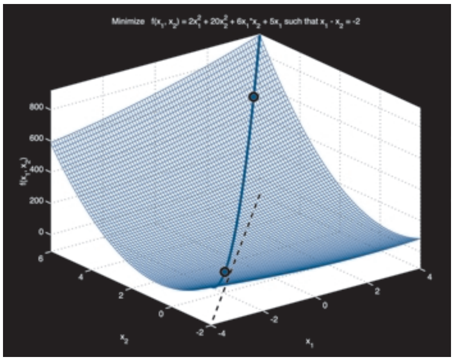


Examples of Programming in MATLAB®

The MATLAB® Environment

MATLAB integrates mathematical computing, visualization, and a powerful language to provide a flexible environment for technical computing. The open architecture makes it easy to use MATLAB and its companion products to explore data, create algorithms, and create custom tools that provide early insights and competitive advantages.

TABLE OF CONTENTS

Introduction to the MATLAB Language of Technical Computing

The MATLAB language is particularly well-suited to designing predictive mathematical models and developing application-specific algorithms.

It provides easy access to a range of both elementary and advanced algorithms for numeric computing. These algorithms include operations for linear algebra, matrix manipulation, differential equation solving, basic statistics, linear data fitting, data reduction, and Fourier analysis. MATLAB Toolboxes are add-ons that extend MATLAB with specialized functions and easy-to-use graphical user interfaces. Toolboxes are accessible directly from the MATLAB interactive programming environment.

MATLAB employs the same language for both interactive computing and structured programming. The intuitive math notation and syntax allow you to express technical ideas just as you would write them mathematically. This familiar style and flexibility make exploration and development with MATLAB very efficient.

The built-in math algorithms, optimized for matrix and vector calculations, allow you to increase efficiency in two areas: more productive programming and optimized code performance. The resulting time savings give you fast development cycles as well as execution speeds from within the MATLAB environment that are comparable to compiled code.

The two examples on the following pages illustrate MATLAB in use:

1) The first example compares MATLAB to C using three approaches to a quadratic minimization problem.

2) The second example describes one user's application of the M-file performance profiler to increase M-file code performance.

These examples demonstrate how MATLAB's straightforward syntax and built-in math algorithms enable development of programs that are shorter, easier to read and maintain, and quicker to develop. The embedded code samples, developed by MATLAB users in the MATLAB language, illustrate how MATLAB application specific toolbox functionality can be easily accessed from the MATLAB language.

COMPARING MATLAB TO C: THREE PROGRAMMING APPROACHES TO QUADRATIC MINIMIZATION

Introduction

Quadratic minimization is a specific form of nonlinear minimization. When a problem has a quadratic objective function instead of a general nonlinear function (such as in standard linear least squares), we can find a minimizer more accurately and efficiently by taking advantage of the quadratic form. This is particularly true when the quadratic problem is convex.

Some examples of these types of applications include:

- Solving contact and friction problems in rigid body mechanics, typical of the kind encountered in aerospace applications
- Solving journal bearing lubrication problems, useful for machine design and maintenance planning
- Computing flow through a porous medium, a common application in chemistry
- Selecting the best portfolio of financial investments

In this example we will use quadratic programming to solve a minimization problem. This example demonstrates the use of MATLAB to simplify an optimization task and compares that solution to the alternative C programming approach. The three following code examples compare three approaches to the minimization problem:

- MATLAB
- The MATLAB Optimization Toolbox
- C code including calls to LAPACK, the library that makes up much of the mathematical core of MATLAB

Problem Statement

Minimize a quadratic equation such as

$$y = 2x_1^2 + 20x_2^2 + 6x_1x_2 + 5x_1$$

with the constraint that

$$x_1 - x_2 = -2$$

Solution

Express the original equation in standard quadratic form. First, transform the original equation into matrix notation

$$y = \ast \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \ast \begin{bmatrix} 4 & 6 \\ 6 & 40 \end{bmatrix} \ast \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}^T \ast \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

such that

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \ast \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2$$

