m-6}\,,\qquad \lambda^*=\frac{1}{6}\frac{m}{m-6}\,. \tag{16.9}$$

In terms of the parameter m, the dimension of the adjoint reperesentation is given by

$$N=-122+10m+\frac{360}{m} \tag{16.10}$$

As N is an integer, allowed m are rationals built from any combination of subfactors of $360=2^3\cdot3^2\cdot5$ in the numerator, and 1, 2, 5 or 10 in denominator, 45 distinct rationals in all. The existence of the pair of roots λ, λ^* is reflected in the symmetry of (16.10) under interchange $m/6\leftrightarrow 6/m$, so we need to check only the 27 rationals $m>6$; we postpone the Diophantine analysis to sect. 16.4.

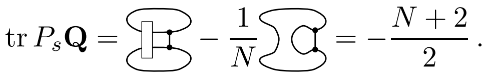
The associated projection operators are



$$P_{|??|??|}\;=\;=\frac{1}{\lambda-\lambda^*}\left\{\;-\lambda^*\;-\frac{1-\lambda^*}{N}\right\} \tag{16.11}$$

$$P_{|??|??|^*}\;=\;=\frac{1}{\lambda^*-\lambda}\left\{\;-\lambda\;-\frac{1-\lambda}{N}\right\}. \tag{16.12}$$

To compute the dimensions of the two subspaces we first evaluate



$$\mathrm{tr}\,P_s\mathbf{Q}=\;-\frac{1}{N}\;=-\frac{N+2}{2}\,. \tag{16.13}$$

The dimension of $|??\|??|$ is then given by

$$d_{|??|??|}=\mathrm{tr}\,P_{|??|??|}=\frac{(N+2)(1/\lambda+N-1)}{2(1-\lambda^*/\lambda)}\,, \tag{16.14}$$

		=		+	$\frac{1}{\alpha}$	+	$(-\frac{1}{\alpha})$
E_7	$(10000000) \otimes (10000000)$	=	(10000000)	+	(01000000)	+	(00000000)
E_7	$(1000000) \otimes (1000000)$	=	(1000000)	+	(0100000)	+	(0000000)
E_6	$(000001) \otimes (000001)$	=	(000001)	+	(001000)	+	(000000)
F_4	$(1000) \otimes (1000)$	=	(1000)	+	(0100)	+	(0000)
D_4	$(0100) \otimes (0100)$	=	(0100)	+	(1010)	+	(0000)
G_2	$(10) \otimes (10)$	=	(10)	+	(03)	+	(00)
A_2	$(11) \otimes (11)$	=	(11)	+	$(12) + (21)$	+	(00)
A_1	$(2) \otimes (2)$	=	(2)	+	(0)	+	(4)
Dimensions	N^2	=	N	+	$\frac{N(N-3)}{2}$	+	1
E_8	27^2	=	351	+	27	+	351
E_7	27^2	=	351	+	27	+	351
E_6	27^2	=	351	+	27	+	351
F_4	15^2	=	105	+	15	+	105
D_4	15^2	=	105	+	15	+	105
G_2	9^2	=	36	+	9	+	36
A_2	6^2	=	15	+	6	+	15
A_1	$8\ \ 3^2$	=	3	+	0	+	1

Table 16.1: E_8 family Clebsch-Gordon series for $\bigotimes A^2$. The corresponding projection operators are listed in (16.3), (16.11) and (16.12). The parameter m is defined in (16.9). FILL THIS TABLE OUT!

and $d_{|??|??|^*}$ is obtained by interchanging λ and λ^*. Substituting (16.10), (16.15) leads to

$$\begin{aligned} d_{|??|??|} &= \frac{5(m-6)^2(5m-36)(2m-9)}{m(m+6)} \\ d_{|??|??|^*} &= \frac{270(m-6)^2(m-5)(m-8)}{m^2(m+6)} . \end{aligned} \tag{16.15}$$

The solutions that survive the Diophantine conditions form the E_8 *family*, listed in table 16.1.

To summarize, in absence of a primitive 4-index invariant, $A \otimes A$ decomposes into 5 irreducible representations

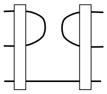
$$\mathbf{1} = P_{|??|} + P_{\substack{|??|\\|??|}} + P_{\bullet} + P_{|??|??|} + P_{|??|??|^*} \tag{16.16}$$

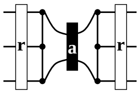
The decomposition is parametrized by integer m and is possible only if N and $d_{|??|??|}$ satisfy Diophantine conditions (16.10), (16.15).

The general strategy for decomposition of higher tensor products is as follows; the equation (16.6) reduces $\mathbf{Q}^2$ to $\mathbf{Q}$, P_r weighted by the eigenvalues λ, λ^*. For higher tensor products we shall use the same result to decompose symmetric subspaces. We shall refer to a decomposition as "boring" if it brings no new Diophantine condition. As $\mathbf{Q}$ acts only on the symmetric subspaces, decompositions of antisymmetric subspaces will in general be boring, as was already the case in (16.3). We illustrate the technique by working out the decomposition of $\mathrm{Sym}^3 A$ and $|??\|??| \otimes |??|$ in the next two sections.

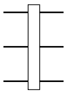
16.2 Decomposition of Sym^3A

Consider $SymA$ fully symmetrized subspace of $\otimes^3 A$. As the first step, project out the A and $A \otimes A$ content of Sym^3A:

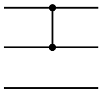
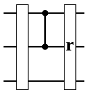
$P_{|}??| \;=\; \dfrac{3}{N+2}$  (16.17)

$P_{\substack{|??|\\|??|}} \;=\; \dfrac{6(N+1)(N^2-4)}{5(N^2+2N-5)}$  (16.18)

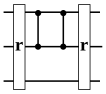
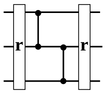
$P_{|}??|$ projects out Sym$^3A \to A$, and $P_{\substack{|??|\\|??|}}$ projects out the antisymmetric subspace (16.3) Sym$^3A \to \wedge^2 A$. The ugly prefactor is a normalization, and will play no role in what follows. We shall decompose the remainder of the Sym^3A space

$P_r = S - P_{|}??| - P_{\substack{|??|\\|??|}} =$  (16.19)

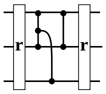
by the invariant tensor $\mathbf{Q}$ restricted to the P_r remainder subspace

$\mathbf{Q} =$  , $\hat{\mathbf{Q}} =$  $\hat{\mathbf{Q}} = P_r \mathbf{Q} P_r$ (16.20)

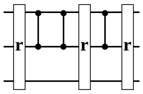
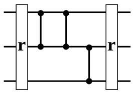
We can partially reduce $\hat{\mathbf{Q}}^2$ using (16.6) but symmetrization leads also to a new invariant tensor

$\hat{\mathbf{Q}}^2 = \dfrac{1}{3}$  $+\; \dfrac{2}{3}$  . (16.21)

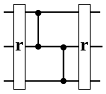
A calculation that requires applications of the Jacobi relation (3.105), symmetry identities such as

 $= 0$ (16.22)

and relies on the fact that P_r contains no $A \otimes 2A$ subspaces yields

$\hat{\mathbf{Q}}^3 = \dfrac{1}{3}$  $+\; \dfrac{2}{3}$  (16.23)

Reducing by (16.7) leads to

$\hat{\mathbf{Q}}^3 = (\lambda + \lambda^*) \left\{ \dfrac{1}{3}\hat{\mathbf{Q}}^2 + \dfrac{2}{3}\right.$  $\left.\right\} - \lambda\lambda^* \hat{\mathbf{Q}}$ (16.24)

The extra tensor can be eliminated by (16.21), and the result is a cubic equation for $\hat{\mathbf{Q}}$ (where we have substituted $\lambda + \lambda^* = 1/6$):

$$0 = \left(\hat{\mathbf{Q}} - 1/18\right)\left(\hat{\mathbf{Q}} - \lambda\right)\left(\hat{\mathbf{Q}} - \lambda^*\right) P_r \tag{16.25}$$

