CHAPTER 10 INTRODUCING HOMOLOGICAL ALGEBRA

Roughly speaking, homological algebra consists of (A) that part of algebra that is fundamental in building the foundations of algebraic topology, and (B) areas that arise naturally in studying (A).

10.1 Categories

We have now encountered many algebraic structures and maps between these structures. There are ideas that seem to occur regardless of the particular structure under consideration. Category theory focuses on principles that are common to all algebraic systems.

10.1.1 Definitions and Comments A *category* $\mathcal{C}$ consists of *objects* $A, B, C, \dots$ and *morphisms* $f : A \to B$ (where A and B are objects). If $f : A \to B$ and $g : B \to C$ are morphisms, we have a notion of *composition*, in other words, there is a morphism $gf = g \circ f : A \to C$, such that the following axioms are satisfied.

(i) *Associativity*: If $f : A \to B,\ g : B \to C,\ h : C \to D$, then (hg)f=h(gf);

(ii) *Identity*: For each object A there is a morphism $1_A : A \to A$ such that for each morphism $f : A \to B$, we have $f1_A = 1_B f = f$.

A remark for those familiar with set theory: For each pair (A, B) of objects, the collection of morphisms $f : A \to B$ is required to be a set rather than a proper class.

We have seen many examples:

1. **Sets**: The objects are sets and the morphisms are functions.
2. **Groups**: The objects are groups and the morphisms are group homomorphisms.
3. **Rings**: The objects are rings and the morphisms are ring homomorphisms.
4. **Fields**: The objects are fields and the morphisms are field homomorphisms [= field monomorphisms; see (3.1.2)].
5. **R-mod**: The objects are left R-modules and the morphisms are R-module homomorphisms. If we use right R-modules, the corresponding category is called **mod-R**.
6. **Top**: The objects are topological spaces and the morphisms are continuous maps.
7. **Ab**: The objects are abelian groups and the the morphisms are homomorphisms from one abelian group to another.

A morphism $f : A \to B$ is said to be an *isomorphism* if there is an inverse morphism $g : B \to A$, that is, $gf = 1_A$ and $fg = 1_B$. In **Sets**, isomorphisms are bijections, and in **Top**, isomorphisms are homeomorphisms. For the other examples, an isomorphism is a bijective homomorphism, as usual.

In the category of sets, a function f is injective iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$. But in an abstract category, we don't have any elements to work with; a morphism $f : A \to B$ can be regarded as simply an arrow from A to B. How do we generalize injectivity to an arbitrary category? We must give a definition that does not depend on elements of a set. Now in **Sets**, f is injective iff it has a left inverse; equivalently, f is *left cancellable*, i.e. if $fh_1 = fh_2$, then $h_1 = h_2$. This is exactly what we need, and a similar idea works for surjectivity, since f is surjective iff f is *right cancellable*, i.e., $h_1 f = h_2 f$ implies $h_1 = h_2$.

10.1.2 Definitions and Comments A morphism f is said to be *monic* if it is left cancellable, *epic* if it is right cancellable.

In all the categories listed in (10.1.1), a morphism f is monic iff f is injective as a mapping of sets. If f is surjective, then it is epic, but the converse can fail. See Problems 2 and 7-10 for some of the details.

In the category **R-mod**, the zero module $\{0\}$ has the property that for any R-module M, there is a unique module homomorphism from M to $\{0\}$ and a unique module homomorphism from $\{0\}$ to M. Here is a generalization of this idea.

10.1.3 Definitions and Comments Let A be an object in a category. If for every object B, there is a unique morphism from A to B, then A is said to be an *initial object*. If for every object B there is a unique morphism from B to A, then A is said to be a *terminal object*. A *zero object* is both initial and terminal.

In the category of sets, there is only one initial object, the empty set. The terminal objects are singletons $\{x\}$, and consequently there are no zero objects. In the category of groups, the trivial group consisting of the identity alone is a zero object. We are going to prove that any two initial objects are isomorphic, and similarly for terminal objects. This will be a good illustration of the duality principle, to be discussed next.

10.1.4 Duality If $\mathcal{C}$ is a category, the *opposite* or *dual* category $\mathcal{C}^{\text{op}}$ has the same objects as $\mathcal{C}$. The morphisms are those of $\mathcal{C}$ with arrows reversed; thus $f : A \to B$ is a morphism of $\mathcal{C}^{\text{op}}$ iff $f : B \to A$ is a morphism of $\mathcal{C}$. If the composition gf is permissible in $\mathcal{C}$, then fg is permissible in $\mathcal{C}^{\text{op}}$. To see how the duality principle works, let us first prove that if A and B are initial objects of $\mathcal{C}$, then A and B are isomorphic. There is a unique morphism $f : A \to B$ and a unique morphism $g : B \to A$. But $1_A : A \to A$ and $1_B : B \to B$, and it follows that $gf = 1_A$ and $fg = 1_B$. The point is that we need not give a separate proof that any two terminal objects are isomorphic. We have just proved the following:

If A and B are objects in a category $\mathcal{C}$, and for every object D of $\mathcal{C}$, there is a unique morphism from A to D and there is a unique morphism from B to D, then A and B are isomorphic.

Our statement is completely general; it does not involve the properties of any specific category. If we go through the entire statement and reverse all the arrows, equivalently, if we replace $\mathcal{C}$ by $\mathcal{C}^{\text{op}}$, we get:

If A and B are objects in a category $\mathcal{C}$, and for every object D of $\mathcal{C}$, there is a unique morphism from D to A and there is a unique morphism from D to B, then A and B are isomorphic.

In other words, any two terminal objects are isomorphic. If this is unconvincing, just go through the previous proof, reverse all the arrows, and interchange fg and gf. We say that *initial and terminal objects are dual*. Similarly, monic and epic morphisms are dual.

If zero objects exist in a category, then we have zero morphisms as well. If Z is a zero object and A and B arbitrary objects, there is a unique $f : A \to Z$ and a unique $g : Z \to B$. The *zero morphism* from A to B, denoted by 0_{AB}, is defined as gf, and it is independent of the particular zero object chosen (Problem 3). Note that since a zero morphism goes through a zero object, it follows that for an arbitrary morphism h, we have $h0 = 0h = 0$.

10.1.5 Kernels and Cokernels If $f : A \to B$ is an R-module homomorphism, then its kernel is, as we know, $\{x \in A : f(x) = 0\}$. The *cokernel* of f is defined as the quotient group $B/\text{im}(f)$. Thus f is injective iff its kernel is 0, and f is surjective iff its cokernel is 0. We will generalize these notions to an arbitrary category that contains zero objects. The following diagram indicates the setup for kernels.

$$\begin{array}{ccccc} & i & & f & \\ C & \rightarrow & A & \rightarrow & B \\ & h \nwarrow & g \uparrow & \nearrow 0 & \\ & & D & & \end{array}$$

We take C to be the kernel of the module homomorphism f, with i the inclusion map. If $fg = 0$, then the image of g is contained in the kernel of f, so that g actually maps into C. Thus there is a unique module homomorphism $h : D \to C$ such that $g = ih$; simply take $h(x) = g(x)$ for all x. The key to the generalization is to think of the kernel as the *morphism* i. This is reasonable because C and i essentially encode the same information. Thus a *kernel* of the morphism $f : A \to B$ is a morphism $i : C \to A$ such that:

(1) $fi = 0$.

(2) If $g : D \to A$ and $fg = 0$, then there is a unique morphism $h : D \to C$ such that $g = ih$.

Thus any map killed by f can be factored through i.

If we reverse all the arrows in the above diagram and change labels for convenience, we get an appropriate diagram for cokernels.

$$\begin{array}{ccccc} & f & & p & \\ A & \to & B & \to & C \\ & 0 \searrow & g \downarrow & \swarrow h & \\ & & D & & \end{array}$$

We take p to be the canonical map of B onto the cokernel of f, so that $C = B/\text{im}\ (f)$. If $gf = 0$, then the image of f is contained in the kernel of g, so by the factor theorem, there is a unique homomorphism h such that $g = hp$. In general, a *cokernel* of a morphism $f : A \to B$ is a morphism $p : B \to C$ such that:

($1'$) $pf = 0$.

($2'$) If $g : B \to D$ and $gf = 0$, then there is a unique morphism $h : C \to D$ such that $g = hp$.

Thus any map that kills f can be factored through p.

Since going from kernels to cokernels simply involves reversing arrows, kernels and cokernels are dual. Note, however, that in an arbitrary category with 0, kernels and cokernels need not exist for arbitrary morphisms. But every monic has a kernel and (by duality) every epic has a cokernel; see Problem 5.

Problems For Section 10.1

1. Show that in any category, the identity and inverse are unique.
2. In the category of rings, the inclusion map $i : \mathbb{Z} \to \mathbb{Q}$ is not surjective. Show, however, that i is epic.
3. Show that the zero morphism 0_{AB} is independent of the particular zero object chosen in the definition.
4. Show that a kernel must be monic (and by duality, a cokernel must be epic). [In the definition of kernel, we can assume that i is monic in (1) of (10.1.5), and drop the uniqueness assumption on h. For i monic forces uniqueness of h, by definition of monic. Conversely, uniqueness of h forces i to be monic, by Problem 4.]
5. Show that in a category with 0, every monic has a kernel and every epic has a cokernel.
6. Show that if $i : C \to A$ and $j : D \to A$ are kernels of $f : A \to B$, then C and D are isomorphic. (By duality, a similar statement holds for cokernels.)
7. Let $f : A \to B$ be a group homomorphism with kernel K, and assume f not injective, so that $K \neq \{1\}$. Let g be the inclusion map of K into A. Find a homomorphism h such that $fg = fh$ but $g \neq h$.
8. It follows from Problem 7 that in the category of groups, f monic is equivalent to f injective as a mapping of sets, and a similar proof works in the category of modules. Why does the argument fail in the category of rings?
9. Continue from Problem 8 and give a proof that does work in the category of rings.
10. Let $f : M \to N$ be a module homomorphism with nonzero cokernel, so that f is not surjective. Show that f is not epic; it follows that epic is equivalent to surjective in the category of modules.

10.2 Products and Coproducts

We have studied the direct product of groups, rings, and modules. It is natural to try to generalize the idea to an arbitrary category, and a profitable approach is to forget (temporarily) the algebraic structure and just look at the cartesian product $A = \prod_i A_i$ of a family of sets $A_i, i \in I$. The key property of a product is that if we are given maps f_i from a set S *into* the factors A_i, we can *lift* the f_i into a single map $f : S \to \prod_i A_i$. The commutative diagram below will explain the terminology.

$$\begin{array}{ccc} & & A_i \\ & f_i \nearrow & \uparrow p_i \\ S & \underset{f}{\to} & A \end{array} \tag{1}$$

In the picture, p_i is the projection of A onto the i^{th} factor A_i. If $f_i(x) = a_i, i \in I$, we take $f(x) = (a_i, i \in I)$. It follows that $p_i \circ f = f_i$ for all i; this is what we mean by lifting the f_i to f. (Notice that there is only one possible lifting, i.e., f is unique.) If A is the direct product of groups A_i and the f_i are group homomorphisms, then f will also be a group homomorphism. Similar statements can be made for rings and modules. We can now give a generalization to an arbitrary category.

10.2.1 Definition A *product* of objects A_i in a category $\mathcal{C}$ is an object A, along with morphisms $p_i : A \to A_i$, with the following *universal mapping property.* Given any object S of $\mathcal{C}$ and morphisms $f_i : S \to A_i$, there is a unique morphism $f : S \to A$ such that $p_i f = f_i$ for all i.