Second, substitute variable names for the vectors and matrices

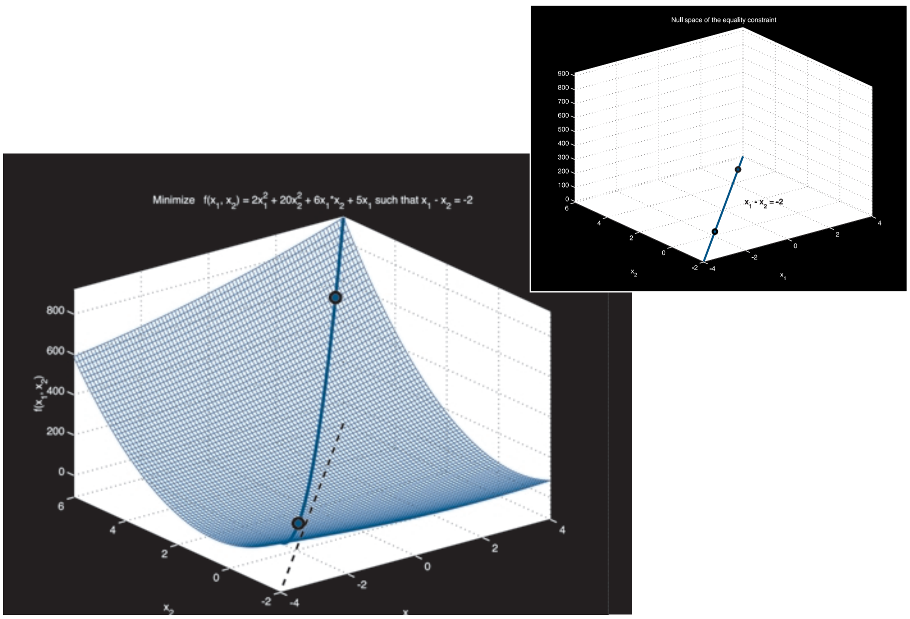
$$H = \begin{bmatrix} 4 & 6 \\ 6 & 40 \end{bmatrix},\ f = \begin{bmatrix} 5 \\ 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & -1 \end{bmatrix},\ b = \begin{bmatrix} -2 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Third, rewrite the quadratic equation as

y 5   * x T * H * x 1f T* x

and the constraint equation as

A * x =b.



The surface is the graphical representation of the quadratic function that we are minimizing, subject to the constraint that x1 and x2 lie on the line x1 – x2 = –2 (represented by the blue line in the small graphic and the dashed black line in the large graphic). This blue line represents the null space of the equality constraint. The black curve represents the values on the quadratic surface where the equality constraint is satisfied. We start from a point on the surface (the upper cyan circle) and the quadratic minimization takes us to the solution (the lower cyan circle).

The quadratic form of the equation is easier to understand and to solve using MATLAB's matrix-oriented computing language.

Having transformed the original equation, we're ready to compare the three programming approaches.

Example 1: Quadratic Minimization with MATLAB M code

MATLAB's matrix manipulation and equation solving capabilities make it particularly well-suited to quadratic programming problems. The constraint stated above is that *A * x =b*. In the example below we specify four values of b for which x must be solved, corresponding to the four rows of matrix A. In general, the data is determined by the known parameters of the problem. For example, to solve a financial optimization problem, such as portfolio optimization, the data would be determined by the expected return of the investments in the portfolio and the covariance of the returns. Then we would solve for the optimal weighting of the investments by minimizing the variance in the portfolio while requiring the sum of the weights to be unity.

In this example, we use synthetic data for *H, f, A,* and *b*. Once we set up these variables, the next step is to solve for the unknown, x. The code below roughly follows these steps:

1) Find a point in six-dimensional space that satisfies the equality constraints *A * x =b* (6 is the number of columns in *H* and in *A*).