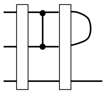
The projection operators for the corresponding three subspaces

$$\begin{aligned}
P_3 &= \frac{\left(\hat{\mathbf{Q}} - \lambda\right)\left(\hat{\mathbf{Q}} - \lambda^*\right)}{(1/18 - \lambda)(1/18 - \lambda^*)} P_r \\
&= -\frac{162\,(m-6)^2}{(m+3)(m+12)}\left\{\hat{\mathbf{Q}}^2 - \frac{1}{6}\hat{\mathbf{Q}} - \frac{m}{6(m-6)^2}\right\} P_r \qquad (16.26)\\
P_4 &= \frac{\left(\hat{\mathbf{Q}} - 1/18\right)\left(\hat{\mathbf{Q}} - \lambda^*\right)}{(\lambda - 1/18)(\lambda - \lambda^*)} P_r \\
&= \frac{54\,(m-6)^2}{(m+3)(m+6)}\left\{\hat{\mathbf{Q}}^2 - \frac{m-24}{18(m-6)}\hat{\mathbf{Q}} + \frac{1}{18(m-6)}\right\} P_r \qquad (16.27)\\
P_{4^*} &= \frac{\left(\hat{\mathbf{Q}} - 1/18\right)\left(\hat{\mathbf{Q}} - \lambda\right)}{(\lambda^* - 1/18)(\lambda^* - \lambda)} P_r \\
&= \frac{108\,(m-6)^2}{(m+6)(m+12)}\left\{\hat{\mathbf{Q}}^2 - \frac{2(m-3)}{9(m-6)}\hat{\mathbf{Q}} + \frac{m}{108(m-6)}\right\} P_r \qquad (16.28)
\end{aligned}$$

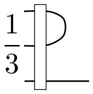
The presumption is (still to be proven for a general tensor product) that only reductions occur in the symmetric subspaces, always via the $\mathbf{Q}$ characteristic equation. As overall scale of $\mathbf{Q}$ is arbitrary, there is only one rational parameter in the problem, either λ/λ^* or m, or whatever seems conveninent. Hence all dimensions and any coefficients will be in m.

To proceed, we follow the method outlined in appendix A. On $P_{|}??|$, $P_{\substack{|??|\\|??|}}$ subspaces $S\mathbf{Q}$ has eigenvalues

$S\mathbf{Q}P_{|}??| \; =$

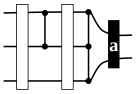


$= \frac{1}{3}$

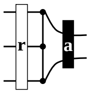


$\rightarrow \lambda_{|}??| = 1/3 \qquad (16.29)$

$S\mathbf{Q}P_{\substack{|??|\\|??|}} \; =$



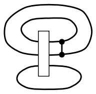
$= (\lambda + \lambda^*)$



$\rightarrow \lambda_{\substack{|??|\\|??|}} = 1/6\,, \qquad (16.30)$

so the eigenvalues are $\lambda_{|}??| = 1/3$, $\lambda_{\substack{|??|\\|??|}} = 1/6$, $\lambda_3 = 1/18$, $\lambda_4 = \lambda$, $\lambda_{4^*} = \lambda^*$. The dimension formulas (A.8) now require evaluation of

$\mathrm{tr}\, S\mathbf{Q} \; =$



$= -\frac{N(N+2)}{6} \qquad (16.31)$

$$\mathrm{tr}\,(S\mathbf{Q})^2 = \texttt{birdTrack} = \frac{N(3N+16)}{36}\,. \tag{16.32}$$

Substituting into (A.8) we obtain the dimensions of the three representations:

$$\begin{aligned} d_3 &= \frac{27\,(m-5)\,(m-8)\,(2m-15)\,(2m-9)\,(5m-36)\,(5m-24)}{m^2(3+m)(12+m)} && (16.33)\\ d_4 &= \frac{10(m-6)^2\,(m-5)\,(m-1)\,(2\,m-9)\,(5\,m-36)\,(5\,m-24)}{3m^2(6+m)(12+m)} && (16.34)\\ d_{4^*} &= \frac{5\,(m-5)\,(m-8)\,(m-6)^2\,(2m-15)\,(5m-36)}{m^3(3+m)(6+m)}(36-m) && (16.35) \end{aligned}$$

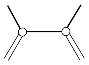
The integer solutions of the above Diophantine conditions are listed in table 16.4.

The main result of all this heavy birdtracking is that $N > 248$ is excluded by the positivity of d_{4^*}, and $N = 248$ is special, as $P_{4^*} = 0$ implies existence of a tensorial identity on the $\mathrm{Sym}^3 A$ subspace. That dimensions should all factor into terms linear in m is althogehter not obvious at this point.

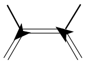
16.3 Decomposition of $|??| \otimes |??\|??|^*$

The decomposition of $\otimes A^2$ tensors has split the traceless symmetric subspace into a pair of representations which we denoted by $|??\|??|$, $|??\|??|^*$. Now we turn to the decomposition of $|??| \otimes |??\|??|$ Kronecker product. We commence by identifying the A and $\otimes A^2$ content of the $|??| \otimes |??\|??| \in \otimes A^3$ Kronecker product. The $|??|$, $|??\|??|$ and $\substack{|??|\\|??|}$ components of $|??| \otimes |??\|??|$ are projected out by

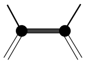
$$P_{|??|} \;=\; K_{|??|} \qquad (16.36)$$

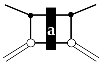


$$P_{|??|??|} \;=\; K_{|??|??|} \qquad (16.37)$$



$$P_{\substack{|??|\\|??|}} \;=\; K_{\substack{|??|\\|??|}} \qquad = \qquad (16.38)$$

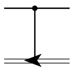




where the $|??| \otimes |??\|??|$ vertex is given by (16.11), and $\substack{|??|\\|??|}$ is the not-adjoint antisymmetric representation in (16.3). In this section double line denotes $|??\|??|$ representation, and K_α are normalization factors given by ratios of dimensions and appropriate Dynkin indices (4.7) (or $3j$ coefficients (????)). As we shall not need them here, we do not write them out explicitly.

We shall use the invariant tensor

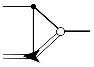
$$R = \qquad (16.39)$$

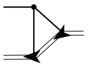


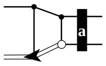
to decompose the remainder subspace

$$P_r = 1 - P_{|??|} - P_{|??|??|} - P_{\substack{|??|\\|??|}}\,. \qquad (16.40)$$

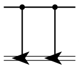
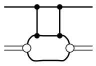
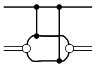
The eigenvalue of R on each of the above subspaces follows from invariance conditions (**??**) and the eigenvalue equation (3.50) $QP_{|??|??|} = \lambda P_{|??|??|}$ (16.11):

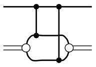
$$RP_{|??|} \quad = $$

$$= (1-\lambda)P_{|??|} \qquad (16.41)$$

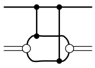
$$RP_{|??|??|} \quad = $$

$$= \frac{1}{2}P_{|??|??|} \qquad (16.42)$$

$$RP_{\substack{|??|\\|??|}} \quad = $$

$$= (1/2-\lambda)P_{\substack{|??|\\|??|}} \,. \qquad (16.43)$$

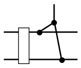
The characteristic equation for R projected to the remained subspace (cf. (3.54)) is obtained by evaluating R^2 and R^3:

$$R^2P_r \quad = $$

$$P_r = 2\left\{\right.$$

$$+$$

$$\left.\right\} P_r$$

$$= \left\{(\lambda+\lambda^*)\hat{R} - 2\lambda\lambda^* + 2\right.$$

$$\left.\right\} P_r \qquad (16.44)$$

$$R^3P_r \quad = \quad (\lambda+\lambda^*)\hat{R}^2 - 4\lambda\lambda^*\hat{R} + 4(\lambda+\lambda^*)$$

$$P_r \qquad (16.45)$$

We have used (16.7), invariance, and the symmetry identity (16.22).


$$= 0 \qquad (16.46)$$

Eliminating the extra invariant tensor in (16.45) by (16.44) we find that R satisfies a cubic equation symmetric under interchange $\lambda \leftrightarrow \lambda^*$

$$0 = (R-(\lambda+\lambda^*))(R-2\lambda)(R-2\lambda^*)P_r\,, \qquad (16.47)$$

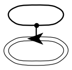
so the eigenvalues of R on the six subspaces of $|??|\otimes|??\|??|$ are $\lambda_{|??|}, \lambda_{|??|??|}, \lambda_{BB}, \lambda_5, \lambda_6, \lambda_7 = 1-\lambda, 1/2, 1/2-\lambda, 1/6, 2\lambda, 2\lambda^*$. As in the preceeding section, this leads to decomposition of the remainder subspace P_r into three subspaces:

