Embedding the diamond graph in L_p and dimension reduction in L_1

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Abstract

We show that any embedding of the level k diamond graph of Newman and Rabinovich [6] into L_p, $1 < p \leq 2$, requires distortion at least $\sqrt{k(p-1)+1}$. An immediate corollary is that there exist arbitrarily large n-point sets $X \subseteq L_1$ such that any D-embedding of X into ℓ_1^d requires $d \geq n^{\Omega(1/D^2)}$. This gives a simple proof of a recent result of Brinkman and Charikar [2] which settles the long standing question of whether there is an L_1 analogue of the Johnson-Lindenstrauss dimension reduction lemma [4].

1 The diamond graphs, distortion, and dimension

We recall the definition of the diamond graphs $\{G_k\}_{k=0}^{\infty}$ whose shortest path metrics are known to be uniformly bi-lipschitz equivalent to a subset of L_1 (see [3] for the L_1 embeddability of general series-parallel graphs). The diamond graphs were used in [6] to obtain lower bounds for the Euclidean distortion of planar graphs and similar arguments were previously used in a different context by Laakso [5].

G_0 consists of a single edge of length 1. G_i is obtained from G_{i-1} as follows. Given an edge $(u,v) \in E(G_{i-1})$, it is replaced by a quadrilateral u, a, v, b with edge lengths 2^{-i}. In what follows, (u,v) is called an edge of level $i-1$, and (a,b) is called the level i anti-edge corresponding to (u,v). Our main result is a lower bound on the distortion necessary to embed G_k into L_p, for $1 < p \leq 2$.

Theorem 1.1. *For every $1 < p \leq 2$, any embedding of G_k into L_p incurs distortion at least $\sqrt{1+(p-1)k}$.*

The following corollary shows that the diamond graphs cannot be well-embedded into low-dimensional ℓ_1 spaces. In particular, an L_1 analogue of the Johnson-Lindenstrauss dimension reduction lemma does not exist. The same graphs were used in [2] as an example which shows the impossibility of dimension reduction in L_1. Our proof is different and, unlike the linear programming based argument appearing there, relies on geometric intuition. We proceed by observing that a lower bound on the rate of decay of the distortion as $p \to 1$ yields a lower bound on the required dimension in ℓ_1.

Corollary 1.2. *For every $n \in \mathbb{N}$, there exists an n-point subset $X \subseteq L_1$ such that for every $D > 1$, if X D-embeds into ℓ_1^d, then $d \geq n^{\Omega(1/D^2)}$.*

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Proof. Since ℓ_1^d is $O(1)$-isomorphic to ℓ_p^d when $p = 1 + \frac{1}{\log d}$ and G_k is $O(1)$-equivalent to a subset $X \subseteq L_1$, it follows that $\sqrt{1 + \frac{k}{\log d}} = O(D)$. Noting that $k = \Omega(\log n)$ completes the proof. □

2 Proof

The proof is based on the following inequality. The case $p = 2$ is the well known "short diagonals lemma" which was central to the argument in [5, 6].

Lemma 2.1. *Fix $1 < p \leq 2$ and $x, y, z, w \in L_p$. Then,*

$$\|y - z\|_p^2 + (p-1)\|x - w\|_p^2 \leq \|x - y\|_p^2 + \|y - w\|_p^2 + \|w - z\|_p^2 + \|z - x\|_p^2.$$

Proof. For every $a, b \in L_p$, $||a + b||_p^2 + (p-1)||a - b||_p^2 \leq 2(||a||_p^2 + ||b||_p^2)$. A simple proof of this classical fact can be found, for example, in [1]. Now,

$$||y - z||_p^2 + (p-1)||y - 2x + z||_p^2 \leq 2||y - x||_p^2 + 2||x - z||_p^2$$

and

$$||y - z||_p^2 + (p-1)||y - 2w + z||_p^2 \leq 2||y - w||_p^2 + 2||w - z||_p^2.$$

Averaging these two inequalities yields

$$||y - z||_p^2 + (p-1)\frac{||y - 2x + z||_p^2 + ||y - 2w + z||_p^2}{2} \leq \|x - y\|_p^2 + \|y - w\|_p^2 + \|w - z\|_p^2 + \|z - x\|_p^2.$$

The required inequality follows by convexity. □

Lemma 2.2. *Let A_i denote the set of anti-edges at level i and let $\{s, t\} = V(G_0)$, then for $1 < p \leq 2$ and any $f : G_k \to L_p$,*

$$||f(s) - f(t)||_p^2 + (p-1)\sum_{i=1}^{k}\sum_{(x,y)\in A_i} ||f(x) - f(y)||_p^2 \leq \sum_{(x,y)\in E(G_k)} ||f(x) - f(y)||_p^2.$$

Proof. Let (a, b) be an edge of level i and (c, d) its corresponding anti-edge. By Lemma 2.1, $||f(a) - f(b)||_p^2 + (p-1)||f(c) - f(d)||_p^2 \leq ||f(a) - f(c)||_p^2 + ||f(b) - f(c)||_p^2 + ||f(d) - f(a)||_p^2 + ||f(d) - f(b)||_p^2$. Summing over all such edges and all $i = 0, \ldots, k-1$ yields the desired result by noting that the terms $||f(x) - f(y)||_p^2$ corresponding to $(x, y) \in E(G_i)$ cancel for $i = 1, \ldots, k-1$. □

The main theorem now follows easily.

Proof of Theorem 1.1. Let $f : G_k \to L_p$ be a non-expansive D-embedding. Since $|A_i| = 4^{i-1}$ and the length of a level i anti-edge is 2^{1-i}, applying Lemma 2.2 yields $\frac{1+(p-1)k}{D^2} \leq 1$. □

References

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