2) Project the problem into the null space of A, that is, project into a subspace where movement in that subspace will not violate these constraints.

3) Minimize in this subspace. As our objective is a convex quadratic, we find the minimizer by stepping to the point at which the gradient is equal to 0, one Newton step.

4) Calculate the final answer by projecting the step to the minimizer and back into the original full space.

The following MATLAB M code, solves the type of minimization problem described above. This example has 6 variables and 4 constraints. The following assignment statements assign the data to the variables H (matrix), f (vector), A (matrix), and b (vector), respectively. The dimensionality of the problem, in this case 6, is specified implicitly by the number of columns in H and A. Note that in MATLAB code, lines preceded by "%" are comments.

```
%
% minimize  1/2*x'*H*x +f'*x   such that A*x=b
%  x

% 6 dimensional problem with 4 constraints:
%
H = [36 17 19 12 8 15; 17 33 18 11 7 14; 19 18 43 13 8 16;
   12 11 13 18 6 11; 8 7 8 6 9 8; 15 14 16 11 8 29];
f = [ 20 15 21 18 29 24 ]';
A = [ 7 1 8 3 3 3; 5 0 5 1 5 8; 2 6 7 1 1 8; 1 0 0 0 0 0 ];
b = [ 84 62 65 1 ]';

Z = null(A);          % Find the null space of constraint matrix A
x = A\b;              % Determine an x satisfying the equality constraints A*x = b
g = H*x +f;           % Compute the gradient (first derivative vector) of the quadratic
                      % equation at x
Zg = Z'*g;            % Find the projected gradient
ZHZ = Z'*H*Z;         % Compute the projected Hessian matrix (second derivative matrix)
t = -ZHZ \ Zg;        % Solve for the projected Newton step
xsol = x + Z*t;       % Project the step back into the full space and add to x to find
                      % the solution xsol
```

Example 2: Quadratic Minimization with the Optimization Toolbox

We can also solve the problem in Example 1 using the MATLAB Optimization Toolbox. It turns out that `quadprog`, the quadratic programming function in the Optimization Toolbox, can solve the entire problem described above. The following code could be typed in at the MATLAB command line or saved in a "script" file and run from MATLAB. Once the four MATLAB variable assignments are executed (the same ones as in Example 1), and some options are set, the call to `quadprog` is all that is required to solve the above problem.

```
H = [36 17 19 12 8 15; 17 33 18 11 7 14; 19 18 43 13 8 16;
   12 11 13 18 6 11; 8 7 8 6 9 8; 15 14 16 11 8 29];
f = [ 20 15 21 18 29 24 ]';
A = [ 7 1 8 3 3 3; 5 0 5 1 5 8; 2 6 7 1 1 8; 1 0 0 0 0 0 ];
b = [ 84 62 65 1 ]';
options = optimset ('largescale','off');
xsol = quadprog(H,f,[],[],A,b,[],[],[],options);
```

Example 3: Quadratic Minimization in C and LAPACK

What would the equivalent C code implementation look like for the same problem?