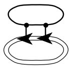
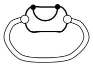
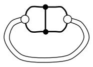
$$P_5 \quad = \quad -\frac{1}{(\lambda-\lambda^*)^2}(R-2\lambda)(R-2\lambda^*)P_r \qquad (16.48)$$

$$P_6 \quad = \quad \frac{1}{2(\lambda-\lambda^*)^2}(R-(\lambda+\lambda^*))(R-2\lambda^*)P_r \qquad (16.49)$$

$$P_7 \quad = \quad \frac{1}{2(\lambda-\lambda^*)^2}(R-(\lambda+\lambda^*))(R-2\lambda)P_r \qquad (16.50)$$

Dimension formulas of sect. **??** require that we evaluate

$$\mathrm{tr}\,\mathbf{1} \quad = \quad Nd_{\substack{|??|\\|??|}}\,, \mathrm{tr}\,R = $$

$$= 0$$

$$\mathrm{tr}\,R^2 \quad = $$

$$= 2\left\{\right.$$

$$-$$

$$\left.\right\} = 2(1-\lambda)d_{|??|??|} \qquad (16.51)$$

	5	8	9	10	12	15	18	24	30	36
N	0	3	8	14	28	52	78	133	190	248
d_1	0	0	1	7	56	273	650	1463	1520	0
d_2	0	-3	0	64	700	4096	11648	40755	87040	147250
d_3	0	0	27	189	1701	10829	34749	152152	392445	779247

Table 16.2: All solutions of Diophantine conditions (16.52–16.54); a bogus $m = 30$ solution still survives this set of conditions. This solution will be DG: MANGLER ET ORD HER, SOM JEG IKKE KAN LAESE eliminated by (**??**) which says that it does not exist for the F_4 subgroup of E_8.

Substituting into (A.8) we obtain the dimensions of the three representations

$$d_5 = \frac{27(m-15)(2m-15)(m-8)(2m-9)(5m-24)(5m-36)}{m^2(m+3)(m+12)} \tag{16.52}$$

$$d_6 = \frac{5(m-5)(2m-15)(m-6)^2(m-8)(5m-36)}{m^3(m+3)(m+6)}(36-m) \tag{16.53}$$

$$d_7 = \frac{5120(m-5)(2m-15)(m-6)^2(m-9)(2m-9)}{m^3(m+6)(m+12)} \tag{16.54}$$

We see that nothing significant is gained beyond the decomposition of $\mathrm{Sym}^3 A$ of the preceeding section; we have recovered representations (**??**), (**??**). Representation (16.54) is new, but yields no new Diophantine condition. If we consider $|??|\times|??\|??|^*$ instead, the only difference is that (16.51) changes to $2(1-\lambda^*)d^*_{|??|??|}$ and we obtain 2 conjugate representations corresponding to $m/6 \leftrightarrow 6/m$ exchange;

$$d_6^* = \tag{16.55}$$

$$d_7^* = \tag{16.56}$$

16.4 Diophantine conditions

This Diophantine condition is satisfied only for $m = 8, 9, 10, 12, 18, 20, 24, 30, 36, 40$ and 45. As we shall show later, $m = 20, 30$ and 40 are bogus solutions, which do not survive further Diophantine conditions.

The integer solutions of the above Diophantine conditions are listed in table 16.4. The formulas (16.52)-(16.54) yield, upon substitution of N, λ, λ^* the correct Clebsch-Gordan series for all members of the E_8 family, table 16.4.

Remark **16.1** Remains to be done: $P_1 = 0 \Rightarrow$ what special E_8 relation? (reduction of 6-loops `birdTrack`?)

16.5 Generalized Young tableaux for E_8

A very tedious table goes in here...?

16.6 Conjectures of Deligne

The construction of the E_8 family outlined in this chapter dates from early 1980's and was partially published in ref. [11]. In a 1995 paper Deligne [19] attributed to Vogel [18] the observation that for the 5 exceptional groups the antisymmetric $\wedge^2 A$ and the symmetric $\mathrm{Sym}^2 A$ adjoint representation tensor products $P_{|}??| + P_{\substack{|??|\\|??|}}$ and $P_\bullet + P_s$ in (16.3), respectively) can be decomposed into irreducible representations in a uniform way, and that their dimensions and casimirs are rational functions of parameter a, related to the parameter m of (16.9) by

$$a = \frac{1}{m-6} \,. \tag{16.57}$$

Here a is $a = \Phi(\alpha, \alpha)$, where α is the largest weight of the representation, and Φ the canonical bilinear form for the Lie algebra, in the notation of Bourbaki. Deligne conjectured that for A_1, A_2, G_2, F_4, E_6, E_7 and E_8, the dimensions of higher tensor representations $\bigotimes^k A$ could likewise be expressed as rational functions of parameter a.

The conjecture was checked on computer by Cohen and de Man [20] for dimensions and quadratic casimirs for all representations up to $\bigotimes^4 A$. They note that "miraculously for all these rational functions both numerator and denominator factor in $Q[a]$ as a product of linear factors". That can perhaps be deduced from the method of decomposing symmetric subspaces outlined in this chapter.

Cohen and de Man have also observed that D_4 should be added to Deligne's list, in agreement with the definition of the E_8 family construction here, consisting of A_1, A_2, G_2, D_4, F_4, E_6, E_7 and E_8. Their algebra goes way beyond the results given in this chapter, which were originally obtained by paper and pencil birdtrack computations performed on trains while commuting between Gothenburg and Copenhagen. Cohen and de Man give formulas for 25 representations, 7 of which are also computed here.

In current context $-a = \lambda^* = 1/6 - \lambda$ is the symmetric space eigenvalue of the invariant tensor $\mathbf{Q}$, my (16.9). The only role of the tensor $\mathbf{Q}$ is to split the traceless symmetric subspace, and its overall scale is arbitrary. In this chapter scale was fixed by setting the adjoint representation quadratic casimir equal to unity, $C_A = 1$ in (16.1). Deligne [19] and Cohen and de Man [20] fix the scale of their λ, λ^* by setting $\lambda + \lambda^* = 1$, so their dimension formulas are stated in terms of a parameter related to the λ used here by $\lambda_{CdM} = 6\lambda$.

They refer to the interchange of the roots $\lambda \leftrightarrow \lambda^*$ as "involution ".

My (16.52) is their A

My (16.53) is their Y_3^*

My (16.54) is their C^*

Chapter 17

E_6 family of invariance groups

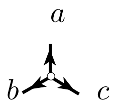
In this chapter we construct all invariance groups whose primitive invariant tensors are δ_b^a and fully symmetric d_{abc}, d^{abc}. The reduction of $\otimes V^2$ space yields a rule for evaluation of the loop contraction of four d-invariants (**??**). The reduction of $V \otimes \overline{V}$ yields the first Diophantine condition (**??**) on the allowed dimensions of the defining representation. The reduction of $\otimes V^3$ tensors is straightforward, but the reduction of $A \otimes V$ space yields the second Diophantine condition (d_4 in table **??**), and limits n to $n \leq 27$. The solutions of Diophantine conditions form the E_6 *family* consisting of $E_6, A_5, A_2 + A_2$ and A_2. For the interesting $E_6(n = 27)$ case the cubic casimir (**??**) vanishes. This property of E_6 enables us to evaluate loop contractions of six d-invariants (**??**), reduce $\otimes A^2$ tensors (table **??**) and investigate relations among the higher order casimirs of E_6 (sect. **??**). In sect. 17.7 we introduce a Young tableaux notation for any representation of E_6 and exemplify its use in construction of Clebsch-Gordan series, table **??**.

17.1 Reduction of two-index tensors

By assumption the primitive invariants set that we shall study

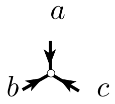
$$\delta_a^b = a \longleftarrow b$$

$$d_{abc} =$$



$$= d_{bac} = d_{acb}$$

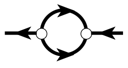
$$d^{abc} =$$



(17.1)

Irreducibility of the defining n-dimensional representation implies

$$d_{abc}d^{bcd} = \alpha \delta_a^d$$



$$= \alpha \longleftarrow \qquad (17.2)$$

The value of α depends on the normalisation convention.[1] We find it convenient to set it to $\alpha = 1$.