In a definition via a universal mapping property, we use a condition involving morphisms, along with a uniqueness statement, to specify an object and morphisms associated with that object. We have already seen this idea in connection with kernels and cokernels in the previous section, and in the construction of the tensor product in Section 8.7.

Not every category has products (see Problems 1 and 2), but if they do exist, they are essentially unique. (The technique for proving uniqueness is also essentially unique.)

10.2.2 Proposition If $(A, p_i, i \in I)$ and $(B, q_i, i \in I)$ are products of the objects A_i, then A and B are isomorphic.

Proof. We use the above diagram (1) with $S = B$ and $f_i = q_i$ to get a morphism $f : B \to A$ such that $p_i f = q_i$ for all i. We use the diagram with $S = A$, A replaced by B, p_i replaced by q_i, and $f_i = p_i$, to get a morphism $h : A \to B$ such that $q_i h = p_i$. Thus

$$p_i f h = q_i h = p_i \text{ and } q_i h f = p_i f = q_i.$$

But

$$p_i 1_A = p_i \text{ and } q_i 1_B = q_i$$

and it follows from the uniqueness condition in (10.2.1) that $fh = 1_A$ and $hf = 1_B$. Formally, we are using the diagram two more times, once with $S = A$ and $f_i = p_i$, and once with $S = B$, A replaced by B, p_i replaced by q_i, and $f_i = q_i$. Thus A and B are isomorphic. ♣

The discussion of diagram (1) indicates that in the categories of groups, abelian groups, rings, and R-modules, products coincide with direct products. But a category can have products that have no connection with a cartesian product of sets; see Problems 1 and 2. Also, in the category of torsion abelian groups (torsion means that every element has finite order), products exist but do not coincide with direct products; see Problem 5.

The dual of a product is a coproduct, and to apply duality, all we need to do is reverse all the arrows in (1). The following diagram results.

$$\begin{array}{ccc} & & M_j \\ f_j \swarrow & & \downarrow i_j \\ N & \underset{f}{\leftarrow} & M \end{array} \tag{2}$$

We have changed the notation because it is now profitable to think about modules. Suppose that M is the direct sum of the submodules M_j, and i_j is the inclusion map of M_j into M. If the f_j are module homomorphisms *out of* the factors M_j and into a module N, the f_j can be lifted to a single map f. If $x_{j1} \in M_{j1}, \dots, x_{jr} \in M_{jr}$, we take

$$f(x_{j1} + \cdots + x_{jr)} = f_{j1}(x_{j1}) + \cdots + f_{jr}(x_{jr}).$$

Lifting means that $f \circ i_j = f_j$ for all j. We can now give the general definition of coproduct.

10.2.3 Definition A *coproduct* of objects M_j in a category $\mathcal{C}$ is an object M, along with morphisms $i_j : M_j \to M$, with the following universal mapping property. Given any object N of $\mathcal{C}$ and morphisms $f_j : M_j \to N$, there is a unique morphism $f : M \to N$ such that $fi_j = f_j$ for all j

Exactly as in (10.2.2), any two coproducts of a given collection objects are isomorphic.

The discussion of diagram (2) shows that in the category of R-modules, the coproduct is the direct sum, which is isomorphic to the direct product if there are only finitely many factors. In the category of sets, the coproduct is the *disjoint union.* To explain what this means, suppose we have sets $A_j, j \in J$. We can disjointize the A_j by replacing A_j by $A'_j = \{(x,j) : x \in A_j\}$. The coproduct is $A = \bigcup_{j \in J} A'_j$, with morphisms $i_j : A_j \to A$ given by $i_j(a_j) = (a_j, j)$. If for each j we have $f_j : A_j \to B$, we define $f : A \to B$ by $f(a_j, j) = f_j(a_j)$.

The coproduct in the category of groups will be considered in the exercises.

Problems For Section 10.2

1. Let S be a preordered set, that is, there is a reflexive and transitive relation $\le$ on S. Then S can be regarded as a category whose objects are the elements of S. If $x \le y$, there is a unique morphism from x to y, and if $x \not\le y$, there are no morphisms from x to y. Reflexivity implies that there is an identity morphism on x, and transitivity implies that associativity holds. Show that a product of the objects x_i, if it exists, must be a greatest lower bound of the x_i. The greatest lower bound will be unique (not just essentially unique) if S is a partially ordered set, so that $\le$ is antisymmetric.
2. Continuing Problem 1, do products always exist?
3. Continuing Problem 2, what can be said about coproducts?
4. If A is an abelian group, let $T(A)$ be the set of torsion elements of A. Show that $T(A)$ is a subgroup of A.
5. Show that in the category of torsion abelian groups, the product of groups A_i is $T(\prod A_i)$, the subgroup of torsion elements of the direct product.
6. Assume that we have a collection of groups G_i, pairwise disjoint except for a common identity 1. The *free product* of the G_i (notation $*_i G_i$) consists of all words (finite sequences) $a_1 \cdots a_n$ where the a_j belong to distinct groups. Multiplication is by concatenation with cancellation. For example, with the subscript j indicating membership in G_j,

$$(a_1 a_2 a_3 a_4)(b_4 b_2 b_6 b_1 b_3) = a_1 a_2 a_3 (a_4 b_4) b_2 b_6 b_1 b_3$$

and if $b_4 = a_4^{-1}$, this becomes $a_1a_2a_3b_2b_6b_1b_3$. The empty word is the identity, and inverses are calculated in the usual way, as with free groups (Section 5.8). In fact a free group on S is a free product of infinite cyclic groups, one for each element of S. Show that in the category of groups, the coproduct of the G_i is the free product.

7. Suppose that products exist in the category of finite cyclic groups, and suppose that the cyclic group C with generator a is the product of the cyclic groups C_1 and C_2 with generators a_1 and a_2 respectively. Show that the projections p_1 and p_2 associated with the product of C_1 and C_2 are surjective.

8. By Problem 7, we may assume without loss of generality that $p_i(a) = a_i,\ i = 1, 2$. Show that for some positive integer n, $na_1 = a_1$ and $na_2 = 0$. [Take $f_1 : C_1 \to C_1$ to be the identity map, and let $f_2 : C_1 \to C_2$ be the zero map (using additive notation). Lift f_1 and f_2 to $f : C_1 \to C$.]

9. Exhibit groups C_1 and C_2 that can have no product in the category of finite cyclic groups.

10.3 Functors

We will introduce this fundamental concept with a concrete example. Let $\text{Hom}_R(M, N)$ be the set of R-module homomorphisms from M to N. As pointed out at the beginning of Section 4.4, $\text{Hom}_R(M, N)$ is an abelian group. It will also be an R-module if R is a commutative ring, but not in general. We are going to look at $\text{Hom}_R(M, N)$ as a function of N, with M fixed.

10.3.1 The Functor $\text{Hom}_R(M, -)$

We are going to construct a mapping from the category of R-modules to the category of abelian groups. Since a category consists of both objects and morphisms, our map will have two parts:

(i) Associate with each R-module N the abelian group $\text{Hom}_R(M, N)$.

(ii) Associate with each R-module homomorphism $h : N \to P$ a homomorphism h_* from the abelian group $\text{Hom}_R(M, N)$ to the abelian group $\text{Hom}_R(M, P)$. The following diagram suggests how h_* should be defined.

$$M \xrightarrow{f} N \xrightarrow{h} P$$

Take

$$h_*(f) = hf.$$

Note that if h is the identity on N, then h_* is the identity on $\text{Hom}_R(M, N)$.

Now suppose we have the following situation:

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} Q$$

Then $(hg)_*(f) = (hg)f = h(gf) = h_*(g_*(f))$, so that

$$(hg)_* = h_*g_* \ (= h_* \circ g_*).$$

To summarize, we have a mapping F called a *functor* that takes an object A in a category $\mathcal{C}$ to an object $F(A)$ in a category $\mathcal{D}$; F also takes a morphism $h : A \to B$ in $\mathcal{C}$ to a morphism $h_* = F(h) : F(A) \to F(B)$ in $\mathcal{D}$. The key feature of F is the *functorial property*:

$$F(hg) = F(h)F(g) \text{ and } F(1_A) = 1_{F(A)}.$$

Thus a functor may be regarded as a homomorphism of categories.

10.3.2 The Functor $\text{Hom}_R(-, N)$

We now look at $\text{Hom}_R(M,N)$ as a function of M, with N fixed. Here is an appropriate diagram:

$$K \overset{g}{\rightarrow} L \overset{h}{\rightarrow} M \overset{f}{\rightarrow} N$$

If M is an R-module, we take $F(M)$ to be the abelian group $\text{Hom}_R(M,N)$. If $h : L \to M$ is an R-module homomorphism, we take $h^* = F(h)$ to be a homomorphism from the abelian group $\text{Hom}_R(M,N)$ to the abelian group $\text{Hom}_R(L,N)$, given by

$$h^*(f) = fh.$$

It follows that

$$(hg)^*(f) = f(hg) = (fh)g = g^*(fh) = g^*(h^*(f))$$

hence

$$(hg)^* = g^*h^*,$$

and if h is the identity on M, then h^* is the identity on $\text{Hom}_R(M,N)$.

Thus F does not quite obey the functorial property; we have $F(hg) = F(g)F(h)$ instead of $F(hg) = F(h)F(g)$. However, F is a legal functor on the *opposite category* of **R-mod**. In the literature, $\text{Hom}_R(-,N)$ is frequently referred to as a *contravariant functor* on the original category **R-mod**, and $\text{Hom}_R(M,-)$ as a *covariant functor* on **R-mod**.

If we replace the category of R-modules by an arbitrary category, we can still define functors (called *hom functors*) as in (10.3.1) and (10.3.2). But we must replace the category of abelian groups by the category of sets.

10.3.3 The Functors $\mathbf{M} \otimes_R -$ and $- \otimes_R \mathbf{N}$

To avoid technical complications, we consider tensor products of modules over a commutative ring R. First we discuss the tensor functor $T = M \otimes_R -$ The relevant diagram is given below.

$$N \overset{g}{\rightarrow} P \overset{f}{\rightarrow} Q$$

If N is an R-module, we take $T(N) = M \otimes_R N$. If $g : N \to P$ is an R-module homomorphism, we set $T(g) = 1_M \otimes g : M \otimes_R N \to M \otimes_R P$, where 1_M is the identity mapping on M [Recall that $(1_M \otimes g)(x \otimes y) = x \otimes g(y)$.] Then

$$T(fg) = 1_M \otimes fg = (1_M \otimes f)(1_M \otimes g) = T(f)T(g)$$

and

$$T(1_N) = 1_{T(N)}$$

so T is a functor from **R-mod** to **R-mod**.

The functor $S = - \otimes_R N$ is defined in a symmetrical way. If M is an R-module, then $S(M) = M \otimes_R N$, and if $f : L \to M$ is an R-module homomorphism, then $S(f) : L \otimes_R N \to M \otimes_R N$ is given by $S(f) = f \otimes 1_N$.