Assume that you have the C code version of the LAPACK linear algebra subroutine library and the BLAS subroutine library at your disposal. The following code uses the LAPACK subroutines `dgesv`, `dgelsx`, `dgesvd`, and the BLAS routines `daxpy`, `dgemm`, and `dgemv`. (Without these functions, you would need to write the numerical linear algebra routines in C directly). Here is the resulting C program, totaling 70 lines:

```
#include <stdio.h>
#include <math.h>
#include "f2c.h"

main()
{
  integer m = 4;
  integer n = 6;
  integer nrhs = 1;
  integer max1 = max(min(m,n)+3*n,2*min(m,n)+nrhs);          /* for dgelsx_ */
  integer max2 = max(3*min(m,n)+max(m,n),5*min(m,n)-4);      /* for dgesvd_ */
  integer lwork = max(max1,max2);                            /* for both */

  /* Problem definition arrays (column order) */
  double A[] = {7,5,2,1, 1,0,6,0, 8,5,7,0, 3,1,1,0, 3,5,1,0, 3,8,8,0};
  double b[] = {84,62,65,1};
  double H[] = {36,17,19,12,8,15, 17,33,18,11,7,14, 19,18,43,13,8,16,
                12,11,13,18,6,11, 8,7,8,6,9,8,  15,14,16,11,8,29};
  double f[] = {20,15,21,18,29,24};

  double *QR, *x, *work, *g, *s, *U, *VT, *Z, *ZH, *ZHZ, *Zg;
  double *p1, *p2, rcond, one = 1.0, zero = 0.0, minusone = -1.0;
  integer *jpvt, *ipiv, rankA, nullA, info, inc = 1, i, j;
  char jobu = 'N', jobvt = 'A', transn = 'N', transt = 'T';

  /* Allocate contiguous memory for work arrays */
  QR = malloc( m*n*sizeof(double) );
  x = malloc( n*sizeof(double) );
  work = malloc( lwork*sizeof(double) );
  g = malloc( n*sizeof(double) );
  s = malloc( m*sizeof(double) );
  U = NULL;
  VT = malloc( n*n*sizeof(double) );
  Z = malloc( n*n*sizeof(double) );
  ZH = malloc( n*n*sizeof(double) );
  ZHZ = malloc( n*n*sizeof(double) );
  Zg = malloc( n*sizeof(double) );
  jpvt = malloc( n*sizeof(int) );
  ipiv = malloc( n*sizeof(int) );

  /* QR = A: since it will be overwritten by dgelsx_ */
  for (i=0, p1=QR, p2=A; i<m*n; i++)
   *p1++ = *p2++;

  /* x = b: since it will be overwritten by dgelsx_ */
         for (i=0, p1=x, p2=b; i<m; i++)
```

```
     *p1++ = *p2++;

    /* x = A \ b: solve min||b - A*x|| for x, using a QR factorization of A */
    dgelsx_(&m, &n, &nrhs, QR, &m, x, &n, jpvt, &rcond, &rankA, work, &info);

    /* g = H * x */
    dgemv_(&transn, &n, &n, &one, H, &n, x, &inc, &zero, g, &inc);

    /* g = g + f */
    daxpy_(&n, &one, f, &inc, g, &inc);

    /* Z = null(A) */
    /* [U,S,V] = svd(A) */
    dgesvd_(&jobu, &jobvt, &m, &n, A, &m,
        s, U, &m, VT, &n, work, &lwork, &info);

    /* dimension of null space of A */
    nullA = max(m,n) - rankA;

    /* Z = last nullA columns of V (last nullA rows of VT) */
    for (j=0; j<nullA; j++)
     for (i=0; i<n; i++)
      Z[i+j*n] = VT[(i+1)*n-nullA+j];

    /* ZHZ = Z' * H * Z */
    dgemm_(&transt, &transn, &nullA, &n, &n,
       &one, Z, &n, H, &n, &zero, ZH, &n);
    dgemm_(&transn, &transn, &nullA, &nullA, &n,
       &one, ZH, &n, Z, &n, &zero, ZHZ, &n);

    /* Zg = Z' * g */
    dgemv_(&transt, &n, &nullA, &one, Z, &n, g, &inc, &zero, Zg, &inc);

    /* Zg = ZHZ \ Zg */
    dgesv_(&nullA, &nrhs, ZHZ, &n, ipiv, Zg, &n, &info);

    /* g = - Z * Zg; */
    dgemv_(&transn, &n, &nullA, &minusone, Z, &n, Zg, &inc, &zero, g, &inc);

    /* x = x + g */
    daxpy_(&n, &one, g, &inc, x, &inc);

    printf("=============================\n");
    for (i=0; i<n; i++) {
     printf("%f\n",x[i]);
    }
    printf("=============================\n");
  }
```

As you can see, except for the calls to the LAPACK routines and the comments, the remaining C code consists almost exclusively of variable declarations and memory allocation via calls to `malloc`.