We can immediately write a Clebsch-Gordon series for the two-index tensors. The symmetric subspace in (8.5) is reduced by the $d_{abc}d_{cde}$ invariant:

17.2 Mixed two-index tensors

17.3 Diophantine conditions and the E_6 family

The expressions for the dimensions of various representations (see tables in this chapter) are ratios of polynomials in n, the dimension of the defining representation. As the dimension of a representation should be a non-negative integer, these relations are the Diophantine conditions on the allowed values of n. The dimensions of the adjoint representation (**??**) is one such condition; the dimension of λ_4 from table **??** another. Furthermore, the positivity of the dimension λ_4 restricts the solutions to $n \leq 27$. This leaves us with six solutions $n = 3, 6, 9, 15, 21, 27$. As we shall show this in chapter **??**. Of these solutions only $n = 21$ is spurious - the remaining five solutions are realized as the E_6 row of the magic triangle, table **??**.

In the Cartan notation, the corresponding Lie algebras are A_2, $A_2 + A_2$, A_5 and E_6. We do not need to prove this, as for E_6 Springer (see sect. **??**). We call these invariance groups the E_6 *family*, and list the corresponding dimensions, Dynkin labels and Dynkin indices in the tables of this chapter.

17.4 Three-index tensors

The $\otimes V^3$ tensor subspaces cf. $SU(n)$ listed in table **??** are decomposed by invariant matrices constructed from the cubic primitive d_{abc} in the following manner.

17.4.1 Fully symmetric $\otimes V^3$ tensors

17.4.2 Mixed symmetry $\otimes V^3$ tensors

17.4.3 Fully antisymmetric $\otimes V^3$ tensors

17.5 Defining $\otimes$ adjoint tensors

We turn next to the determination of the Clebsch-Gordan series for $V \otimes A$ representations. As always, this series contains the n-dimensional representation

17.6 Two-index adjoint tensors

---

[1] *Freudenthal (1954)* takes $\alpha = 5/2$. *Konstein (1977)* and *Kephart (1981)* take $\alpha = 10$.

17.6.1 Reduction of antisymmetric 3-index tensors

17.7 Dynkin labels and Young tableaux for E_6

17.8 Casimirs for E_6

17.9 Subgroups of E_6

Why is $A_2(6)$ in E_6 family?

The symmetric two-index representation (**??**) of $SU(3)$ is 6 dimensional. The symmetric cubic invariant (17.2) can be constructed using a pair of Levi-Civita tensors

Contractions of several d_{abc}'s can be reduced using the projection operator properties (5.32) of Levi-Civita tensors, yielding expressions like

etc. The reader can check that, for example, the Springer relation (**??**) is satisfied.

Why is $A_5(15)$ in E_6 family?

The antisymmetric two-quark representation (**??**) of $A_5 = SU(6)$ is 15 dimensional. The symmetric cubic invariant (17.2) is constructed using the Levi-Civita invariant (5.28) for $SU(6)$

The reader is invited to check the correctness of the primitivity assumption (**??**). All the other results of this chapter then follow.

Is $A_2 + A_2(9)$ in E_6 family?

Exercise for the reader: unravel the $A_2 + A_2$ 9-dimensional representation, construct the d_{abc} invariant.

17.10 Springer relation

Substituting P_A into the invariance condition (5.58) for d_{abc} one obtains the *Springer* (1959, 1962) *relation*

The Springer relation can be used to eliminate one of the three possible contractions of the three possible contractions of three d_{abc}'s.

17.10.1 Springer's construction of E_6

In the preceding sections we have given a self-contained derivation of the E_6 family, in a form unfamiliar to most experts. Here we shall translate our results into more established notations, and identify those relations which have already been given by other authors.

Consider the exceptional simple Jordan algebra A of Hermitian [3 × 3] matrices x with octonian matrix elements *Freudenthal 1954, 1964)*, and its dual $\overline{A}$ (complex conjugate of A). Following *Springer (1959, 1962)*, define products

$$\begin{aligned} (\overline{x}, y) &= \mathrm{tr}\,(\overline{x}y), \\ x \times y &= \overline{z} \\ 3 < x, y, z > &= (x \times y, z), \end{aligned} \tag{17.3}$$

and assume that they satisfy

$$(x \times x) \times (x \times x) = < x, x, x > x \tag{17.4}$$

Expanding $x, \overline{x}$ in ?? , we chose a normalization

$$(\mathbf{e}_a, \mathbf{e}^b) = \delta^b aa, b = 1, 2, ..., 27 \tag{17.5}$$

and define

$$\mathbf{e}_a \times \mathbf{e}_b = d_{abc}\mathbf{e}^c. \tag{17.6}$$

Substituting into ( ), we obtain ( ), with $\alpha = \frac{5}{2}$. Freudenthal and Springer prove that ( ) is satisfied if d_{abc} is related to the usual Jordan product

$$\mathbf{e}_a \cdot \mathbf{e}_b = ??d_{abc}\mathbf{e}_c, \tag{17.7}$$

by

$$\begin{aligned} d_{abc} \equiv ??d_{abc} - \frac{1}{2}(\delta_{ab}\mathrm{tr}\,(\mathbf{e}_c) + \delta_{ac}\mathrm{tr}\,(\mathbf{e}_b) + \delta_{bc}\mathrm{tr}\,(\mathbf{e}_a)) \\ + \frac{1}{2}\mathrm{tr}\,(\mathbf{e}_a) \cdot \mathrm{tr}\,(\mathbf{e}_b) \cdot \mathrm{tr}\,(\mathbf{e}_c) \end{aligned} \tag{17.8}$$

$E_6(27)$ is the group of isomorphisms which leave $\overline{x}, y = \delta^b_a x^a y_b$ and $< x, y, z >= d^{abc}x_a y_b z_c$ invariant. The derivation was constructed by Freudenthal (1954) (his equation (1.21)):

$$Dz \equiv < x, \overline{y} > z = 2\overline{y}\mathbf{x}(\mathbf{xxz}) - \frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{6}} < \mathbf{x}, \overline{\mathbf{y}} > \mathbf{z}. \tag{17.9}$$

Substituting ( ) we obtain the projector ( ):

$$(Dz)_d = -3x_a y^b P^{ac}_{bd} z_c. \tag{17.10}$$

The object $< z, \overline{y} >$ considered by Freudenthal is in our notation and the above factor -3 is the normalization ( ), Freudenthal's equation (1.26). The invariance of the x-product is given by Freudenthal as

$$< x, x \times x >>= 0. \tag{17.11}$$

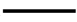
Substituting ( ) we obtain ( ) for d_{abc}.

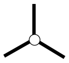
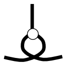
Chapter 18

F_4 family of invariance groups

In this chapter we classify and construct all invariance groups whose primitive invariant tensors are a symmetric bilinear d_{ab}, and a symmetric cubic d_{abc}, satisfying the relation (18.15). The result are summarized in table **??**.

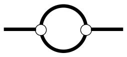
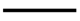
Take as primitives a symmetric quadratic invariant d_{ab} and a symmetric cubic invariant d_{abc}. As explained in chapter **??**, we can use d_{ab} to lower all indices. In the birdtrack notation, we drop the circles denoting d^{ab}, and we drop arrows on all lines:

$d^{ab} = a$  b

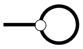
$d_{abc} = d_{bac} = d_{acb} =$  $=$  (18.1)

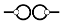
The defining n-dimensional representation is by assumption irreducible, so

$d_{abc}d_{bcd} = \alpha\delta_{ad}$

 $= \alpha$  (18.2)

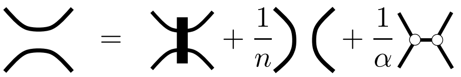
$d_{abb} = 0$

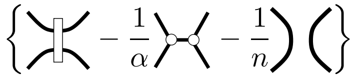
 $= 0$ (18.3)

(Otherwise, we could use  to project out a 1-dimensional subspace). The value of α depends on the normalisation convention (Schafer (1966) takes $\alpha = 7/3$).

18.1 Two-index tensors

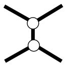
d_{abc} is a Clebsch-Gordan coefficient for $V \otimes V \to V$, so $V \otimes V$ space is decomposed into at least four subspaces:



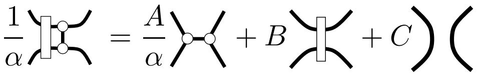


$$1 = A + P_3 + P_1 \qquad (18.4)$$

We turn next to the decompositions induced by the invariant matrix

$$\mathbf{Q}_{ab,cd} = \frac{1}{\alpha}$$

(18.5)

We shall assume that **Q** does not decompose the symmetric subspace, *ie.* that its symmetrized projection can be expressed as



(18.6)

Together with the list of primitives (18.1), this assumption *defines* the F_4 family[1]. This corresponds to the assumption (15.3) in the construction of G_2. In the present case, we have not been able to construct the F_4 family without assuming (18.6).

