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# Introduction

In the summer of 1988 a group of us from the Five Colleges—Amherst, Hampshire, Mount Holyoke, and Smith Colleges, and the University of Massachusetts—in western Massachusetts began working on a new calculus curriculum under a five-year grant from the National Science Foundation. We had two broad goals in mind: 1) to develop the concepts of calculus in the context of substantial problems from the other sciences, and 2) to incorporate the visual and computational power of computers into the exploration of these concepts.

For the first five years, at the end of each semester the faculty who had taught the material got together for a daylong session to share experiences of what had worked well and what difficulties had been encountered. Each summer we then worked to revise the materials and write new sections, ironing out those spots that had been unclear or where the purity of our initial conception ran into the realities of the classroom. We also ran a number of workshops around the country for other faculty thinking of using the materials. This *Handbook* is a distillation of the topics and issues which regularly arose during the debriefing sessions and workshops. As more teachers use these materials in a wider range of settings, the *Handbook* will undoubtedly continue to be revised. We would therefore like very much to hear from you about things that didn't work for you, pitfalls or opportunities that developed in your class, or about suggested improvements in either the text or this *Handbook*.

## Why Read this Handbook?

There are a number of snares lying in wait for those teaching this material for the first time. The choice, order, and development of topics are substantially different from the way most of us were taught, and have ourselves been teaching. This provides many points at which new users of these materials can anticipate how they think the material will or should unfold, only to be left off-balance when a very different tack is taken. In this *Handbook* we flag a number of these points and explain the choices made.

Moreover, this course in places draws upon examples from domains like epidemiology or ecology that are not part of the standard training of most mathematics teachers. While the examples are meant to be self-contained, a number of teachers have expressed an interest in having more of the background available. This *Handbook* therefore includes supplemental discussion and references for those wishing to explore the topics further.

Finally, this course is technology-dependent to a much greater extent than the courses many of us are used to teaching. We identify a number of technical and pedagogical issues that have come up relating to the use of computers or graphing calculators, and include some suggestions for dealing with them.

## How to Use this Handbook

The body of the *Handbook* consists of two main sections. The first 25 pages on Course Structure lay out the general pedagogical and curricular concerns underlying the choice and presentation of topics, with some suggestions regarding their implementation. The remainder of the body of the *Handbook* is a section-by-section commentary on the text.

Throughout, this *Handbook* and the main text should be viewed as a guide only. The arrangement of the topics in the text is a suggestion, an order which has worked well for us. Even we don't go through every item in order every time. You undoubtedly have your own pet examples and ways of covering some topics, and you should certainly feel free to customize the materials. Nevertheless, if you are trying the material for the first time, we urge you to stay fairly close to the order and style given to get a good overall feel for the novel features of this approach. For purposes of future versions of the text and this *Handbook*, if you do try a variation which works well, please let us know.

# Course Structure

## The Audience

The original users of these materials were undergraduates at four-year liberal arts colleges, majoring in everything from mathematics to the arts. By and large, they are not unusually gifted mathematically and have as much difficulty as most undergraduates in remembering the quadratic formula. When we began, we thought a separate course for mathematics and physics majors might be needed, but we have come to feel that this course's view of mathematics and its relation to the other disciplines is an important one for these students to cultivate as well, and we now have them all in one course. Now, many students at high schools, two-year colleges, and universities are also using this text.

As is increasingly the case at a number of schools, many of our students arrive having already completed a calculus course elsewhere, and the usual problems of deciding where to place them arise. For those whose background is quite strong, we have found that they can generally acquire the ideas and tools of this course in their other work, without taking this course. For those not quite this strong, moving directly into the second semester material works well, requiring only a bit of scrambling on their part at the beginning of the term to become familiar with some of the numerical concepts involved.

The less strong students who have had calculus before, though, or those looking for an easy course (since they think they already know the material), pose particular problems. These students usually sign up for the first semester. For most of them, this works out well—the material is different enough from what they've seen so they don't get bored, and the new perspective often helps them understand the concepts more profoundly. Some, though, feel betrayed when they see how different this course is from what they've had before, responding with "When are we going to get to the calculus?" and resenting the time they have to put into the course when they thought they knew it already. It helps to point out the kinds of problems they are able to solve in this course that they couldn't have dealt with in their previous course.

