Complex Numbers and Complex Arithmetic

Complex Numbers

As a set of mathematical objects, the *complex numbers* can be considered to coincide exactly with the points in the standard two dimensional real vector space $\mathbf{R}^2$. What is new in designating the points of R^2 as complex numbers is the arithmetic and algebraic structure thereby imposed on that set of points. Complex numbers, therefore, consist of pairs :

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x,\, y \text{ real numbers.}$$

So written, x is called the *real part* of the complex number z and y is called the *imaginary part* of z; we write

$$x = \mathbf{Re}\, z, \quad y = \mathbf{Im}\, z.$$

We single out particular pairs for special emphasis. Any pair of the form $(r, 0)$ is considered to be essentially the same as the real number r and will henceforth be denoted simply as r. In particular the pairs $(0, 0)$ and $(1, 0)$ will simply be referred to as 0 and 1, respectively. The pair $(0, 1)$ is denoted by i and any pair $(0, s)$ is written as $s\, i$ or $i\, s$; thus we can write

$$z = \begin{pmatrix} x \\ y \end{pmatrix} = x + i\, y,$$

which is the standard representation of complex numbers. In engineering and some of the natural sciences it is not uncommon to find the i notation replaced by j, simply reflecting the fact that in R^2 as a vector space we often find the vector (x, y) written as $x\, \mathbf{i} + y\, \mathbf{j}$. It is immaterial which choice is made but here we will use the symbol i for $(0, 1)$.

An important alternative representation of the complex numbers is obtained by going to *polar coordinates*. *Cartesian coordinates* (x, y) for points in the plane are related to polar coordinates r, θ for those same points by the relationships

$$x = r\cos\theta,\ y = r\sin\theta,\ \text{if } r \neq 0,$$
$$x = 0,\ y = 0,\ \text{if } r = 0;$$
$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(y/x),\ x, y \text{ not both } 0.$$

Correspondingly we have the polar representation of the complex number

$$z = x + i\,y = r\,(\cos\theta + i\,\sin\theta).$$

This is sometimes written as $z = r\,\mathbf{cis}\,\theta$ - but not here; later, after we introduce the complex exponential function, we will use the notation $z = r\,e^{i\theta}$. The angle θ, normally given in radian measure, is called the *argument* of the complex number z as shown, while $r = \sqrt{x^2 + y^2}$ is the *absolute value*, or *modulus* of z.

Complex Arithmetic

Additive Operations on Complex Numbers *Addition* and *subtraction* of complex numbers agrees exactly with addition and subtraction of vectors in R^2; it is performed *componentwise*. Thus

$$(u + i\,v) \pm (x + i\,y) = (u \pm x) + (v \pm y)\,i.$$

As a result of this we can see immediately that the *real numbers*, $r = (r, 0)$ and the *(purely) imaginary* numbers $s\,i = (0, s)$ are closed under addition or subtraction and that $0 = (0, 0)$ continues to play the role of the additive identity for complex numbers just as it does for real numbers, i.e., $z + 0 = z$ for all complex numbers z. Thus all complex numbers with a given modulus

r lie on the circle of radius r in $\mathbf{R}^2$ with center at the origin and all complex numbers with a given argument θ lie on the ray emanating from the origin which forms the angle θ with the positive real axis. The angle θ is not unique; two angles θ and ψ differing by an integer multiple of 2π are equivalent (but sometimes have to be distinguished as in, for example, computation of roots, discussed subsequently).

Complex Multiplication It is fair to say that in addition to the introduction of the symbol $i = (0, 1)$, the "essence" of the complex number system lies in the definition of **complex multiplication**. Historically, complex numbers were not originally introduced as two dimensional vectors, but rather as "ideal", "imaginary", or "impossible" numbers to provide solutions for equations like $y^2 + 1 = 0$. The symbol i was first introduced as that "ideal", "imaginary" or "impossible" quantity such that $i^2 = -1$, which then clearly solves $y^2 + 1 = 0$. What is remarkable is that, as it later turned out, by introducing this one number and its extensions to $x + i\,y$, a system was created in which all polynomial equations, of any degree n, would have n solutions, admitting multiplicity. Starting with $i^2 = -1$ and assuming we want a system in which multiplication is *distributive*, we necessarily have (using juxtaposition to indicate the product)

$$z\,w \equiv (x + i\,y)\,(u + i\,v) = x\,u + i\,x\,v + i\,y\,u + i^2\,y\,v$$

$$= (x\,u - y\,v) + i\,(x\,v + y\,u).$$

As we readily check, this is the same as $(u + i\,v)(x + i\,y)$, so complex multiplication turns out to be *commutative* in the sense that for any two complex numbers z and w we have $z\,w = w\,z$. It is also true that this product is *associative*; if we have three complex numbers w,z and ζ, we have