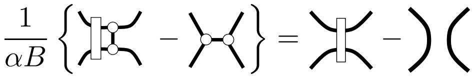
Symmetrizing (18.6) in all legs we obtain

$$\frac{1-A}{\alpha} = (B+C) \qquad (18.7)$$

Neither of the tensors can vanish, as contractions with δ's would lead to

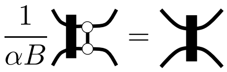
$$\begin{aligned} 0 &= \quad \Rightarrow n+2=0 \\ 0 &= \quad \Rightarrow \alpha=0 \end{aligned} \qquad (18.8)$$

If the coefficients were to vanish, $1-A = B+C = 0$, we would have



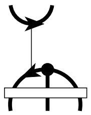
(18.9)

Antisymmetrizing the top two legs we find that in this case also the antisymmetric part of the invariant matrix **Q** (18.5) is reducible:



(18.10)

This would imply that the adjoint representation of $SO(n)$ would also be the adjoint representation for the invariance group of d_{abc}. However, the invariance condition

$$0 =$$

(18.11)

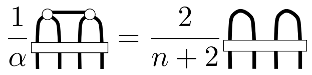
---

[1] Invariance groups with primitives d_{ab}, d_{abc} which do not satisfy (18.6) also exist. The most familiar example is the adjoint representation of $SU(n), n \geq 4$, where d_{abc} is the Gell-Mann (?!) symmetric tensor.

cannot be satisfied for any positive dimension n :

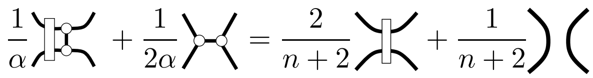
$$\begin{aligned} 0 = \texttt{birdTrack} \quad &\Rightarrow \quad 0 = \texttt{birdTrack} - \texttt{birdTrack} \\ &\Rightarrow \quad n+1=0 \end{aligned} \qquad (18.12)$$

Hence the coefficients in (18.7) are non-vanishing, and are fixed by tracing with δ_{ab}:



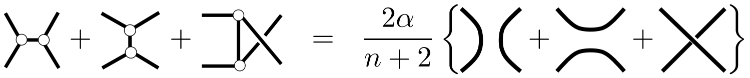
(18.13)

Expanding the symmetrization operator we can write this relation as



(18.14)

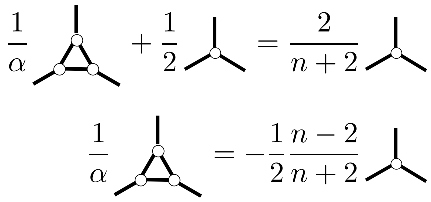
(this fixes $A = -1/2, B = 2/(n+2), C = 1/(n+2)$ in (18.6)), or as



$$d_{abe}d_{ecd} + d_{ade}d_{ebc} + d_{ace}d_{ebd} \quad = \quad \frac{2\alpha}{n+2}(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd})\,. \qquad (18.15)$$

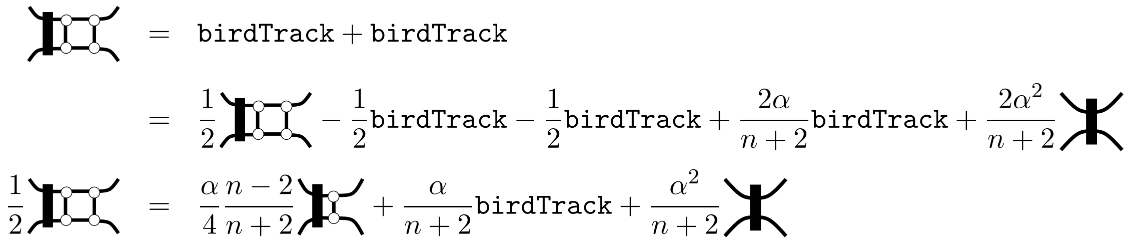
In sect. 18.3 we shall show that this relation can be interpreted as the characteristic equation for [3×3] octonian matrices. This is the *defining relation* for the F_4 family.

The eigenvalue of the invariant matrix $\mathbf{Q}$ on the n-dimensional subspace can now be computed from (18.14)



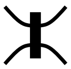
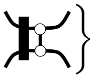
(18.16)

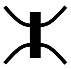
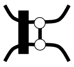
Let us now turn to the action of the invariant matrix $\mathbf{Q}$ on the antisymmetric subspace in (18.4). We evaluate $\mathbf{Q}^2$ with the help of the characteristic equation (18.14):



$$0 \quad = \quad A\left(\mathbf{Q}^2 - \frac{1}{2}\frac{n-6}{n+2}\mathbf{Q} - \frac{2}{n+a}\right) \qquad (18.17)$$

The roots are $\lambda_A = -1/2, \lambda_5 = 4/(n+2)$, and the associated projectors are

$P_A = \frac{8}{n+10}\Big\{$  $+ \frac{n+2}{4\alpha}$  $\Big\}$ (18.18)

$P_5 = \frac{n+2}{n+10}\Big\{$  $- \frac{2}{\alpha}$  $\Big\}$ (18.19)

The dimensions and Dynkin indices are listed in P_A is the projector for the adjoint representation, as it satisfies the invariance condition (18.11):

$$\begin{aligned} \texttt{birdTrack} &= -\frac{1}{2}\texttt{birdTrack} \\ P_A\mathbf{Q} &= -\frac{1}{2}P_A \end{aligned} \tag{18.20}$$

18.2 Defining ⊗ adjoint tensors

18.2.1 Two-index adjoint tensors

18.3 Jordan algebra and $F_4(26)$

Consider the exceptional simple Jordan algebra of traceless Hermitian [3×3] matrices x with octonion matrix elements (Freudenthal 1964, Schafer 1966). The nonassociative multiplication rule for elements x can be written in a basis $x = x_a\mathbf{e}_a$ as

$$\begin{aligned} \mathbf{e}_a\mathbf{e}_b &= \mathbf{e}_b\mathbf{e}_a = \frac{\delta_{ab}}{3}\mathbf{I} + \mathbf{d_{abc}e_c} \\ a, b, c &= 1, 2.......26. \end{aligned} \tag{18.21}$$

where $\mathrm{tr}\,(\mathbf{e}_a) = 0$ and $\mathbf{I}$ is the [3×3] unit matrix. Traceless [3×3] matrices satisfy a characteristic equation

$$x^3 - \frac{1}{2}\mathrm{tr}\,(x^2)x - \frac{1}{3}\mathrm{tr}\,(x^3)\mathbf{I} = \mathbf{0} \tag{18.22}$$

Substituting (?!) we obtain (?!) with normalization $\alpha = \frac{7}{3}$. Substituting (?!) into the Jordan identity (Schafer 1966)

$$(xy)x^2 = x(yx^2) \tag{18.23}$$

we obtain (?!). It is interesting to note that the Jordan identity (which defines Jordan algebra in the way Jacobi identity defines Lie algebra) is a trivial consequence of (?!). $F_4(28)$ is the group of isomorphisms which leave forms $\mathrm{tr}\,(xy) = \delta_{ab}x_ax_b$ and $\mathrm{tr}\,(xyz) = d_{abc}x_ay_bz_c$ invariant. The derivation is given by Tits (1966) as

$$Dz = (xz)y - x(zy)\,. \qquad \text{Tits 1966, equation(28)} \tag{18.24}$$

Substituting (18.21) we obtain the adjoint representation projection operator (18.18)

$$(Dz)_d = -3x_ay_b\left(\frac{\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}}{9} + \frac{d_{bce}d_{ead} - d_{ace}d_{ebd}}{3}\right)z_c \tag{18.25}$$

Chapter 19

E_7 family of invariance groups

E_7 family of invariance groups, negative dimensions: published (birdtrack free) as *"Negative dimensions and E_7symmetry"*, *Nucl. Phys.* **B188**, 373 (1981).