10.3.4 Natural Transformations

Again we will introduce this idea with an explicit example. The diagram below summarizes the data.

$$\begin{array}{ccc} F(A) & \overset{t_A}{\rightarrow} & G(A) \\ Ff \downarrow & & \downarrow Gf \\ F(B) & \underset{t_B}{\rightarrow} & G(B) \end{array} \qquad (1)$$

We start with abelian groups A and B and a homomorphism $f : A \to B$. We apply the *forgetful functor*, also called the *underlying functor* $\mathcal{U}$. This is a fancy way of saying that we forget the algebraic structure and regard A and B simply as sets, and f as a mapping between sets. Now we apply the *free abelian group functor* $\mathcal{F}$ to produce $F(A) = \mathcal{F}\mathcal{U}(A)$, the free abelian group with A as basis (and similarly for $F(B)$). Thus $F(A)$ is the direct sum of copies of $\mathbb{Z}$, one copy for each $a \in A$. The elements of $F(A)$ can be represented as $\sum_a n(a)x(a),\ n(a) \in \mathbb{Z}$, where $x(a)$ is the member of the direct sum that is 1 in the a^{th} position and 0 elsewhere. [Similarly, we represent elements of B as $\sum_b n(b)y(b)$.] The mapping f determines a homomorphism $Ff : F(A) \to F(B)$, via $\sum n(a)x(a) \to \sum n(a)y(f(a))$.

Now let G be the identity functor, so that $G(A) = A, G(B) = B, Gf = f$. We define an abelian group homomorphism $t_A : F(A) \to A$ by $t_A(\sum n(a)x(a)) = \sum n(a)a$, and similarly we define $t_B(\sum n(b)y(b)) = \sum n(b)b$. (Remember that we began with *abelian groups* A and B.) The diagram (1) is then commutative, because

$$ft_A(x(a)) = f(a) \text{ and } t_B[(Ff)(x(a))] = t_B(y(f(a)) = f(a).$$

To summarize, we have two functors, F and G from the category $\mathcal{C}$ to the category $\mathcal{D}$. (In this case, $\mathcal{C} = \mathcal{D} =$ the category of abelian groups.) For all objects $A, B \in \mathcal{C}$ and morphisms $f : A \to B$, we have morphisms $t_A : F(A) \to G(A)$ and $t_B : F(B) \to G(B)$ such that the diagram (1) is commutative. We say that t is a *natural transformation* from F to G. If for every object $C \in \mathcal{C}$, t_C is an isomorphism (not the case in this example), t is said to be a *natural equivalence*.

The key intuitive point is that the process of going from $F(A)$ to $G(A)$ is "natural" in the sense that as we move from an object A to an object B, the essential features of the process remains the same.

Problems For Section 10.3

1. Let $F : S \to T$, where S and T are preordered sets. If we regard S and T as categories, as in Section 10.2, Problem 1, what property must F have in order to be a functor?
2. A group may be regarded as a category with a single object 0, with a morphism for each element $g \in G$. The composition of two morphisms is the morphism associated with the product of the elements. If $F : G \to H$ is a function from a group G to a group H, and we regard G and H as categories, what property must F have in order to be a functor?

3. We now look at one of the examples that provided the original motivation for the concept of a natural transformation. We work in the category of vector spaces (over a given field) and linear transformations. If V is a vector space, let V^* be the dual space, that is, the space of linear maps from V to the field of scalars, and let V^{**} be the dual of V^*. If $v \in V$, let $\overline{v} \in V^{**}$ be defined by $\overline{v}(f) = f(v), f \in V^*$. The mapping from v to $\overline{v}$ is a linear transformation, and in fact an isomorphism if V is finite-dimensional.

Now suppose that $f : V \to W$ and $g : W \to X$ are linear transformations. Define $f^* : W^* \to V^*$ by $f^*(\alpha) = \alpha f, \alpha \in W^*$. Show that $(gf)^* = f^*g^*$.

4. The *double dual functor* takes a vector space V into its double dual V^{**}, and takes a linear transformation $f : V \to W$ to $f^{**} : V^{**} \to W^{**}$, where $f^{**}(v^{**}) = v^{**}f^*$. Show that the double dual functor is indeed a functor.

5. Now consider the following diagram.

$$\begin{array}{ccc} & t_V & \\ V & \rightarrow & V^{**} \\ & & \\ f \downarrow & & \downarrow f^{**} \\ & & \\ W & \rightarrow & W^{**} \\ & t_W & \end{array}$$

We take $t_V(v) = \overline{v}$, and similarly for t_W. Show that the diagram is commutative, so that t is a natural transformation from the identity functor to the double dual functor.

In the finite-dimensional case, we say that there is a natural isomorphism between a vector space and its double dual. "Natural" means coordinate-free in the sense that it is not necessary to choose a specific basis. In contrast, the isomorphism of V and its single dual V^* is not natural.

6. We say that $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if the objects of $\mathcal{D}$ are also objects of $\mathcal{C}$, and similarly for morphisms (and composition of morphisms). The subcategory $\mathcal{D}$ is *full* if every $\mathcal{C}$-morphism $f : A \to B$, where A and B are *objects of* $\mathcal{D}$ (the key point) is also a $\mathcal{D}$-morphism. Show that the category of groups is a full subcategory of the category of monoids.

7. A functor $F : \mathcal{C} \to \mathcal{D}$ induces a map from $\mathcal{C}$-morphisms to $\mathcal{D}$-morphisms; $f : A \to B$ is mapped to $Ff : FA \to FB$. If this map is injective for all objects A, B of $\mathcal{C}$, we say that F is *faithful*. If the map is surjective for all objects A, B of $\mathcal{C}$, we say that F is *full*.

(a) The *forgetful functor* from groups to sets assigns to each group its underlying set, and to each group homomorphism its associated map of sets. Is the forgetful functor faithful? full?

(b) We can form the product $\mathcal{C} \times \mathcal{D}$ of two arbitrary categories; objects in the product are pairs (A, A') of objects, with $A \in \mathcal{C}$ and $A' \in \mathcal{D}$. A morphism from (A, A') to (B, B') is a pair (f, g), where $f : A \to B$ and $g : A' \to B'$. The *projection functor* from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{C}$ takes (A, A') to A and (f, g) to f. Is the projection functor faithful? full?

10.4 Exact Functors

10.4.1 Definitions and Comments We are going to investigate the behavior of the hom and tensor functors when presented with an exact sequence. We will be working in the categories of modules and abelian groups, but exactness properties can be studied in the more general setting of *abelian categories*, which we now describe very informally.

In any category $\mathcal{C}$, let $\text{Hom}_{\mathcal{C}}(A, B)$ (called a "hom set") be the set of morphisms in $\mathcal{C}$ from A to B. [As remarked in (10.1.1), the formal definition of a category requires that $\text{Hom}_{\mathcal{C}}(A, B)$ be a set for all objects A and B. The collection of all objects of $\mathcal{C}$ is a class but need not be a set.] For $\mathcal{C}$ to be an abelian category, the following conditions must be satisfied.

1. Each hom set is an abelian group.
2. The distributive laws $f(g + h) = fg + fh, \quad (f + g)h = fh + gh$ hold.
3. $\mathcal{C}$ has a zero object.
4. Every finite set of objects has a product and a coproduct. (The existence of finite coproducts can be deduced from the existence of finite products, along with the requirements listed so far.)
5. Every morphism has a kernel and a cokernel.
6. Every monic is the kernel of its cokernel.
7. Every epic is the cokernel of its kernel.
8. Every morphism can be factored as an epic followed by a monic.

Exactness of functors can be formalized in an abelian category, but we are going to return to familiar ground by assuming that each category that we encounter is **R-mod** for some R. When $R = \mathbb{Z}$, we have the category of abelian groups.

10.4.2 Left Exactness of Hom$_R(M, -)$ Suppose that we have a short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \tag{1}$$

We apply the covariant hom functor $F=\text{Hom}_R(M, -)$ to the sequence, dropping the last term on the right. We will show that the sequence

$$0 \rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \tag{2}$$

is exact. A functor that behaves in this manner is said to be *left exact*.

We must show that the transformed sequence is exact at FA and FB. We do this in three steps.

(a) Ff is monic.

Suppose that $(Ff)(\alpha) = f\alpha = 0$. Since f is monic (by exactness of the sequence (1)), $\alpha = 0$ and the result follows.

(b) im $Ff \subseteq$ ker Fg.

If $\beta \in$ im Ff, then $\beta = f\alpha$ for some $\alpha \in \mathrm{Hom}_R(M, A)$. By exactness of (1), im $f \subseteq$ ker g, so $g\beta = gf\alpha = 0\alpha = 0$. Thus $\beta \in$ ker g.

(c) ker $Fg \subseteq$ im Ff.

If $\beta \in$ ker Fg, then $g\beta = 0$, with $\beta \in \mathrm{Hom}_R(M, B)$. Thus if $y \in M$, then $\beta(y) \in$ ker $g =$ im f, so $\beta(y) = f(x)$ for some $x = \alpha(y) \in A$. Note that x is unique since f is monic, and $\alpha \in \mathrm{Hom}_R(M, A)$. Thus $\beta = f\alpha \in$ im Ff. ♣

10.4.3 Left Exactness of Hom$_R(-, N)$ The contravariant hom functor $G = \mathrm{Hom}_R(-, N)$ is a functor on the opposite category, so before applying it to the sequence (1), we must reverse all the arrows. Thus left-exactness of G means that the sequence

$$0 \rightarrow GC \xrightarrow{Gg} GB \xrightarrow{Gf} GA \tag{3}$$

is exact. Again we have three steps.

(a) Gg is monic.

If $(Gg)\alpha = \alpha g = 0$, then $\alpha = 0$ since g is epic.

(b) im $Gg \subseteq$ ker Gf.

If $\beta \in$ im Gg, then $\beta = \alpha g$ for some $\alpha \in \mathrm{Hom}_R(C, N)$. Thus $(Gf)\beta = \beta f = \alpha g f = 0$, so $\beta \in$ ker Gf.

(c) ker $Gf \subseteq$ im Gg.

Let $\beta \in \mathrm{Hom}_R(B, N)$ with $\beta \in$ ker Gf, that is, $\beta f = 0$. If $y \in C$, then since g is epic, we have $y = g(x)$ for some $x \in B$. If $g(x_1) = g(x_2)$, then $x_1 - x_2 \in$ ker $g =$ im f, hence $x_1 - x_2 = f(z)$ for some $z \in A$. Therefore $\beta(x_1) - \beta(x_2) = \beta(f(z)) = 0$, so it makes sense to define $\alpha(y) = \beta(x)$. Then $\alpha \in \mathrm{Hom}_R(C, N)$ and $\alpha g = \beta$, that is, $(Gg)\alpha = \beta$. ♣

10.4.4 Right Exactness of the Functors M$\otimes_R-$ and $-\otimes_R$N

If we apply the functor $H = M \otimes_R -$ to the exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

[see (1) of (10.4.2)], we will show that the sequence

$$HA \xrightarrow{Hf} HB \xrightarrow{Hg} HC \rightarrow 0 \tag{4}$$

is exact. A similar result holds for $- \otimes_R N$. A functor that behaves in this way is said to be *right exact*. Once again, there are three items to prove.

(i) Hg is epic.

An element of $M \otimes C$ is of the form $t = \sum_i x_i \otimes y_i$ with $x_i \in M$ and $y_i \in C$. Since g is epic, there exists $z_i \in B$ such that $g(z_i) = y_i$. Thus $(1 \otimes g)(\sum_i x_i \otimes z_i) = \sum_i x_i \otimes g(z_i) = t$.

(ii) im $Hf \subseteq$ ker Hg.

This is a brief computation: $(1 \otimes g)(1 \otimes f) = 1 \otimes gf = 1 \otimes 0 = 0$.

(iii) ker $Hg \subseteq$ im Hf.