$(w\,z)\zeta \;=\; w(z\,\zeta)$. We can also check that

$$0\,w \;=\; 0, \quad 1\,w \;=\; w$$

for any complex number w. If two complex numbers are given in polar form

$$z \;=\; r\,(\cos\theta + i\,\sin\theta)\,, \quad w \;=\; \rho\,(\cos\psi + i\,\sin\psi)\,,$$

then we have, using standard trigonometric identities,

$$z\,w \;=\; (r\,\cos\theta \;+\; i\,r\,\sin\theta)(\rho\,\cos\psi \;+\; i\,\rho\,\sin\psi)$$

$$= r\,\rho\,(\cos\theta\cos\psi \;-\; \sin\theta\sin\psi) \;+\; i\,(\sin\theta\cos\psi \;+\; \cos\theta\sin\psi)$$

$$= r\,\rho\;(\cos(\theta+\psi) \;+\; i\,\sin(\theta+\psi))\,.$$

Thus we see that in complex multiplication the modulus of the product is the product of the moduli of the factors while the argument of the product argument is the sum of the arguments of the factors (modulo 2π).

The Complex Conjugate If $z \;=\; x \;+\; i\,y$ is a complex number, the *conjugate* of z is $\overline{z} \;=\; x \;-\; i\,y$; one simply changes the sign of the imaginary part. One of the more important relationships is expressed by

$$z\,\overline{z} \;=\; (x + i\,y)(x - i\,y) \;=\; (xx - y(-y)) + i\,(xy + x(-y)) \;=\; x^2 + y^2 \;=\; |z|^2\,.$$

Since

$$\cos\theta \;=\; \cos(-\theta), \quad -\sin\theta \;=\; \sin(-\theta),$$

it is easy to see that, if $z \;=\; r\,(\cos\theta + \sin\theta)$, the polar representation of $\overline{z}$ is $z \;=\; r\,(\cos(-\theta) + \sin(-\theta))$, so the argument of $\overline{z}$ is the negative of the argument of z.

Complex Division Assuming that division and multiplication for complex numbers should be related in much the same sense as they are for real numbers, the complex number q represented by the quotient z/w, for z and w complex numbers such that $w \neq 0$, should have the property $q\,w = z$. From the earlier definition of multiplication, if $q = r + i\,s$, we have

$$(r\,u - s\,v) + i\,(r\,v + s\,u) = x + i\,y,$$

so that

$$\begin{matrix} r\,u - s\,v = x \\ r\,v + s\,u = y \end{matrix}.$$

This is a system of two linear algebraic equations in two unknowns, r and s. The determinant of the system is $u^2 + v^2 = |w|^2$, so there is a unique solution if, as we have already stipulated, $w \neq 0$. We can solve this system for r and w, hence for q, but it is easier to make the following observation:

$$q\,w\overline{w} = q(u^2 + v^2) = |w|^2\,q = z\,\overline{w} \longrightarrow q = \frac{z\,\overline{w}}{|w|^2}$$

and therefore

$$q = \frac{z}{w} = \frac{(x\,u + y\,v) + i\,(y\,u - x\,v)}{u^2 + v^2}.$$

Using the appropriate trigonometric identities again we can readily see that if $z = r\,(\cos\theta + \sin\theta)$ and $w = \rho\,(\cos\psi + \sin\psi)$, then

$$\frac{z}{w} = \frac{r}{\rho}\,(\cos(\theta - \psi) + i\,\sin(\theta - \psi))\,,$$

so the modulus of the quotient is the quotient of the moduli while the argument of the quotient is the difference of the moduli.