Comparing the Three Programing Approaches

Clearly, it is possible to solve the minimization problem using MATLAB, the Optimization Toolbox, or standard C. All three approaches run and produce correct answers. The primary differences between them are code length and overall complexity.

The following table summarizes the number of lines of code required for each solution

MATLAB code example:	11 lines (including 4 lines of data definitions)
Optimization Toolbox example:	6 lines (including 4 lines of data definitions)
C code example:	70 lines (including 14 lines of declarations and 13 lines of memory allocation)

Both the MATLAB and Optimization Toolbox examples are fairly short and straightforward to follow. MATLAB accomplishes the task in 6 lines or 11 lines, depending on the approach. It requires 70 lines of C code to do what MATLAB could handle in 1/10 of that. In this case, the LAPACK industry-tested library contained the required routines to perform the necessary foundation numeric computations. Corresponding routines to those available in MATLAB and the complementary application specific toolboxes may not always be so reliable and easily accessible when solving other types of problems.

The C code version is much more difficult to read and understand than the corresponding MATLAB code that performs essentially the same steps. In addition, C, like many high-level languages, requires memory management, variable initializations, and variable declarations. These additional steps not only take up space, but require additional programming, debugging, and testing time. By comparison, MATLAB handles these overhead tasks behind the scenes, thereby saving time and allowing you to create more readable and maintainable code.

APPLICATION DEVELOPMENT IN MATLAB: TUNING M-FILES WITH THE MATLAB PERFORMANCE PROFILER

Many users have found MATLAB to be a very productive environment for experimenting with and comparing different approaches to application development. The three examples in the preceding section illustrate how MATLAB can help you shorten your development cycles and thereby save you time.

A number of MATLAB programming tools make it possible to not only develop solutions, but also to refine your prototypes and iterate on your designs. These tools include the MATLAB color-coded visual editor/debugger, the online hot-linked reference documentation, and the M-file performance profiler.

The performance profiler monitors the computing time spent on each line of code for a particular MATLAB M-file function. This information can provide you with insight into which operations are the most time-intensive, perhaps giving implicit clues about which computing tasks to tune or even replace with different, faster alternatives.

In the following example we apply the M-file performance profiler to a digital image processing application involving object measurement.

A number of image processing applications rely on the ability to measure and compare the characteristics of similar objects. This type of object measurement is useful in applications such as:

- Medical research, for example the study of cancer cells
- Quality control applications involving inspection sampling and process monitoring
- Analysis of satellite imagery, such as the object enhancement and measurement of rocks in photographs during the Sojourner Mars mission

In the following profiler summary report, the first few lines list the total execution time (2.18 seconds) and the lines that consumed the most computing time, in descending order of time. Three numbers precede the code on each of the individual code lines: the computing time in seconds, the percentage of the total time, and the number of lines of code in the original M-file routine.

The complete function, `imfeature.m`, is 220+ lines long. In the interest of space, the entire function is not included here but the relevant portions are shown. After profiling an execution of `imfeature.m`, the profiler generates this summary report:

```
profile report
Total time in "h:\ipt2.1\imfeature.m": 2.18 seconds
79% of the total time was spent on lines:
      [25 219 20 40 218 90 3 220 214 209]
             2:
0.05s, 2%    3:  numObjs = round(max(L(:)));
             4:  if (numObjs < 0)

            19: elementValues = L(idx);
0.24s, 11%  20: S = sparse(idx, elementValues, ones(size(elementValues)));
            21: objRowCoordinates = cells(1,numObjs);

24: for k = 1:numObjs
0.52s, 24%   25:   [objRowCoordinates{k},objColCoordinates{k}] = ind2sub(sizeL . . .
             26: end

             39:   stats(k).EulerNumber = eulerNumber;
0.14s, 6%    40:   stats(k).Centroid = [mean(r) mean(c)];
             41: end

             89: lut = aglut;
0.07s, 3%    90: markers = applylut(BW, lut);
             91: numQ1 = length(find(markers == 1));

            208:
0.04s, 2%   209: minR = min(r);
            210: maxR = max(r);

            213: r = r - minR + 2;
0.04s, 2%   214: c = c - minC + 2;
            215: M = maxR - minR + 1;

            217: BW = uint8(0);
0.09s, 4%   218: BW = repmat(BW, M+2, N+2);
0.49s, 22%  219: idx = sub2ind([M+2, N+2], r, c);
0.04s, 2%   220: BW(idx) = 1;
```