By (ii), the kernel of $1 \otimes g$ contains $L = \text{im } (1 \otimes f)$, so by the factor theorem, there is a homomorphism $\overline{g} : (M \otimes_R B)/L \to M \otimes_R C$ such that $\overline{g}(m \otimes b + L) = m \otimes g(b), m \in M, b \in B$.

Let π be the canonical map of $M \otimes_R B$ onto $(M \otimes_R B)/L$. Then $\overline{g}\pi(m \otimes b) = \overline{g}(m \otimes b + L) = m \otimes g(b)$, so

$$\overline{g}\pi = 1 \otimes g.$$

If we can show that $\overline{g}$ is an isomorphism, then

$$\ker(1 \otimes g) = \ker(\overline{g}\pi) = \ker \pi = L = \text{im}(1 \otimes f)$$

and we are finished. To show that $\overline{g}$ is an isomorphism, we will display its inverse. First let h be the bilinear map from $M \times C$ to $(M \otimes_R B)/L$ given by $h(m, c) = m \otimes b + L$, where $g(b) = c$. [Such a b exists because g is epic. If $g(b) = g(b') = c$, then $b - b' \in \ker g = \text{im } f$, so $b - b' = f(a)$ for some $a \in A$. Then $m \otimes b - m \otimes b' = m \otimes f(a) = (1 \otimes f)(m \otimes a) \in L$, and h is well-defined.] By the universal mapping property of the tensor product, there is a homomorphism $\overline{h} : M \otimes_R C \to (M \otimes_R B)/L$ such that

$$\overline{h}(m \otimes c) = h(m, c) = m \otimes b + L, \text{where } g(b) = c.$$

But $\overline{g} : (M \otimes_R B)/L \to M \otimes_R C$ and

$$\overline{g}(m \otimes b + L) = m \otimes g(b) = m \otimes c.$$

Thus $\overline{h}$ is the inverse of $\overline{g}$. ♣

10.4.5 Definition A functor that is both left and right exact is said to be *exact*. Thus an exact functor is one that maps exact sequences to exact sequences. We have already seen one example, the localization functor (Section 8.5, Problems 4 and 5).

If we ask under what conditions the hom and tensor functors become exact, we are led to the study of projective, injective and flat modules, to be considered later in the chapter.

Problems For Section 10.4

In Problems 1-3, we consider the exact sequence (1) of (10.4.2) with $R = \mathbb{Z}$, so that we are in the category of abelian groups. Take $A = \mathbb{Z}$, $B = \mathbb{Q}$, the additive group of rational numbers, and $C = \mathbb{Q}/\mathbb{Z}$, the additive group of rationals mod 1. Let f be inclusion, and g the canonical map. We apply the functor $F = \text{Hom}_R(M, -)$ with $M = \mathbb{Z}_2$. [We will omit the subscript R when $R = \mathbb{Z}$, and simply refer to $\text{Hom}(M, -)$.]

1. Show that $\text{Hom}(\mathbb{Z}_2, \mathbb{Q}) = 0$.
2. Show that $\text{Hom}(\mathbb{Z}_2, \mathbb{Q}/\mathbb{Z}) \neq 0$.
3. Show that $\text{Hom}(\mathbb{Z}_2, -)$ is not right exact.

In Problems 4 and 5, we apply the functor $G = \text{Hom}(-, N)$ to the above exact sequence, with $N = \mathbb{Z}$.

4. Show that $\text{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$.
5. Show that $\text{Hom}(-, \mathbb{Z})$ is not right exact.

Finally, in Problem 6 we apply the functor $H = M \otimes -$ to the above exact sequence, with $M = \mathbb{Z}_2$.

6. Show that $\mathbb{Z}_2 \otimes -$ (and similarly $- \otimes \mathbb{Z}_2$) is not left exact.
7. Refer to the sequences (1) and (2) of (10.4.2). If (2) is exact for all possible R-modules M, show that (1) is exact.
8. State an analogous result for the sequence (3) of (10.4.3), and indicate how the result is proved.

10.5 Projective Modules

Projective modules are direct summands of free modules, and are therefore images of natural projections. Free modules are projective, and projective modules are sometimes but not always free. There are many equivalent ways to describe projective modules, and we must choose one of them as the definition. In the diagram below and the definition to follow, all maps are R-module homomorphisms. The bottom row is exact, that is, g is surjective.

$$\begin{array}{ccccc} & & P & & \\ & h \swarrow & \downarrow f & & \\ M & \underset{g}{\rightarrow} & N & \rightarrow & 0 \end{array}$$

10.5.1 Definition The R-module P is *projective* if given $f : P \rightarrow N$, and $g : M \rightarrow N$ surjective, there exists $h : P \rightarrow M$ (not necessarily unique) such that the diagram is commutative, that is, $f = gh$. We sometimes say that we have "lifted" f to h.

The definition may look obscure, but the condition described is a familiar property of free modules.

10.5.2 Proposition Every free module is projective.

Proof. Let S be a basis for the free module P. By (4.3.6), f is determined by its behavior on basis elements $s \in S$. Since g is surjective, there exists $a \in M$ such that $g(a) = f(s)$. Take $h(s) = a$ and extend by linearity from S to P. Since $f = gh$ on S, the same must be true on all of P. ♣

Here is the list of equivalences.

10.5.3 Theorem The following conditions on the R-module P are equivalent.

(1) P is projective.
(2) The functor $\text{Hom}_R(P, -)$ is exact.
(3) Every short exact sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ splits.
(4) P is a direct summand of a free module.

Proof.
(1) is equivalent to (2). In view of the left exactness of $F = \text{Hom}_R(P, -)$ (see (10.4.2)), (2) says that if $g : M \rightarrow N$ is surjective, then so is $Fg : FM \rightarrow FN$. But Fg maps $h : P \rightarrow M$ to $gh : P \rightarrow N$, so what we must prove is that for an arbitrary morphism $f : P \rightarrow N$, there exists $h : P \rightarrow M$ such that $gh = f$. This is precisely the definition of projectivity of P.

(2) implies (3). Let $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ be a short exact sequence, with $g : N \rightarrow P$ (necessarily surjective). Since P is projective, we have the following diagram.

$$\begin{array}{ccccc} & & P & & \\ & h \swarrow & \downarrow 1 & & \\ N & \underset{g}{\rightarrow} & P & \rightarrow & 0 \end{array}$$

Thus there exists $h : P \to N$ such that $gh = 1_P$, which means that the exact sequence splits (see (4.7.1)).

(3) implies (4). By (4.3.6), P is a quotient of a free module, so there is an exact sequence $0 \to M \to N \to P \to 0$ with N free. By (3), the sequence splits, so by (4.7.4), P is a direct summand of N.

(4) implies (1). Let P be a direct summand of the free module F, and let π be the natural projection of F on P; see the diagram below.

$$\begin{array}{ccccc} & \pi & & & \\ F & \to & P & & \\ h\downarrow & & \downarrow f & & \\ M & \to & N & \to & 0 \\ & g & & & \end{array}$$

Given $f : P \to N$, we have $f\pi : F \to N$, so by (10.5.2) there exists $h : F \to M$ such that $f\pi = gh$. If h' is the restriction of h to P, then $f = gh'$. ♣

10.5.4 Corollary The direct sum $P = \oplus P_j$ is projective if and only if each P_j is projective.

Proof. If P is a direct summand of a free module, so is each P_j, and therefore the P_j are projective by (4) of (10.5.3). Conversely, assume that each P_j is projective. Let $f : P \to N$ and $g : M \to N$, with g surjective. If i_j is the inclusion map of P_j into P, then $fi_j : P_j \to N$ can be lifted to $h_j : P_j \to M$ such that $fi_j = gh_j$. By the universal mapping property of direct sum (Section 10.2), there is a morphism $h : P \to M$ such that $hi_j = h_j$ for all j. Thus $fi_j = ghi_j$ for every j, and it follows from the uniqueness part of the universal mapping property that $f = gh$. ♣

If we are searching for projective modules that are not free, the following result tells us where not to look.

10.5.5 Theorem A module M over a principal ideal domain R is projective if and only if it is free.

Proof. By (10.5.2), free implies projective. If M is projective, then by (4) of (10.5.3), M is a direct summand of a free module. In particular, M is a submodule of a free module, hence is free by (4.6.2) and the discussion following it. ♣

10.5.6 Examples

1. A vector space over a field k is a free k-module, hence is projective.

2. A finite abelian group G is not a projective $\mathbb{Z}$-module, because it is not free. [If $g \in G$ and $n = |G|$, then $ng = 0$, so g can never be part of a basis.]

3. If p and q are distinct primes, then $R = \mathbb{Z}_{pq} = \mathbb{Z}_p \oplus \mathbb{Z}_q$. We claim that $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are projective but not free R-modules. (As in Example 2, they are not projective $\mathbb{Z}$-modules.) This follows from (4) of (10.5.3) and the fact that any ring R is a free R-module (with basis $\{1\}$).

Problems For Section 10.5

In Problems 1-5, we are going to prove the *projective basis lemma*, which states that an R-module P is projective if and only if there are elements $x_i \in P$ $(i \in I)$ and homomorphisms $f_i : P \to R$ such that for every $x \in P$, $f_i(x) = 0$ for all but finitely many i and

$$x = \sum_i f_i(x)x_i.$$

The set of x_i's is referred to as the projective basis.

1. To prove the "only if" part, let P be a direct summand of the free module F with basis $\{e_i\}$. Take f to be the inclusion map of P into F, and π the natural projection of F onto P. Show how to define the f_i and x_i so that the desired results are obtained.

2. To prove the "if" part, let F be a free module with basis $\{e_i, i \in I\}$, and define $\pi : F \to P$ by $\pi(e_i) = x_i$. Define $f : P \to F$ by $f(x) = \sum_i f_i(x)e_i$. Show that πf is the identity on P.

3. Continuing Problem 2, show that P is projective.

4. Assume that P is finitely generated by n elements. If R^n is the direct sum of n copies of R, show that P is projective iff P is a direct summand of R^n.
5. Continuing Problem 4, show that if P is projective and generated by n elements, then P has a projective basis with n elements.
6. Show that a module P is projective iff P is a direct summand of every module of which it is a quotient. In other words, if $P \cong M/N$, then P is isomorphic to a direct summand of M.
7. In the definition (10.5.1) of a projective module, give an explicit example to show that the mapping h need not be unique.

10.6 Injective Modules

If we reverse all arrows in the mapping diagram that defines a projective module, we obtain the dual notion, an injective module. In the diagram below, the top row is exact, that is, f is injective.

$$\begin{array}{ccccc} & & & f & \\ 0 & \to & N & \to & M \\ & & g \downarrow & \swarrow h & \\ & & E & & \end{array}$$

10.6.1 Definition The R-module E is *injective* if given $g : N \to E$, and $f : N \to M$ injective, there exists $h : M \to E$ (not necessarily unique) such that $g = hf$. We sometimes say that we have "lifted" g to h.

As with projectives, there are several equivalent ways to characterize an injective module.

10.6.2 Theorem The following conditions on the R-module E are equivalent.

(1) E is injective.
(2) The functor $\text{Hom}_R(-, E)$ is exact.
(3) Every exact sequence $0 \to E \to M \to N \to 0$ splits.

Proof.
(1) is equivalent to (2). Refer to (3) of (10.4.3), (1) of (10.4.2) and the definition of the contravariant hom functor in (10.3.2) to see what (2) says. We are to show that if $f : N \to M$ is injective, then $f^* : \text{Hom}_R(M, E) \to \text{Hom}_R(N, E)$ is surjective. But $f^*(h) = hf$, so given $g : N \to E$, we must produce $h : M \to E$ such that $g = hf$. This is precisely the definition of injectivity.