Powers and Roots

Powers of Complex Numbers These are defined much as they are for real numbers. Thus if z is complex, $z^n = z \cdot z \cdot z ... \cdot z$, n factors altogether. The main innovation lies in the polar representation. Thus if $z = r(\cos\theta + i\,\sin\theta)$, application of the product rule above gives

$$z^2 = r^2(\cos\theta + i\,\sin\theta)^2 = r^2(\cos(2\theta) + i\,\sin(2\theta)).$$

Then we proceed inductively. Assuming that $z^{n-1} = r^n(\cos((n-1)\theta) + i\,\sin((n-1)\theta))$, a further multiplication by z gives

$$\begin{aligned} z^n = z^{n-1}\cdot z &= r^{n-1}(\cos((n-1)\theta) + i\,\sin((n-1)\theta))\cdot(\cos\theta + i\,\sin\theta) \\ &= r^n(\cos n\theta + i\,\sin n\theta) \end{aligned}$$

and the *principle of mathematical induction* then leads us to conclude that the formula

$$z^n = r^n(\cos n\theta + i\,\sin n\theta)$$

holds for all positive integer values of the power n.

Example 1 We compute $(1+i)^7$. Changing to polar form we have

$$\begin{aligned} 1+i &= \sqrt{1^2 + 1^2}\Big(\cos(\tan^{-1}(1/1)) + i\,\sin(\tan^{-1}(1/1))\Big) \\ &= \sqrt{2}\,(\cos(\pi/4) + i\,\sin(\pi/4))\,. \end{aligned}$$

Then, since in general $\cos(\theta + 2\pi) + i\,\sin(\theta + 2\pi) = \cos\theta + i\,\sin\theta$, we have

$$\begin{aligned} (1+i)^7 = (\sqrt{2})^7(\cos(7\pi/4) + i\,\sin(7\pi/4)) &= 8\sqrt{2}\,(\cos(-\pi/4) + i\,\sin(-\pi/4)) \\ &= 8\sqrt{2}\,(\cos(-\pi/4) + i\,\sin(-\pi/4)) = 8(1-i). \end{aligned}$$

Roots and Rational Powers of Complex Numbers The property

$$(\cos\theta + i\,\sin\theta)^n = \cos(n\theta) + i\,\sin(n\theta), \cos(\theta+2\pi) + i\,\sin(\theta+2\pi) = \cos\theta + i\,\sin\theta$$

become particularly important in finding integer roots of non-zero complex numbers. Suppose $z = r(\cos\theta + i\sin\theta)$ and we want to find $z^{1/n}$, i.e., the n-th root of z. We seek, then, one or more complex numbers w such that $w^n = z$. If $w = \rho(\cos\psi + i\sin\psi)$, we must have

$$\rho^n(\cos(n\psi) + i\sin(n\psi)) = r(\cos\theta + i\sin\theta).$$

This immediately gives $\rho = r^{1/n}$, the standard n-th root of the positive number r. The condition $\cos(n\psi) + i\sin(n\psi) = \cos\theta + i\sin\theta$ needs to be treated a little more carefully. Since for any integer k we have $\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi) = \cos\theta + i\sin\theta$, that requirement implies that

$$\psi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

which gives us

$$w = r^{1/n}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right).$$

Because of the condition $\cos(\theta + 2\pi) + i\sin(\theta + 2\pi) = \cos\theta + i\sin\theta$ it is only necessary to consider the cases $k = 0, 1, 2, ..., n-1$; proceeding further simply leads to repetition.

Example 2 Compute the fifth roots of $1 + i$. Here $r = \sqrt{2} = 2^{1/2}$ so $\rho = (2^{1/2})^{1/5} = 2^{1/10}$. To obtain the possible values of ψ we use

$$\psi = \frac{\pi}{4 \cdot 5} + \frac{2k\pi}{5}, \quad k = 0, 1, 2, 3, 4.$$

So the collection of all 5-th roots of $1 + i$ is

$$2^{1/10}\left(\cos\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right) + i\cos\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right)\right), \quad k = 0, 1, 2, 3, 4.$$

QuickCheck Exercises

1. Compute the product $(2+3\,i)(1-2\,i)$ and verify that the modulus of the product is the product of the moduli of the factors.

2. Convert $3 - i\,3\sqrt{3}$ to polar form and then compute its square.

3. Compute $(2+2\,i)^6$.

4. Find all values of:

$$\text{(a)}\ \sqrt[4]{1+i\sqrt{3}};\quad \text{(b)}\ \sqrt[3]{i}.$$