Chapter 20

Exceptional magic

The study of invariance algebras pursued in the preceding chapters might appear to be a rather haphazard affair. Given a set of primitives, one gets some Diophantine equations, constructs the family of invariance algebras and moves onto the next set of primitives. However, a closer scrutiny of the Diophantine conditions leads to a surprise: most of the Diophantine equations are special cases of one and the same Diophantine equation, and they magically arrange all exceptional families into a single triangular pattern which we shall call the "magic triangle".

20.1 Magic triangle

Our construction of invariance algebras has generated a series of Diophantine conditions which we now summarize. The adjoint representation conditions are:

$$\begin{array}{ll} F_4 \text{ family} & N = 3n - 36 + \dfrac{360}{n+10} \\ E_6 \text{ family} & N = 4n - 40 + \dfrac{360}{n+9} \\ E_7 \text{ family} & N = 3n - 45 + \dfrac{360}{n/2+8} \\ E_8 \text{ family} & N = 10m - 122 + \dfrac{360}{m} \end{array} \tag{20.1}$$

There is a striking similarity between the conditions for different families. If we define

$$\begin{array}{ll} F_4 \text{ family} & m = n + 10 \\ E_6 \text{ family} & m = n + 9 \\ E_7 \text{ family} & m = n/2 + 8 \end{array} \tag{20.2}$$

we can parametrize all the solutions of the above Diophantine conditions with a single integer m, see table 20.1. The Clebsch-Gordan series for $A \otimes V$ Kronecker

m	8	9	10	12	15	18	20	24	30	36	40	···	360
F_4			0	0	3	8	.	21	.	52	.	···	.
E_6		0	0	2	8	16	.	35	36	78	.	···	.
E_7	0	1	3	9	21	35	.	66	99	133	.	···	.
E_8	3	8	14	28	52	78	.	133	190	248	.	···	.

Table 20.1: All solutions of Diophantine conditions (20.1) not eliminated by other Diophantine conditions of chapter 15 through 18; those are marked by "·".

products also show a striking similarity. The characteristic equations (16.7), (**??**), (**??**) and (**??**) are the one and the same equation

$$(Q-1)\left(Q+\frac{6}{m}\right)P_r=0 \tag{20.3}$$

Here P_r removes the defining and $\otimes V^2$ subspaces, and we have rescaled the E_8 operator Q (16.7) by factor 2. (Role of the Q operator is only to distinguish between two subspaces - we are free to rescale it as we wish).

In the dimensions of the associated representations, eigenvalue $6/m$ introduces a new Diophantine denominator $m+6$. For example, from (16.15), table **??**, (**??**) and (**??**), the highest dimensional representation in $V\otimes A$ has dimension (in terms of parametrization (20.2)):

$$\begin{aligned}
&F_4 \text{ family} && 3(m+6)^2-156(m+6)+2673-\frac{15120}{m+6}\\
&E_6 \text{ family} && 4(m+6)^2-188(m+6)+2928-\frac{15120}{m+6}\\
&E_7 \text{ family} && 2\left\{6(m+6)^2-246(m+6)+3348-\frac{15120}{m+6}\right\}\\
&E_8 \text{ family} && 50m^2-1485m+19350+\frac{27\cdot 360}{m}-\frac{11\cdot 15120}{m+6}
\end{aligned} \tag{20.4}$$

These Diophantine conditions eliminate most of the spurious solutions of (20.1); only the $m=30$, 60, 90 and 120 spurious solutions survive, but are in turn eliminated by further conditions. For the E_8 family $V\otimes V=V\otimes A=A\otimes A$ (the defining representation is the adjoint representation), hence the Diophantine condition (20.4) includes both $1/m$ and $1/(m+6)$ terms.

Not only can the four Diophantine conditions (20.1) be parametrized by a single integer m; the list of solutions table 20.1 turns out to be symmetric under the flip across the diagonal. F_4 solutions are the same as those in the $m=15$ column, and so on. This suggests that the rows be parametrized by an integer ℓ, in a fashion symmetric to the column parametrization by m. Indeed, the requirement of $m\leftrightarrow\ell$ symmetry leads to a unique expression which contains the four Diophantine conditions (20.1) as special cases:

$$N=\frac{(\ell-6)(m-6)}{3}-72+\frac{360}{\ell}+\frac{360}{m} \tag{20.5}$$

We take $m = 8, 9, 10, 12, 15, 24$ and 36 as all the solutions allowed in table 20.1. By symmetry ℓ takes the same values. All the solutions fill up the *magic triangle*, table 20.1. Within each entry the number in the upper left corner is N, the dimension of the corresponding Lie algebra, and the number in the lower left corner is n, the dimension of the defining representation. The expressions for n for the top four rows are guesses. The triangle is called magic, partly because we arrived to it by magic, and partly because it contains Freudenthal's (1964) magic square, marked by the dotted line in table 20.1.

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $\mathbf{2A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $\mathbf{3A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Table 20.2: *Magic triangle.* All exceptional Lie groups defining and adjoint representations form an array of the solutions of the Diophantine condition (20.5). Within each entry the number in the upper left corner is N, the dimension of the corresponding Lie algebra, and the number in the lower left corner is n, the dimension of the defining representation.

Chapter 21

Magic negative dimensions

21.1 E_7 and $SO(4)$

21.2 E_6 and $SU(3)$

Appendix A

Recursive decomposition

This appendix deals with practicalities of computing eigenvalues, and is best skipped on first reading.

Let P stand for a projection onto a subspace, or the entire space (in which case $P = \mathbf{1}$). Assume that the subspace has already been reduced into m irreducible subspaces and a reminder

$$P = \sum_{\gamma=1}^{m} P_\gamma + P_r \tag{A.1}$$

Now adjoin a new invariant matrix Q to the set of invariants. By assumption, Q does not reduce further the $\gamma = 1, 2, \ldots, m$ subspaces, *ie.* has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$

$$QP_\gamma = \lambda_\gamma P_\gamma \qquad \text{(no sum).} \tag{A.2}$$

on the γth subspace. We construct an invariant matrix $\hat{Q}$ restricted to the remaining (as yet not decomposed) subspace by

$$\hat{Q} := P_r Q P_r = PQP - \sum_{\gamma=1}^{m} \lambda_\gamma P_\gamma \tag{A.3}$$

As P_r is a finite dimensional subspace, $\hat{Q}$ satisfies a *minimal* characteristic equation of order $n \geq 2$

$$\sum_{k=0}^{n} a_k \hat{Q}^k = \prod_{\alpha=m+1}^{m+n} (\hat{Q} - \lambda_\alpha P_r) = 0 \tag{A.4}$$

with the corresponding projection operators (3.45).

$$P_\alpha = \prod_{\beta \neq \alpha} \frac{\hat{Q} - \lambda_\beta}{\lambda_\alpha - \lambda_\beta} P_r \,, \qquad \alpha = \{m+1, \ldots, m+n\} \tag{A.5}$$

"Minimal" in the above means that we drop repeated roots, so all eigenvalues are distinct. $\hat{Q}$ is an awkward object in computations, so we reexpress the projection operator in terms of Q as follows.

Define first the polynomial obtained by deleting the $(\hat{Q} - \lambda_\alpha)$ factor from (A.4)

$$\prod_{\beta \neq \alpha} (x - \lambda_\beta) = \sum_{k=0}^{n-1} b_k x^k \,, \qquad \alpha, \beta = m+1, \dots m+n \,, \tag{A.6}$$

where the expansion coefficient $b_k = b_k^{(\alpha)}$ depends on the choice of the subspace α. Substituting $P_r = P - \sum_{\alpha=1}^{m} P_\alpha$ and using the orthonomality of P_α we obtain an alternative formula for the projection operators

$$P_\alpha = \frac{1}{\sum b_k \lambda_\alpha^k} \sum_{k=0}^{n-1} b_k \left\{ (PQ)^k - \sum_{\gamma=1}^{m} \lambda_\alpha^k P_\gamma \right\} P \tag{A.7}$$

and dimensions

$$d_\alpha = \mathrm{tr}\, P_\alpha = \frac{1}{\sum b_k \lambda_\alpha^k} \sum_{k=0}^{n-1} b_k \left\{ \mathrm{tr}\, (PQ)^k - \sum_{\gamma=1}^{m} \lambda_\gamma^k d_\gamma \right\} . \tag{A.8}$$

The utility of this formula lies in the fact that once the polynomial (A.6) is given only new data it requires are the traces $\mathrm{tr}\,(PQ)^k$, and those are simpler to evaluate than $\mathrm{tr}\,\hat{Q}^k$.