(2) implies (3). Let $0 \to E \to M \to N \to 0$ be a short exact sequence, with $f : E \to M$ (necessarily injective). Since E is an injective module, we have the following diagram:

$$\begin{array}{ccccc} & & & f & \\ 0 & \to & E & \to & M \\ & & 1 \downarrow & \swarrow g & \\ & & E & & \end{array}$$

Thus there exists $g : M \to E$ such that $gf = 1_E$, which means that the exact sequence splits.

(3) implies (1). Given $g : N \to E$, and $f : N \to M$ injective, we form the *pushout* of f and g, which is a commutative square as indicated in the diagram below.

$$\begin{array}{ccc} & f & \\ N & \rightarrow & M \\ & & \\ g \downarrow & & \downarrow g' \\ & & \\ E & \rightarrow & Q \\ & f' & \end{array}$$

Detailed properties of pushouts are developed in the exercises. For the present proof, all we need to know is that since f is injective, so is f'. Thus the sequence

$$0 \rightarrow E \overset{f'}{\rightarrow} Q \rightarrow Q/\text{im } f' \rightarrow 0$$

is exact. By (3), there exists $h : Q \to E$ such that $hf' = 1_E$. We now have $hg' : M \to E$ with $hg'f = hf'g = 1_E g = g$, proving that E is injective. ♣

We proved in (10.5.4) that a direct sum of modules is projective iff each component is projective. The dual result holds for injectives.

10.6.3 Proposition A direct product $\prod_j E_j$ of modules is injective iff each E_j is injective. Consequently, a finite direct sum is injective iff each summand is injective.

Proof. If $f : N \to M$ is injective and $g : N \to \prod_i E_i$, let $g_i = p_i g$, where p_i is the projection of the direct product on E_i. Then finding $h : M \to \prod_i E_i$ such that $g = hf$ is equivalent to finding, for each i, a morphism $h_i : M \to E_i$ such that $g_i = h_i f$. [If $p_i g = h_i f = p_i h f$ for every i, then $g = hf$ by the uniqueness part of the universal mapping property for products.] The last assertion holds because the direct sum of finitely many modules coincides with the direct product. ♣

In checking whether an R-module E is injective, we are given $g : N \to E$, and $f : N \to M$, with f injective, and we must lift g to $h : M \to E$ with $g = hf$. The next result drastically reduces the collection of maps f and g that must be examined. We may take $M = R$ and restrict N to a left ideal I of R, with f the inclusion map.

10.6.4 Baer's Criterion The R-module E is injective if and only if every R-homomorphism $f : I \to E$, where I is a left ideal of R, can be extended to an R-homomorphism $h : R \to E$.

Proof. The "only if" part follows from the above discussion, so assume that we are given $g : N \to E$ and $f : N \to M$, where (without loss of generality) f is an inclusion map. We must extend g to $h : M \to E$. A standard Zorn's lemma argument yields a maximal extension g_0 in the sense that the domain M_0 of g_0 cannot be enlarged. [The partial ordering is $(g_1, D_1) \le (g_2, D_2)$ iff $D_1 \subseteq D_2$ and $g_1 = g_2$ on D_1.] If $M_0 = M$, we are finished, so assume $x \in M \setminus M_0$. Let I be the left ideal $\{r \in R : rx \in M_0\}$, and define $h_0 : I \to E$ by $h_0(r) = g_0(rx)$. By hypothesis, h_0 can be extended to $h'_0 : R \to E$. Let $M_1 = M_0 + Rx$ and define $h_1 : M_1 \to E$ by

$$h_1(x_0 + rx) = g_0(x_0) + rh'_0(1).$$

To show that h_1 is well defined, assume $x_0 + rx = y_0 + sx$, with $x_0, y_0 \in M_0$ and $r, s \in R$. Then $(r - s)x = y_0 - x_0 \in M_0$, so $r - s \in I$. Using the fact that h'_0 extends h_0, we have

$$g_0(y_0 - x_0) = g_0((r - s)x) = h_0(r - s) = h'_0(r - s) = (r - s)h'_0(1)$$

and consequently, $g_0(x_0) + rh_0'(1) = g_0(y_0) + sh_0'(1)$ and h_1 is well defined. If $x_0 \in M_0$, take $r = 0$ to get $h_1(x_0) = g_0(x_0)$, so h_1 is an extension of g_0 to $M_1 \supset M_0$, contradicting the maximality of g_0. We conclude that $M_0 = M$. ♣

Since free modules are projective, we can immediately produce many examples of projective modules. The primary source of injective modules lies a bit below the surface.

10.6.5 Definitions and Comments Let R be an integral domain. The R-module M is *divisible* if each $y \in M$ can be divided by any nonzero element $r \in R$, that is, there exists $x \in M$ such that $rx = y$. For example, the additive group of rational numbers is a divisible abelian group, as is $\mathbb{Q}/\mathbb{Z}$, the rationals mod 1. The quotient field of any integral domain (regarded as an abelian group) is divisible. A cyclic group of finite order $n > 1$ can never be divisible, since it is not possible to divide by n. The group of integers $\mathbb{Z}$ is not divisible since the only possible divisors of an arbitrary integer are ± 1. It follows that a nontrivial finitely generated abelian group, a direct sum of cyclic groups by (4.6.3), is not divisible.

It follows from the definition that a homomorphic image of a divisible module is divisible, hence a quotient or a direct summand of a divisible module is divisible. Also, a direct sum of modules is divisible iff each component is divisible.

10.6.6 Proposition If R is any integral domain, then an injective R-module is divisible. If R is a PID, then an R-module is injective if and only if it is divisible.

Proof. Assume E is injective, and let $y \in E$, $r \in R$, $r \neq 0$. Let I be the ideal Rr, and define an R-homomorphism $f : I \to E$ by $f(tr) = ty$. If $tr = 0$, then since R is an integral domain, $t = 0$ and f is well defined. By (10.6.4), f has an extension to an R-homomorphism $h : R \to E$. Thus

$$y = f(r) = h(r) = h(r1) = rh(1)$$

so division by r is possible and E is divisible. Conversely, assume that R is a PID and E is divisible. Let $f : I \to E$, where I is an ideal of R. Since R is a PID, $I = Rr$ for some $r \in R$. We have no trouble extending the zero mapping, so assume $r \neq 0$. Since E is divisible, there exists $x \in E$ such that $rx = f(r)$. Define $h : R \to E$ by $h(t) = tx$. If $t \in R$, then

$$h(tr) = trx = tf(r) = f(tr)$$

so h extends f, proving E injective. ♣

Problems For Section 10.6

We now describe the construction of the pushout of two module homomorphisms $f : A \to C$ and $g : A \to B$; refer to Figure 10.6.1. Take

$$D = (B \oplus C)/W, \text{where } W = \{(g(a), -f(a)) : a \in A\},$$

and

$$g'(c) = (0, c) + W, \quad f'(b) = (b, 0) + W.$$

In Problems 1-6, we study the properties of this construction.

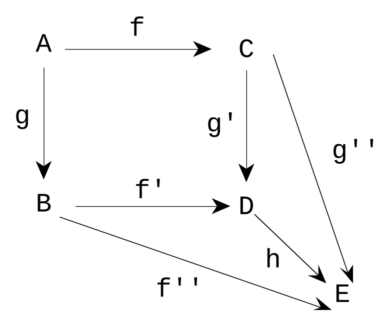

Figure 10.6.1

1. Show that the *pushout square* $ACDB$ is commutative, that is, $f'g = g'f$.

2. Suppose we have another commutative pushout square $ACEB$ with maps $f'' : B \to E$ and $g'' : C \to E$, as indicated in Figure 10.6.1. Define $h : D \to E$ by

$$h((b, c) + W) = g''(c) + f''(b).$$

Show that h is well defined.
3. Show that h makes the diagram commutative, that is, $hg' = g''$ and $hf' = f''$.
4. Show that if $h' : D \to E$ makes the diagram commutative, then $h' = h$.

The requirements stated in Problems 1,3 and 4 can be used to define the pushout via a universal mapping property. The technique of (10.2.2) shows that the pushout object D is unique up to isomorphism.

5. If f is injective, show that f' is also injective. By symmetry, the same is true for g and g'.
6. If f is surjective, show that f' is surjective. By symmetry, the same is true for g and g'.

Problems 7-10 refer to the dual construction, the *pullback*, defined as follows (see Figure 10.6.2). Given $f : A \to B$ and $g : C \to B$, take

$$D = \{(a, c) \in A \oplus C : f(a) = g(c)\}$$

and

$$g'(a, c) = a, \; f'(a, c) = c.$$

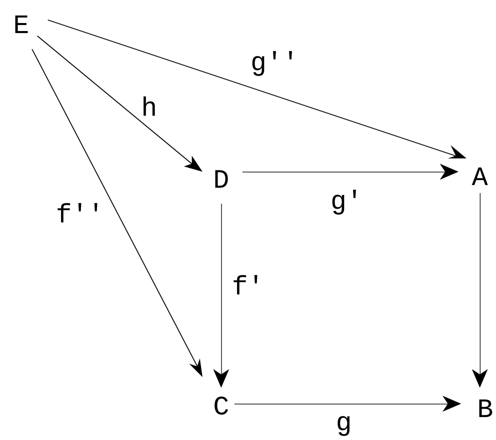

Figure 10.6.2

7. Show that the *pullback square* $DABC$ is commutative, that is, $fg' = gf'$.
8. If we have another commutative pullback square $EABC$ with maps $f'' : E \to C$ and $g'' : E \to A$, show that there is a unique $h : E \to D$ that makes the diagram commutative, that is, $g'h = g''$ and $f'h = f''$.
9. If f is injective, show that f' is injective. By symmetry, the same is true for g and g'.
10. If f is surjective, show that f' is surjective. By symmetry, the same is true for g and g'.
11. Let R be an integral domain with quotient field Q, and let f be an R-homomorphism from an ideal I of R to Q. Show that $f(x)/x$ is constant for all nonzero $x \in I$.
12. Continuing Problem 11, show that Q is an injective R-module.

10.7 Embedding into an Injective Module

We know that every module is a quotient of a projective (in fact free) module. In this section we prove the more difficult dual statement that every module can be embedded in an injective module. (To see that quotients and submodules are dual, reverse all the arrows in a short exact sequence.) First, we consider abelian groups.

10.7.1 Proposition Every abelian group can be embedded in a divisible abelian group.

Proof. If A is an abelian group, then A is a quotient of a free abelian group F, say $A \cong F/B$. Now F is a direct sum of copies of $\mathbb{Z}$, hence F can be embedded in a direct sum D of copies of $\mathbb{Q}$, the additive group of rationals. It follows that F/B can be embedded in D/B; the embedding is just the inclusion map. By (10.6.5), D/B is divisible, and the result follows. ♣

10.7.2 Comments In (10.7.1), we used $\mathbb{Q}$ as a standard divisible abelian group. It would be very desirable to have a canonical injective R-module. First, we consider $H = \operatorname{Hom}_{\mathbb{Z}}(R, A)$, the set of all abelian group homomorphisms from the additive group of the ring R to the abelian group A. If we are careful, we can make this set into a left R-module. The abelian group structure of H presents no difficulties, but we must also define scalar multiplication. If $f \in H$ and $s \in R$, we set

$$(sf)(r) = f(rs),\ \ r \in R.$$

Checking the module properties is routine except for associativity:

$$((ts)f)(r) = f(rts),\ \text{and}\ (t(sf))(r) = (sf)(rt) = f(rts)$$

so $(ts)f = t(sf)$. Notice that sf is an *abelian group* homomorphism, not an R-module homomorphism.