The report shows that `imfeature` is spending 22% of the time in calls to `sub2ind` (on line 219). Looking at the `subind` code, we can see that much of `sub2ind` handles general cases that aren't relevant to this particular use of `sub2ind`. As a result, we inserted the useful lines of code from `sub2ind` directly in the `imfeature` function, eliminating the calls to the `sub2ind` function.

After making the code changes, we ran the profiler on the modified `imfeature`.

```
profile report
Total time in "h:\ipt2.1\imfeature.m": 1.77 seconds
77% of the total time was spent on lines:
      [25 20 40 218 90 93 3 210 221 219]
             2:
0.05s, 3%    3: numObjs = round(max(L(:)));
             4: if (numObjs < 0)

            19: elementValues = L(idx);
0.23s, 13%  20: S = sparse(idx, elementValues, ones(size(elementValues)));
            21: objRowCoordinates = cells(1,numObjs);

            24: for k = 1:numObjs
0.53s, 30%  25:   [objRowCoordinates{k},objColCoordinates{k}] = ind2sub(sizeL . . .)
            26: end

            39:   stats(k).EulerNumber = eulerNumber;
0.22s, 12%  40:   stats(k).Centroid = [mean(r) mean(c)];
            41: end

            89: lut = aglut;
0.07s, 4%   90: markers = applylut(BW, lut);

 91: numQ1 = length(find(markers == 1));
            92: numQ2 = length(find(markers == 2));
0.05s, 3%   93: numQ3 = length(find(markers == 3));
            94: numQ4 = length(find(markers == 4));

           209: minR = min(r);
0.04s, 2%  210: maxR = max(r);
           211: minC = min(c);

           217: BW = uint8(0);
0.11s, 6%  218: BW = repmat(BW, M+2, N+2);
0.03s, 2%  219: idx = (M+2)*(c-1) + r;
           220: %%% idx = sub2ind([M+2, N+2], r, c);
0.03s, 2%  221: BW(idx) = 1;
           222:
```

We see above that the call to `sub2ind` is commented out (line 220) and replaced with an inline assignment statement on line 219.

This second profile report shows that `imfeature` now takes 1.77 seconds, a savings of 19%. Specifically, the inline computation takes 0.03 seconds, compared with 0.44 seconds previously.

Not bad for a single, quick change but we could certainly pursue further optimizations such as investigating whether the code on line 25, now taking up 30% of the total execution time, could be further tuned. This example is just one illustration of the savings possible with the various programming tools available in the MATLAB environment.

For demos, application examples, tutorials, user stories, and pricing:

- Visit **www.mathworks.com**
- Contact The MathWorks directly

US & Canada	508-647-7000
Benelux	+31 (0)182 53 76 44
France	+33 (0)1 41 14 67 14
Germany	+49 (0)89 995901 0
Spain	+34 93 362 13 00
Switzerland	+41 (0)31 954 20 20
UK	+44 (0)1223 423 200

Visit **www.mathworks.com** to obtain contact information for authorized MathWorks representatives in countries throughout Asia Pacific, Latin America, the Middle East, Africa, and the rest of Europe.

www.mathworks.com

The MathWorks

9514v02 08/01