Appendix B

Properties of Young Projections

(H. Elvang and P. Cvitanović)

In this appendix we prove the properties of the Young projections stated in sect. 8.4.

B.1 Uniqueness of Young projection operators

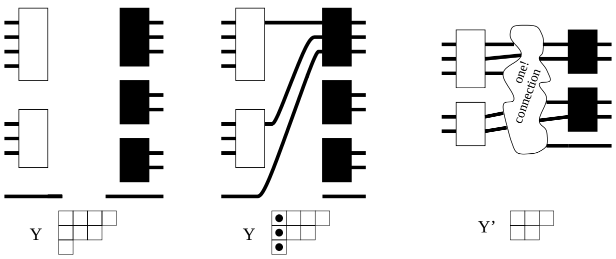
We now show that the Young projection operator P_{Y} is well-defined by proving the existence and uniqueness (up to sign) of a non-vanishing connection between the symmetrizers and antisymmetrizers in P_{Y}.

The proof is induction over the number of columns t in the Young diagram Y. For $t = 1$ the Young projection operator consists of one antisymmetrizer of length s and s symmetrizers of length 1, and clearly the connection can only be made in one way, up to an overall sign.

Assume the result to be valid for Young projections derived from Young diagrams with $t - 1$ columns. Let Y be a Young diagram with t columns. The lines from A_1 in P_{Y} must connect to different symmetrizers for the connection to be non-zero. Since there are exactly $|\mathrm{A}_1|$ symmetrizers in P_{Y}, this can be done in essentially one way, since which line goes to which symmetrizer is only a matter of an overall sign, and where a line enters a symmetrizer is irrelevant due to (5.9).

After having connected A_1, connecting the symmetry operators in the rest of P_{Y} is the problem of connecting symmetrizers to antisymmetrizers in the Young projection $P_{\mathrm{Y}'}$, where Y' is the Young diagram obtained from Y by slicing off the first column. Thus Y' has $k - 1$ columns, so by the induction hypothesis the rest of the symmetry operators in P_{Y} can be connected in exactly one non-vanishing way (up to sign).

The principles are illustrated below:



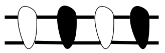
(B.1)

B.2 Normalization

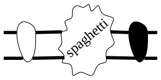
We now derive the formula for the normalization factor α_{Y} such that the Young projection operators are idempotent, $P_{\mathrm{Y}}^2 = P_{\mathrm{Y}}$. By the normalization of the symmetry operators, Young projection operators derived from fully symmetrical or antisymmetrical Young tableaux will be idempotent with $\alpha_{\mathrm{Y}} = 1$.

P_{Y}^2 is simply P_{Y} connected to P_{Y}, hence it may be viewed as a set of *outer* symmetry operators connected by a set of *inner* symmetry operators. Expanding all the inner symmetrisers and using the uniqueness of the non-zero connection between the symmetrizers and antisymmetrizers of the Young projection, we find that each term in the expansion is either 0 or a version of P_{Y}. In fact, the number of non-zero terms — denote it $\|\mathrm{Y}\|$ — is just the number $|\mathrm{Y}|$ defined in sect. 8.4. For a Young diagram with s rows and t columns there will be a factor of $\frac{1}{|\mathrm{S}_i|}$ ($\frac{1}{|\mathrm{A}_i|}$) for expansion of each inner (anti)symmetrizer, thus we find

$$P_{\mathrm{Y}}^2 = \alpha_{\mathrm{Y}}^2$$



$$= \frac{\alpha_{\mathrm{Y}}^2}{\prod_{i=1}^{s}|\mathrm{S}_i|!\prod_{j=1}^{t}|\mathrm{A}_j|!}\sum_{\text{mess}}$$



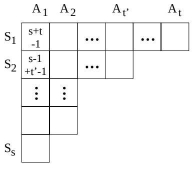
$$= \alpha_{\mathrm{Y}}\frac{|\mathrm{Y}|}{\prod_{i=1}^{s}|\mathrm{S}_i|!\prod_{j=1}^{t}|\mathrm{A}_j|!}P_{\mathrm{Y}}. \tag{B.2}$$

Idempotency is then achieved by taking

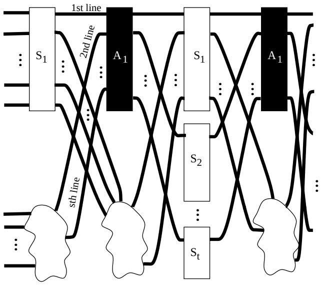
$$\alpha_{\mathrm{Y}} = \frac{\prod_{i=1}^{s}|\mathrm{S}_i|!\prod_{j=1}^{t}|\mathrm{A}_j|!}{|\mathrm{Y}|}. \tag{B.3}$$

Let Y be a Young tableau with $|\mathrm{A}_1| = s$, $|\mathrm{S}_1| = t$, $|\mathrm{S}_2| = t'$ *etc.* We count in how many ways the lines entering the inner A_1 pass through it to yield non-zero connections. We refer to

$$\mathrm{Y} =$$



$$\rightsquigarrow \quad P_{\mathrm{Y}}^2 = \alpha_{\mathrm{Y}}^2$$



(B.4)

in the following. For each of the inner symmetrizers there must be exactly one from A_1. The first line can pass through A_1 in s ways and without loss of generality we may take it to pass straight through, connecting to S_1 where it can pass through in t ways. Thus for the first line, there were $s+t-1$ allowed roads through the inner symmetry operators. The second line may now pass through A_1 in $s-1$ ways, and we can take it to pass straight through to S_2, where it has t' possibilities. Thus we have found $(s-1)+t'-1$ options for the second line. With a similar reasoning we find $(s-2)+t''-1$ allowed ways for the third line *etc.*

Let w_{Y} be the number of ways of passing the m lines entering A_1 through the inner symmetry operators. w_{Y} is then the product of the numbers found above, $w_{\mathrm{Y}} = (s+t-1)(s-1+t'-1)(s-2+t''-1)\cdots$. Note that when calculating $|\mathrm{Y}|$ the product of the numbers in the first column of the Young diagram is w_{Y}.

We show $\|\mathrm{Y}\| = |\mathrm{Y}|$ by induction on the number of columns t in the Young diagram Y.

For a single column Young diagram, $|\mathrm{Y}| = |\mathrm{A}_1|!$, and the number of non-zero ways to connect the A_1 symmetrizers to A_1 in P_{Y} is $|\mathrm{A}_1|!$, hence $\|\mathrm{Y}\| = |\mathrm{Y}|$ for $t = 1$.

Assume that $\|\mathrm{Z}\| = |\mathrm{Z}|$ for any Young diagram Z with $t-1$ columns. Let Y be a Young diagram with t columns and let Y' be the Young diagram obtained form Y by removal of the first column. w_{Y} is the number of ways the lines entering the first inner antisymmetrizer in P^2_{Y} are allowed to pass through the inner symmetry operators. Finding the number of allowed paths for the rest of the lines is the problem of finding the number of allowed paths through the inner symmetry operators of $P^2_{\mathrm{Y}'}$, which is $\|\mathrm{Y}'\| = |\mathrm{Y}'|$. Now we have $\|\mathrm{Y}\| = \|\mathrm{Y}'\| w_{\mathrm{Y}} = |\mathrm{Y}'| w_{\mathrm{Y}} = |\mathrm{Y}|$.

B.3 Orthogonality

If Y and Z denote Young tableaux derived from the same Young diagram, then $P_{\mathrm{Y}} P_{\mathrm{Z}} = P_{\mathrm{Z}} P_{\mathrm{Y}} = \delta_{Y,Z} P_{\mathrm{Y}}$, since there is a non-trivial permutation of the lines connecting the symmetry operators of Y with those of Z and by uniqueness of the non-zero connection the result is either $P^2 = P$ or 0.

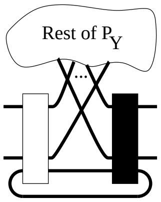
Next, consider two *differently shaped* Young diagrams Y and Z with the same number of boxes. Since at least one column must be bigger in (say) Y than in Z and the p lines from the corresponding antisymmetrizer must connect to different symmetrizers it is not possible to make a non-zero connection between the antisymmetry operators of P_{Y} to the symmetrizers in P_{Z}, and hence $P_{\mathrm{Y}} P_{\mathrm{Z}} = 0$. By a similar argument, $P_{\mathrm{Z}} P_{\mathrm{Y}} = 0$.