Now if A is an R-module and B an abelian group, we claim that

$$\operatorname{Hom}_{\mathbb{Z}}(A, B) \cong \operatorname{Hom}_R(A, \operatorname{Hom}_{\mathbb{Z}}(R, B)), \tag{1}$$

equivalently,

$$\operatorname{Hom}_{\mathbb{Z}}(R \otimes_R A, B) \cong \operatorname{Hom}_R(A, \operatorname{Hom}_{\mathbb{Z}}(R, B)). \tag{2}$$

This is a special case of *adjoint associativity*:

If ${}_SM_R,\ \ {}_RN,\ \ {}_SP$, then

$$\operatorname{Hom}_S(M \otimes_R N, P) \cong \operatorname{Hom}_R(N, \operatorname{Hom}_S(M, P)). \tag{3}$$

Thus if F is the functor $M \otimes_R -$ and G is the functor $\operatorname{Hom}_S(M, -)$, then

$$\operatorname{Hom}_S(FN, P) \cong \operatorname{Hom}_R(N, GP) \tag{4}$$

which is reminiscent of the adjoint of a linear operator on an inner product space. We say that F and G are *adjoint functors*, with F *left adjoint* to G and G *right adjoint* to F. [There is a technical naturality condition that is added to the definition of adjoint functors, but we will not pursue this since the only adjoints we will consider are hom and tensor.]

Before giving the formal proof of (3), we will argue intuitively. The left side of the equation describes all biadditive, R-balanced maps from $M \times N$ to P. If $(x, y) \to f(x, y), x \in M,\ y \in N$, is such a map, then f determines a map g from N to $\operatorname{Hom}_S(M, P)$, namely, $g(y)(x) = f(x, y)$. This is a variation of the familiar fact that a bilinear map amounts to a family of linear maps. Now if $s \in S$, then $g(y)(sx) = f(sx, y)$, which need not equal $sf(x, y)$, but if we factor f through the tensor product $M \otimes_R N$, we can then pull out the s. Thus $g(y) \in \operatorname{Hom}_S(M, P)$. Moreover, g is an R-module homomorphism, because

$g(ry)(x) = f(x, ry)$, and we can factor out the r by the same reasoning as above. Since g determines f, the correspondence between f and g is an isomorphism of abelian groups.

To prove (3), let $f : M \otimes_R N \to P$ be an S-homomorphism. If $y \in N$, define $f_y : M \to P$ by $f_y(x) = f(x \otimes y)$, and define $\psi(f) : N \to \text{Hom}_S(M, P)$ by $y \to f_y$. [$\text{Hom}_S(M, P)$ is a left R-module by Problem 1.]

(a) $\psi(f)$ is an R-homomorphism:

$$\psi(f)(y_1 + y_2) = f_{y_1+y_2} = f_{y_1} + f_{y_2} = \psi(f)(y_1) + \psi(f)(y_2);$$

$$\psi(f)(ry) = f_{ry} = rf_y = (r\psi(f))(y).$$

[$f_{ry}(x) = f(x \otimes ry)$ and $(rf_y)(x) = f_y(xr) = f(xr \otimes y)$.]

(b) ψ is an abelian group homomorphism:

We have $f_y(x) + g_y(x) = f(x \otimes y) + g(x \otimes y) = (f + g)(x \otimes y) = (f + g)_y(x)$ so $\psi(f + g) = \psi(f) + \psi(g)$.

(c) ψ is injective:

If $\psi(f) = 0$, then $f_y = 0$ for all $y \in N$, so $f(x \otimes y) = 0$ for all $x \in M$ and $y \in N$. Thus f is the zero map.

(d) If $g \in \text{Hom}_R(N, \text{Hom}_S(M, P))$, define $\varphi_g : M \times N \to P$ by $\varphi_g(x, y) = g(y)(x)$. Then φ_g is biadditive and R-balanced:

By definition, φ_g is additive in each coordinate, and [see Problem 1] $\varphi_g(xr, y) = g(y)(xr) = (rg(y))(x) = g(ry)(x) = \varphi_g(x, ry)$.

(e) ψ is surjective:

By (d), there is a unique S-homomorphism $\beta(g) : M \otimes_R N \to P$ such that $\beta(g)(x \otimes y) = \varphi_g(x, y) = g(y)(x),\ x \in M,\ y \in N$. It follows that $\psi(\beta(g)) = g$, because $\psi(\beta(g))(y) = \beta(g)_y$, where $\beta(g)_y(x) = \beta(g)(x \otimes y) = g(y)(x)$. Thus $\beta(g)_y = g(y)$ for all y in N, so $\psi\beta$ is the identity and ψ is surjective.

It follows that ψ is an abelian group isomorphism. This completes the proof of (3).

Another adjointness result, which can be justified by similar reasoning, is that if $N_R,\ {}_RM_S,\ P_S$, then

$$\text{Hom}_S(N \otimes_R M, P) \cong \text{Hom}_R(N, \text{Hom}_S(M, P)) \qquad (5)$$

which says that $F = - \otimes_R M$ and $G = \text{Hom}_S(M, -)$ are adjoint functors.

10.7.3 Proposition If E is a divisible abelian group, then $\text{Hom}_{\mathbb{Z}}(R, E)$ is an injective left R-module.

Proof. By (10.6.2), we must prove that $\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(R, E))$ is exact. As in the proof of (1) implies (2) in (10.6.2), if $0 \to N \to M$ is exact, we must show that

$$\text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, E)) \to \text{Hom}_R(N, \text{Hom}_{\mathbb{Z}}(R, E)) \to 0$$

is exact. By (1) of (10.7.2), this is equivalent to showing that

$$\text{Hom}_{\mathbb{Z}}(M, E) \to \text{Hom}_{\mathbb{Z}}(N, E) \to 0$$

is exact. [As indicated in the informal discussion in (10.7.2), this replacement is allowable because a bilinear map can be regarded as a family of linear maps. A formal proof would invoke the naturality condition referred to in (10.7.2).] Since E is an injective $\mathbb{Z}$-module, the result now follows from (10.6.2). ♣

We can now prove the main result.

10.7.4 Theorem If M is an arbitrary left R-module, then M can be embedded in an injective left R-module.

Proof. If we regard M as an abelian group, then by (10.7.1), we can assume that M is a subset of the divisible abelian group E. We will embed M in the injective left R-module $N = \text{Hom}_{\mathbb{Z}}(R, E)$ (see (10.7.3)). If $m \in M$, define $f(m) : R \to E$ by $f(m)(r) = rm$. Then $f : M \to N$, and we claim that f is an injective R-module homomorphism. If $f(m_1) = f(m_2)$, then $rm_1 = rm_2$ for every $r \in R$, and we take $r = 1$ to conclude that $m_1 = m_2$, proving injectivity. To check that f is an R-homomorphism, note that if $r, s \in R$ and $m \in M$, then

$$f(sm)(r) = rsm \text{ and } (sf(m))(r) = f(m)(rs) = rsm$$

by definition of scalar multiplication in the R-module N; see (10.7.2). ♣

It can be shown that every module M has an *injective hull*, that is, there is a smallest injective module containing M.

Problems For Section 10.7

1. If ${}_RM_S$ and ${}_RN$, show that $\text{Hom}_R(M, N)$ is a left S-module via

$$(sf)(m) = f(ms).$$

2. If ${}_RM_S$ and N_S, show that $\text{Hom}_S(M, N)$ is a right R-module via

$$(fr)(m) = f(rm).$$

3. If M_R and ${}_SN_R$, show that $\text{Hom}_R(M, N)$ is a left S-module via

$$(sf)(m) = sf(m).$$

4. If ${}_SM$ and ${}_SN_R$, show that $\text{Hom}_S(M, N)$ is a right R-module via

$$(fr)(m) = f(m)r.$$

5. A useful mnemonic device for remembering the result of Problem 1 is that since M and N are *left* R-modules, we write the function f on the *right* of its argument. The result is $m(sf) = (ms)f$, a form of associativity. Give similar devices for Problems 2,3 and 4.

Note also that in Problem 1, M is a right S-module, but $\text{Hom}_R(M, N)$ is a left S-module. The reversal might be expected, because the hom functor is contravariant in its first argument. A similar situation occurs in Problem 2, but in Problems 3 and 4 there is no reversal. Again, this might be anticipated because the hom functor is covariant in its second argument.

6. Let R be an integral domain with quotient field Q. If M is a vector space over Q, show that M is a divisible R-module.
7. Conversely, if M is a torsion-free divisible R-module, show that M is a vector space over Q.
8. If R is an integral domain that is not a field, and Q is the quotient field of R, show that $\text{Hom}_R(Q, R) = 0$.

10.8 Flat Modules

10.8.1 Definitions and Comments We have seen that an R-module M is projective iff its covariant hom functor is exact, and M is injective iff its contravariant hom functor is exact. It is natural to investigate the exactness of the tensor functor $M \otimes_R -$, and as before we avoid complications by assuming all rings commutative. We say that M is *flat* if $M \otimes_R -$ is exact. Since the tensor functor is right exact by (10.4.4), an equivalent statement is that if $f : A \to B$ is an injective R-module homomorphism, then

$$1 \otimes f : M \otimes A \to M \otimes B$$

is injective. In fact it suffices to consider only R-modules A and B that are finitely generated. This can be deduced from properties of direct limits to be considered in the next section. [Any module is the direct limit of its finitely generated submodules (10.9.3, Example 2). The tensor product commutes with direct limits (Section 10.9, Problem 2). The direct limit is an exact functor (Section 10.9, Problem 4).] A proof that does not involve direct limits can also be given; see Rotman, "An Introduction to Homological Algebra", page 86.

10.8.2 Example Since $\mathbb{Z}_2 \otimes_{\mathbb{Z}} -$ is not exact (Section 10.4, Problem 6), $\mathbb{Z}_2$ is not a flat $\mathbb{Z}$-module.

The next result is the analog for flat modules of property (10.5.4) of projective modules.

10.8.3 Proposition The direct sum $\oplus_i M_i$ is flat if and only if each M_i is flat.

Proof. Let $f : A \to B$ be an injective R-homomorphism. In view of (8.8.6(b)), investigating the flatness of the direct sum amounts to analyzing the injectivity of the mapping

$$g : \oplus_i (M_i \otimes A) \to \oplus_i (M_i \otimes B)$$

given by

$$x_{i_1} \otimes a_1 + \cdots + x_{i_n} \otimes a_n \to x_{i_1} \otimes f(a_1) + \cdots + x_{i_n} \otimes f(a_n).$$

The map g will be injective if and only if all component maps $x_i \otimes a_i \to x_i \otimes f(a_i)$ are injective. This says that the direct sum is flat iff each component is flat. ♣

We now examine the relation between projectivity and flatness.

10.8.4 Proposition R is a flat R-module.

Proof. If $f : A \to B$ is injective, we must show that $(1 \otimes f) : R \otimes_R A \to R \otimes_R B$ is injective. But by (8.7.6), $R \otimes_R M \cong M$ via $r \otimes x \to rx$. Thus the following diagram is commutative.

$$\begin{array}{ccc} R \otimes_R A & \xrightarrow{1 \otimes f} & R \otimes_R B \\ \| & & \| \\ A & \xrightarrow[f]{} & B \end{array}$$

Therefore injectivity of $1 \otimes f$ is equivalent to injectivity of f, and the result follows. ♣

10.8.5 Corollary Every projective module, hence every free module, is flat.

Proof. By (10.8.3) and (10.8.4), every free module is flat. Since a projective module is a direct summand of a free module, it is flat by (10.8.3). ♣

Flat abelian groups can be characterized precisely.

10.8.6 Theorem A $\mathbb{Z}$-module is flat iff it is torsion-free.

Proof. Suppose that M is a $\mathbb{Z}$-module that is not torsion-free. Let $x \in M$ be a nonzero element such that $nx = 0$ for some positive integer n. If $f : \mathbb{Z} \to \mathbb{Z}$ is multiplication by n, then $(1 \otimes f) : M \otimes \mathbb{Z} \to M \otimes \mathbb{Z}$ is given by

$$y \otimes z \to y \otimes nz = ny \otimes z$$

so that $(1 \otimes f)(x \otimes 1) = nx \otimes 1 = 0$. Since $x \otimes 1$ corresponds to x under the isomorphism between $M \otimes \mathbb{Z}$ and M, $x \otimes 1 \neq 0$, and $1 \otimes f$ is not injective. Therefore M is not flat.

The discussion in (10.8.1) shows that in checking flatness of M, we can restrict to finitely generated submodules of M. [We are examining equations of the form $(1 \otimes f)(t) = 0$, where $t = \sum_{i=1}^{n} x_i \otimes y_i,\ x_i \in M,\ y_i \ \in A$.]Thus without loss of generality, we can assume that M is a finitely generated abelian group. If M is torsion-free, then by (4.6.5), M is free and therefore flat by (10.8.5). ♣

10.8.7 Corollary The additive group of rationals $\mathbb{Q}$ is a flat but not projective $\mathbb{Z}$-module.

Proof. Since $\mathbb{Q}$ is torsion-free, it is flat by (10.8.6). If $\mathbb{Q}$ were projective, it would be free by (10.5.5). This is a contradiction (see Section 4.1, Problem 5). ♣

Sometimes it is desirable to change the underlying ring of a module; the term *base change* is used in these situations.

10.8.8 Definitions and Comments If $f : R \to S$ is a ring homomorphism and M is an S-module, we can create an R-module structure on M by $rx = f(r)x,\ r \in R,\ x \in M$. This is a base change by *restriction of scalars.*

If $f : R \to S$ is a ring homomorphism and M is an R-module, we can make $S \otimes_R M$ into an S-module via

$$s(s' \otimes x) = ss' \otimes x,\ \ s, s' \in S,\ \ x \in M.$$

This is a base change by *extension of scalars.* Note that S is an R-module by restriction of scalars, so the tensor product makes sense. What we are doing is allowing linear combinations of elements of M with coefficients in S. This operation is very common in algebraic topology.

In the exercises, we will look at the relation between base change and flatness. There will also be some problems on finitely generated algebras, so let's define these now.

10.8.9 Definition The R-algebra A is *finitely generated* if there are elements $a_1, \dots, a_n \in A$ such that every element of A is a polynomial in the a_i. Equivalently, the algebra homomorphism from the polynomial ring $R[X_1, \dots, X_n] \to A$ determined by $X_i \to a_i,\ i = 1, \dots, n$, is surjective. Thus A is a quotient of the polynomial ring.

It is important to note that if A is finitely generated *as an R-module*, then it is finitely generated as an R-algebra. [If $a = r_1 a_1 + \cdots + r_n a_n$, then a is certainly a polynomial in the a_i.]

Problems For Section 10.8

1. Give an example of a finitely generated R-algebra that is not finitely generated as an R-module.
2. Show that $R[X] \otimes_R R[Y] \cong R[X, Y]$.
3. Show that if A and B are finitely generated R-algebras, so is $A \otimes_R B$.
4. Let $f : R \to S$ be a ring homomorphism, and let M be an S-module, so that M is an R-module by restriction of scalars. If S is a flat R-module and M is a flat S-module, show that M is a flat R-module.
5. Let $f : R \to S$ be a ring homomorphism, and let M be an R-module, so that $S \otimes_R M$ is an S-module by extension of scalars. If M is a flat R-module, show that $S \otimes_R M$ is a flat S-module.
6. Let S be a multiplicative subset of the commutative ring R. Show that for any R-module M, $S^{-1}R \otimes_R M \cong S^{-1}M$ via $\alpha : (r/s) \otimes x \to rx/s$ with inverse $\beta : x/s \to (1/s) \otimes x$.
7. Continuing Problem 6, show that $S^{-1}R$ is a flat R-module.

10.9 Direct and Inverse Limits

If M is the direct sum of R-modules M_i, then R-homomorphisms $f_i : M_i \to N$ can be lifted uniquely to an R-homomorphism $f : M \to N$. The direct limit construction generalizes this idea. [In category theory, there is a further generalization called the *colimit*. The terminology is consistent because the direct sum is the coproduct in the category of modules.]

10.9.1 Direct Systems A *directed set* is a partially ordered set I such that given any $i, j \in I$, there exists $k \in I$ such that $i \le k$ and $j \le k$. A typical example is the collection of finite subsets of a set, ordered by inclusion. If A and B are arbitrary finite subsets, then both A and B are contained in the finite set $A \cup B$.

Now suppose I is a directed set and we have a collection of objects $A_i, i \in I$, in a category $\mathcal{C}$. Assume that whenever $i \le j$, there is a morphism $h(i,j) : A_i \to A_j$. Assume further that the $h(i,j)$ are *compatible* in the sense that if $i \le j \le k$ and we apply $h(i,j)$ followed by $h(j,k)$, we get $h(i,k)$. We also require that for each i, $h(i,i)$ is the identity on A_i. The collection of objects and morphisms is called a *direct system.* As an example, take the objects to be the finitely generated submodules of a module, and the morphisms to be the natural inclusion maps. In this case, the directed set coincides with the set of objects, and the partial ordering is inclusion.

10.9.2 Direct Limits Suppose that $\{A_i, h(i,j), i,j \in I\}$ is a direct system. The *direct limit* of the system will consist of an object A and morphisms $\alpha_i : A_i \to A$. Just as with coproducts, we want to lift morphisms $f_j : A_j \to B$ to a unique $f : A \to B$, that is, $f\alpha_j = f_j$ for all $j \in I$. But we require that the maps α_j be compatible with the $h(i,j)$, in other words, $\alpha_j h(i,j) = \alpha_i$ whenever $i \le j$. A similar constraint is imposed on the f_j, namely, $f_j h(i,j) = f_i$, $i \le j$. Thus the direct limit is an object A along with compatible morphisms $\alpha_j : A_j \to A$ such that given compatible morphisms $f_j : A_j \to B$, there is a unique morphism $f : A \to B$ such that $f\alpha_j = f_j$ for all j. Figure 10.9.1 summarizes the discussion.

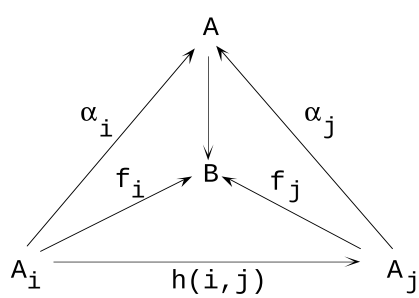

Figure 10.9.1

As in Section 10.2, any two direct limits of a given direct system are isomorphic.

If the ordering on I is the equality relation, then the only element j such that $i \le j$ is i itself. Compatibility is automatic, and the direct limit reduces to a coproduct.

A popular notation for the direct limit is

$$A = \varinjlim A_i.$$

The direct limit is sometimes called an *inductive limit.*

10.9.3 Examples

1. A coproduct is a direct limit, as discussed above. In particular, a direct sum of modules is a direct limit.

2. Any module is the direct limit of its finitely generated submodules. [Use the direct system indicated in (10.9.1).]

3. The algebraic closure of a field F can be constructed (informally) by adjoining roots of all possible polynomials in $F[X]$; see (3.3.7). This suggests that the algebraic closure is the direct limit of the collection of all finite extensions of F. This can be proved with the aid of (3.3.9).

In the category of modules, direct limits always exist.

10.9.4 Theorem If $\{M_i, h(i,j), i,j \in I\}$ is a direct system of R-modules, then the direct limit of the system exists.

Proof. Take M to be $(\oplus_i M_i)/N$, with N the submodule of the direct sum generated by all elements of the form

$$\beta_j h(i,j)x_i - \beta_i x_i,\ i \le j,\ x_i \in M_i \tag{1}$$

where β_i is the inclusion map of M_i into the direct sum. Define $\alpha_i : M_i \to M$ by

$$\alpha_i x_i = \beta_i x_i + N.$$

The α_i are compatible, because

$$\alpha_j h(i,j)x_i = \beta_j h(i,j)x_i + N = \beta_i x_i + N = \alpha_i x_i.$$

Given compatible $f_i : M_i \to B$, we define $f : M \to B$ by

$$f(\beta_i x_i + N) = f_i x_i,$$

the only possible choice. This forces $f\alpha_i = f_i$, provided we show that f is well-defined. But an element of N of the form (1) is mapped by our proposed f to

$$f_j h(i,j)x_i - f_i x_i$$

which is 0 by compatibility of the f_i. Thus f maps everything in N to 0, and the result follows. ♣

10.9.5 Inverse Systems and Inverse Limits Inverse limits are dual to direct limits. An *inverse system* is defined as in (10.9.1), except that if $i \le j$, then $h(i,j)$ maps "backwards" from A_j to A_i. If we apply $h(j,k)$ followed by $h(i,j)$, we get $h(i,k)$; as before, $h(i,i)$ is the identity on A_i. The *inverse limit* of the inverse system $\{A_i, h(i,j), i,j \in I\}$ is an object A along with morphisms $p_i : A \to A_i$. As with products, we want to lift morphisms $f_i : B \to A_i$ to a unique $f : B \to A$. There is a compatibility requirement on the p_i and f_i: if $i \le j$, then $h(i,j)p_j = p_i$, and similarly $h(i,j)f_j = f_i$. Thus the inverse limit is an object A along with compatible morphisms $p_i : A \to A_i$ such that given compatible morphisms $f_i : B \to A_i$, there is a unique morphism $f : B \to A$ such that $p_i f = f_i$ for all i. See Figure 10.9.2.

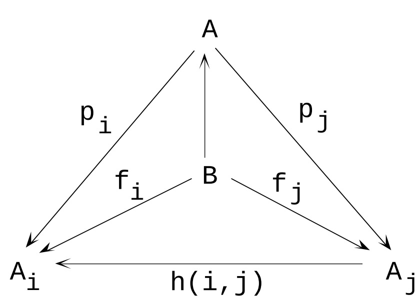

Figure 10.9.2

As in Section 10.2, the universal mapping property determines the inverse limit up to isomorphism.

If the ordering on I is the equality relation, the the inverse limit reduces to a product. In category theory, the *limit* is a generalization of the inverse limit.

A popular notation for the inverse limit is

$$A = \lim_{\leftarrow} A_i.$$

The inverse limit is sometimes called a *projective limit*.

We constructed the direct limit of a family of modules by forming a quotient of the direct sum. By duality, we expect that the inverse limit involves a submodule of the direct product.

10.9.6 Theorem If $\{M_i, h(i,j), i,j \in I\}$ is an inverse system of R-modules, then the inverse limit of the system exists.