B.4 The dimension formula

The dimensions of the irreducible representations can be calculated recursively from the Young projection operators. Here is the recipe:

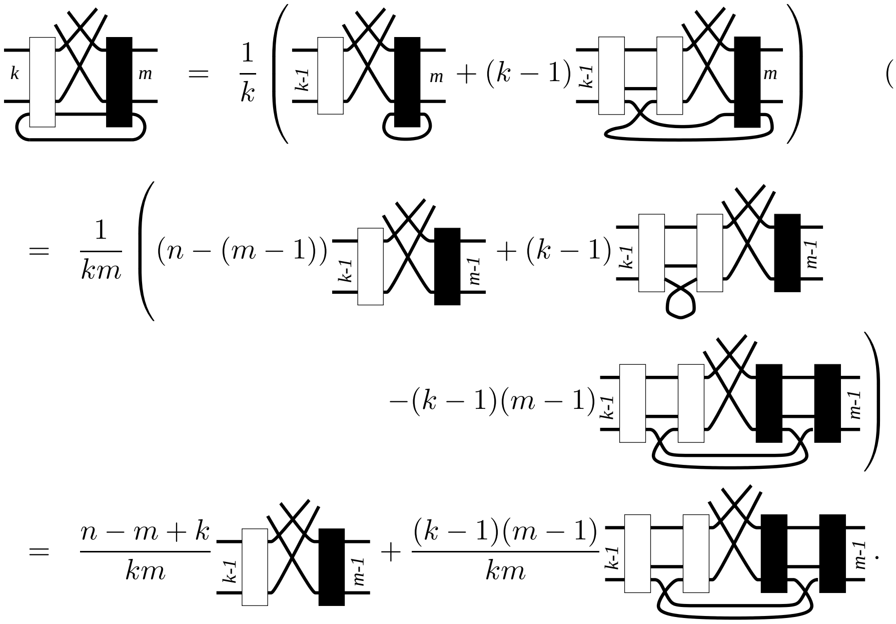
Let Y be a Young diagram and Y′ the Young diagram obtained from Y by removal of the right-most box in the last row. Draw the Young projection operators corresponding to Y and Y′ and note that if we trace the last line of P_{Y} we obtain $P_{\mathrm{Y}'}$ multiplied by a factor.

Quite generally this contraction will look like



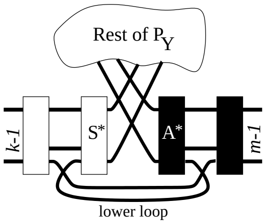
(B.5)

Using (5.11) and (5.20) we have


k
m
$=\frac{1}{k}\Big($
k-1
m
$+(k-1)$
k-1
m
$\Big)$
$=\frac{1}{km}\Big((n-(m-1))$
k-1
m-1
$+(k-1)$
k-1
m-1
$-(k-1)(m-1)$
k-1
m-1
$\Big)$
$=\frac{n-m+k}{km}$
k-1
m-1
$+\frac{(k-1)(m-1)}{km}$
k-1
m-1
.

(B.6)

Inserting (B.6) into (B.5) we see that the first term is proportional to the projection $P_{\mathrm{Y}'}$. The second term vanishes:


Rest of P_Y
k-1
S*
A*
m-1
lower loop

(B.7)

The lines going into S* come from antisymmetrizers in the rest of the P_{Y}-diagram. One of these lines, from A_a, say, must pass from S* through the lower loop to A* and from A^* connect to one of the symmetrizers, say S_S in the rest of the P_Y-diagram. But due to the construction of the connection between symmetrizers and antisymmetrizers in a Young projection, there already is a line connecting S_s to A_a. Hence the diagram vanishes.

The dimensionality formula follows by induction on the number of boxes in the Young diagrams with the dimension of a single box Young diagram being n. Let Y be a Young diagram with p boxes. We assume that the dimensionality formula is valid for any Young diagram with $p-1$ boxes. With $P_{\mathrm{Y}'}$ obtained from P_Y as above, we have (using (B.6) and writing D_Y for the birdtrack diagram of P_Y):

$$\begin{aligned} \dim P_\mathrm{Y} &= \alpha_\mathrm{Y} \mathrm{tr}\, \mathrm{D}_\mathrm{Y} = \frac{\mathrm{n-m+k}}{\mathrm{km}} \alpha_\mathrm{Y} \mathrm{tr}\, \mathrm{D}_{\mathrm{Y}'} && \text{(B.8)} \\ &= (n-m+k)\alpha_{\mathrm{Y}'} \frac{|\mathrm{Y}'|}{|\mathrm{Y}|} \mathrm{tr}\, \mathrm{D}_{\mathrm{Y}'} && \text{(B.9)} \\ &= (n-m+k)\frac{f_{\mathrm{Y}'}}{|\mathrm{Y}|} = \frac{f_\mathrm{Y}}{|\mathrm{Y}|} && \text{(B.10)} \end{aligned}$$

This completes the proof of the dimensionality formula (8.27).

B.5 Literature

- This introduction to the Young tableaux is based on Lichtenberg [4], Hamermesh [5] and van der Waerden [6].
- The rules for reduction of direct products: See Lichtenberg [4]. The rules are stated here as in (Elvang 1999).
- The method of constructing the Young projections directly from the Young tableaux is described in van der Waerden [6], who ascribes the idea to von Neumann. See also Kennedy slides [3].
- Alternative labelling of Young diagrams: Fischler??.
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Index

- ,
  - $mathrmY$72
- $SU(n)$, 74
  - 3-j, 76
  - 3-j evaluation, 77
  - adjoint representation, 84
  - adjoint tensor, 81
  - basis vectors, 75
  - casimirs, 84
  - characters, 79
  - Dynkin labels, 84
  - representation, 69
  - three-vertex, 76
- $SU(n)$ representation
  - dimensionality, 69
- $U(n)$
  - dimensionalities, 72
  - irreducible representations, 72
- $f_{\mathrm{Y}}(n)$, 72
- Young projection operator
  - construction, 71
- Young projection operators
  - basis vectors, 75
- k-standard arrangement, 68
- 3-j
  - $SU(n)$, 76
  - negative dimensions
    - $SU(n)$, 77

- adjoint representation
  - $SU(n)$, 84
- adjoint tensor
  - $SU(n)$, 81
  - two-index $SU(n)$, 83

- casimirs
  - $SU(n)$, 84
- characteristic equation
  - unitary three-index tensor, 67
  - unitary two-index tensor, 65
- characters
  - $SU(n)$, 79
- color algorithm
  - dimensionality, 73
- completeness, 74
- conjugate
  - Young tableaux, 70

- decomposition, 74
- dimensionality
  - $SU(n)$ representation, 69
  - $U(n)$, 72
  - color algorithm, 73
- direct products
  - Young tableaux, 70
- Dynkin indices
  - unitary two-index tensor, 66
- Dynkin labels
  - $SU(n)$, 84

- group
  - unitary, 65, 147

- inner product
  - space, 16
- invariant, 20

- length
  - row/column, 71
- linear space, 16

- mixed two-index tensor, 79
- multi-particle state, 69

negative dimensions
  $SU(n)$
    3-j evaluation, 77
  application, 77
normalization
  Young proejction, 148

orthogonality
  Young projections, 149

representation
  $SU(n)$, 69

standard arrangement, 69
SU(2)
  Young tableaux, 69
sum rule, $SU(n)$ 3-j, 78

tableaux
  Young, 68
tensor
  three-index, 66
  two-index, 65
tensors
  decomposition, 74
three-index tensor, 66
transpose
  Young tableaux, 69
two-index tensor
  adjoint $SU(n)$, 83
  mixed, 79
  unitary, 65
  Young tableaux, 69

unitary
  group, 65, 147

vector space, 16

Young diagram, 68
  k-standard arrangement, 68
  definition, 68
  length, 71
Young projection, 71
  ,
    $mathrmY72$
  3-j, 76
  completeness, 74
  decomposition, 74
  normalization, 72, 148
  orthogonality, 149
  proof of properties, 147
  properties, 72
  three-vertex, 76
  well-defined, 147
Young tableaux, 68
  $SU(n)$, 69
  3-j, 76
  conjugate, 70
  decomposition, 74
  definition, 68
  dimensionality $U(n)$, 72
  direct products, 70
  projection, 71
  standard arrangement, 69
  SU(2), 69
  three-vertex, 76
  transpose, 69
  two-index tensor, 69