Proof. We take M to be the set of all $x = (x_i, i \in I)$ in the direct product $\prod_i M_i$ such that $h(i,j)x_j = x_i$ whenever $i \leq j$. Let p_i be the restriction to M of the projection on the i^{th} factor. Then $h(i,j)p_jx = h(i,j)x_j = x_i = p_ix$, so the p_i are compatible. Given compatible $f_i : N \to M_i$, let f be the product of the f_i, that is, $fx = (f_ix, i \in I)$. By compatibility, $h(i,j)f_jx = f_ix$ for $i \leq j$, so f maps $\prod_i M_i$ into M. By definition of f we have $p_if = f_i$, and the result follows. ♣

10.9.7 Example Recall from Section 7.9 that a p-adic integer can be represented as $a_0 + a_1p + a_2p^2 + \cdots$, where the a_i belong to $\{0, 1, \dots, p-1\}$. If we discard all terms after $a_{r-1}p^{r-1}$, $r = 1, 2, \dots$, we get the ring $\mathbb{Z}_{p^r}$. These rings form an inverse system; if $x \in \mathbb{Z}_{p^s}$ and $r \leq s$, we take $h(r,s)x$ to be the residue of x mod p^r. The inverse limit of this system is the ring of p-adic integers.

Problems For Section 10.9

1. In Theorem (10.9.6), why can't we say "obviously", since direct limits exist in the category of modules, inverse limits must also exist by duality.
2. Show that in the category of modules over a commutative ring, the tensor product commutes with direct limits. In other words,

$$\lim_{\rightarrow} (M \otimes N_i) = M \otimes \lim_{\rightarrow} N_i$$

assuming that the direct limit of the N_i exists.
3. For each $n = 1, 2, \dots$, let A_n be an R-module, with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Take $h(i,j)$, $i \leq j$, to be the inclusion map. What is the direct limit of the A_n? (Be more explicit than in (10.9.4).)
4. Suppose that A, B and C are the direct limits of direct systems $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ of R-modules. Assume that for each i, the sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact. Give an intuitive argument to suggest that the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Thus direct limit is an exact functor.

[A lot of formalism is being suppressed here. We must make the collection of direct systems into a category, and define a morphism in that category. This forces compatibility conditions on the f_i and g_i: $f_j h_A(i,j) = h_B(i,j) f_i$, $g_j h_B(i,j) = h_C(i,j) g_i$. The direct limit functor takes a direct system to its direct limit, but we must also specify what it does to morphisms.]

A possible strategy is to claim that since an element of a direct sum has only finitely many nonzero components, exactness at B is equivalent to exactness at each B_i. This is unconvincing because the direct limit is not simply a direct sum, but a quotient of a direct sum. Suggestions are welcome!

Problems 5 and 6 give some additional properties of direct products.

5. Show that

$$\operatorname{Hom}_R(\oplus A_i, B) \cong \prod_i \operatorname{Hom}_R(A_i, B).$$

6. Show that

$$\operatorname{Hom}_R(A, \prod_i B_i) \cong \prod_i \operatorname{Hom}_R(A, B_i).$$

7. If M is a nonzero R-module that is both projective and injective, where R is an integral domain that is not a field, show that $\operatorname{Hom}_R(M, R) = 0$.
8. Let R be an integral domain that is not a field. If M is an R-module that is both projective and injective, show that $M = 0$.

Appendix to Chapter 10

We have seen that an abelian group is injective if and only if it is divisible. In this appendix we give an explicit characterization of such groups.

A10.1 Definitions and Comments Let G be an abelian group, and T the torsion subgroup of G (the elements of G of finite order). Then G/T is torsion-free, since $n(x+T) = 0$ implies $nx \in T$, hence $x \in T$. If p is a fixed prime, the *primary component* G_p associated with p consists of all elements whose order is a power of p. Note that G_p is a subgroup of G, for if $p^n a = p^m b = 0,\ n \geq m$, then $p^n(a-b) = 0$. (We use the fact that G is abelian; for example, $3(a-b) = a - b + a - b + a - b = a + a + a - b - b - b$.)

A10.2 Proposition The torsion subgroup T is the direct sum of the primary components G_p, p prime.

Proof. Suppose x has order $m = \prod_{j=1}^k p_j^{r_j}$. If $m_i = m/p_i^{r_i}$, then the greatest common divisor of the m_i is 1, so there are integers $a_1, \dots, a_k$ such that $a_1 m_1 + \cdots + a_k m_k = 1$. Thus $x = 1x = \sum_{i=1}^k a_i(m_i x)$, and (by definition of m_i) $m_i x$ has order $p_i^{r_i}$ and therefore belongs to the primary component G_{p_i}. This proves that G is the sum of the G_p. To show that the sum is direct, assume $0 \neq x \in G_p \cap \sum_{q \neq p} G_q$. Then the order of x is a power of p and also a product of prime factors unequal to p, which is impossible. For example, if y has order 9 and z has order 125, then $9(125)(y+z) = 0$, so the order of $y+z$ is of the form $3^r 5^s$. ♣

A10.3 Definitions and Comments A *Prüfer group*, also called a *quasicyclic group* and denoted by $\mathbb{Z}(p^\infty)$, is a p-primary component of $\mathbb{Q}/\mathbb{Z}$, the rationals mod 1. Since every element of $\mathbb{Q}/\mathbb{Z}$ has finite order, it follows from (A10.2) that

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}(p^\infty).$$

Now an element of $\mathbb{Q}/\mathbb{Z}$ whose order is a power of p must be of the form $a/p^r + \mathbb{Z}$ for some integer a and nonnegative integer r. It follows that the elements $a_r = 1/p^r + \mathbb{Z},\ r = 1, 2, \dots$, generate $\mathbb{Z}(p^\infty)$. These elements satisfy the following relations:

$$pa_1 = 0,\ pa_2 = a_1, \dots, pa_{r+1} = a_r, \dots$$

A10.4 Proposition Let H be a group defined by generators $b_1, b_2, \dots$ and relations $pb_1 = 0,\ pb_2 = b_1, \dots, pb_{r+1} = b_r, \dots$. Then H is isomorphic to $\mathbb{Z}(p^\infty)$.

Proof. First note that the relations imply that every element of H is an integer multiple of some b_i. Here is a typical computation:

$$\begin{aligned} 4b_7 + 6b_{10} + 2b_{14} &= 4(pb_8) + 6(pb_{11}) + 2b_{14} \\ &= \cdots = 4(p^7 b_{14}) + 6(p^4 b_{14}) + 2b_{14} = (4p^7 + 6p^4 + 2)b_{14}. \end{aligned}$$

By (5.8.5), there is an epimorphism $f : H \to \mathbb{Z}(p^\infty)$, and by the proof of (5.8.5), we can take $f(b_i) = a_i$ for all i. To show that f is injective, suppose $f(cb_i) = 0$ where $c \in \mathbb{Z}$. Then $cf(b_i) = ca_i = 0$, so $c/p^i \in \mathbb{Z}$, in other words, p^i divides c. (We can reverse this argument to conclude that $f(cb_i) = 0$ iff p^i divides c.) But the relations imply that $p^i b_i = 0$, and since c is a multiple of p^i, we have $cb_i = 0$. ♣

A10.5 Proposition Let G be a divisible abelian group. Then its torsion subgroup T is also divisible. Moreover, G can be written as $T \oplus D$, where D is torsion-free and divisible.

Proof. If $x \in T$ and $0 \neq n \in \mathbb{Z}$, then for some $y \in G$ we have $ny = x$. Thus in the torsion-free group G/T we have $n(y + T) = x + T = 0$. But then $ny \in T$, so (as in (A10.1)) $y \in T$ and T is divisible, hence injective by (10.6.6). By (10.6.2), the exact sequence $0 \to T \to G \to G/T \to 0$ splits, so $G \cong T \oplus G/T$. Since G/T is torsion-free and divisible (see (10.6.5)), the result follows. ♣

We are going to show that an abelian group is divisible iff it is a direct sum of copies of $\mathbb{Q}$ (the additive group of rationals) and quasicyclic groups. To show that every divisible abelian group has this form, it suffices, by (A10.2), (A10.5) and the fact that a direct sum of divisible abelian groups is divisible, to consider only two cases, G torsion-free and G a p-group.

A10.6 Proposition If G is a divisible, torsion-free abelian group, then G is isomorphic to a direct sum of copies of $\mathbb{Q}$.

Proof. The result follows from the observation that G can be regarded as a $\mathbb{Q}$-module, that is, a vector space over $\mathbb{Q}$; see Section 10.7, Problem 7. ♣

For any abelian group G, let $G[n] = \{x \in G : nx = 0\}$.

A10.7 Proposition Let G and H be divisible abelian p-groups. Then any isomorphism φ of $G[p]$ and $H[p]$ can be extended to an isomorphism ψ of G and H.

Proof. Our candidate ψ arises from the injectivity of H, as the diagram below indicates.

$$\begin{array}{ccccc} & & H & & \\ & & \uparrow & \nwarrow \psi & \\ 0 & \to & G[p] & \to & G \end{array}$$

The map from $G[p]$ to G is inclusion, and the map from $G[p]$ to H is the composition of φ and the inclusion from $H[p]$ to H. Suppose that $x \in G$ and the order of x is $|x| = p^n$. We will prove by induction that $\psi(x) = 0$ implies $x = 0$. If $n = 1$, then $x \in G[p]$, so $\psi(x) = \varphi(x)$, and the result follows because φ is injective. For the inductive step, suppose $|x| = p^{n+1}$ and $\psi(x) = 0$. Then $|px| = p^n$ and $\psi(px) = p\psi(x) = 0$. By induction hypothesis, $px = 0$, which contradicts the assumption that x has order p^{n+1}.

Now we prove by induction that ψ is surjective. Explicitly, if $y \in H$ and $|y| = p^n$, then y belongs to the image of ψ. If $n = 1$, then $y \in H[p]$ and the result follows because φ is surjective. If $|y| = p^{n+1}$, then $p^n y \in H[p]$, so for some $x \in G[p]$ we have $\varphi(x) = p^n y$. Since G is divisible, there exists $g \in G$ such that $p^n g = x$. Then

$$p^n(y - \psi(g)) = p^n y - \psi(p^n g) = p^n y - \psi(x) = p^n y - \varphi(x) = 0.$$

By induction hypothesis, there is an element $z \in G$ such that $\psi(z) = y - \psi(g)$. Thus $\psi(g + z) = y$. ♣

A10.8 Theorem An abelian group G is divisible if and only if G is a direct sum of copies of $\mathbb{Q}$ and quasicyclic groups.

Proof. Suppose that G is such a direct sum. Since $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are divisible [$\mathbb{Z}(p^\infty)$ is a direct summand of the divisible group $\mathbb{Q}/\mathbb{Z}$], and a direct sum of divisible abelian groups is divisible, G must be divisible. Conversely, assume G divisible. In view of (A10.6) and the discussion preceding it, we may assume that G is a p-group. But then $G[p]$ is a vector space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$; the scalar multiplication is given by $(n + p\mathbb{Z})g = ng$. Since $pg = 0$, scalar multiplication is well-defined. If the dimension of $G[p]$ over $\mathbb{F}_p$ is d, let H be the direct sum of d copies of $\mathbb{Z}(p^\infty)$. An element of order p in a component of the direct sum is an integer multiple of $1/p + \mathbb{Z}$, and consequently $H[p]$ is also a d-dimensional vector space over $\mathbb{F}_p$. Thus $G[p]$ is isomorphic to $H[p]$, and it follows from (A10.7) that G is isomorphic to H. ♣