As always in launching something new in the classroom, it is important to make sure the students see themselves as our co-experimenters rather than as our guinea pigs. As they compare this course with their own experiences or with other courses their friends are taking, they will need reassurance that we know what we are doing. We have found that one of the most helpful things we as teachers can do is to talk to our students—explain what we are doing and why; tell them how this course compares with a more traditional course, without belittling the standard course; get feedback early and often. The issues can't be adequately addressed in a single conversation at the beginning of the semester, but need to recur throughout the course, more often than our initial intuitions would have suggested.

## The Starting Points

The material in this course is based on five premises:

1. Calculus is fundamentally a way of dealing with functional relationships that occur in scientific contexts. The language, tools, and models of calculus arose through trying to understand these relationships, and the other sciences still provide an ongoing source of new and interesting topics for investigation. An awareness of this connection should be a part of the students' perception of the material from the beginning. Particularly in the initial stages, developing the techniques of calculus must not obscure an overview of the kinds of underlying questions calculus is designed to explore.

2. Computers radically enlarge the range of questions we can explore and the ways we can address them. Computers are much more than a tool for teaching standard calculus; they change the standard. When we can replace sophisticated analytical techniques with conceptually simple computational approaches, some important classes of problems that were formerly considered to be advanced can now be explored at the introductory level. Moreover, numerical approaches often provide a generality in the treatment of topics like integration and differential equations which, in the traditional exposition, can appear to be a miscellany of special cases. Finally, computers encourage a geometric

approach which can substantially enhance the students' mathematical understanding.

3. The concept of a dynamical system is central to science, as any perusal of the current literature will quickly indicate. Therefore, calculus must prepare students, preferably at an early stage, to begin dealing with systems of non-linear differential equations and the kinds of questions that arise about such systems.

4. The concept of derivative is much more fundamental than, and is separable from, the process of differentiation. It has been our experience in the past that students all too often think of the derivative only in terms of a set of differentiation rules. In fact, in many contexts—dynamical systems, for instance—the derivative is given by a model or by a geometric analysis rather than from differentiating some function. Students need a clear geometric and operational understanding of what a derivative is in its own right.

5. The process of successive approximation is a key tool of calculus, even when the outcome of the process—the limit—cannot be explicitly given in closed form. The standard $\epsilon$–$\delta$ approach, assuming as it does that we somehow know the answer, is often a much less useful way of thinking about the limiting process than Cauchy's approach.

## Curricular Themes

We have taken the above starting points and abstracted a small set of themes from them, around which we have organized the curriculum.

### Context

If you ask typical students what mathematics is about, they are likely to deny that it is about anything. They perceive mathematics as existing in a world of its own, with its own rules, having little to do with any questions they might be interested in. The so-called "applications" that are provided, almost always after the mathematics has been completely worked out, are often transparently artificial and do little to convince skeptical students that mathematics has anything to say about the world in which they live. We feel much of the low regard the general public currently has for mathematics arises from treating mathematics as a strictly technical discipline, responsive only to its own internal logic and structure.

Historically, though, much of calculus arose as a tool to explore questions in the sciences—including, of course, other branches of mathematics. Our students need to see this connection throughout as they learn the material, not just as an optional afterthought appended to the mathematics.

Providing this kind of context for the mathematical ideas can be daunting for many teachers. Few of us have the training to claim expertise in any field outside mathematics, and none of us has the time to acquire such expertise now. The main advice is: Don't try to present yourself as an expert. If you are in command of the mathematical component, students can readily accept your role is an intelligent amateur in ecology, physics, or chemistry. They will even enjoy the role reversal, when they know more than you. While the examples in the text are meant to be self-contained, some teachers will want to develop their own examples. Your students can help in this, both during the course and after. They will be glad to enlarge your repertoire by bringing you examples they come across where calculus is used. Your colleagues in the other sciences can be good sources of examples and topics, and they will appreciate being consulted. If you have the time and the inclination, skimming through journals like *Science*, *Nature,* or *The American Naturalist* can suggest possible topics. This *Handbook* contains a modest selection of suggested readings which you can peruse if you are so inclined. View this as a long-term development, not as something which has to be accomplished before you teach this material for the first time.

## Differential Equations

In looking through the scientific journals, the large majority of settings in which calculus occurs take the following form: the investigators have a system of interacting quantities whose behavior they want to analyze, and the constraints acting on the system allow them to model the rates at which the quantities are changing. That is, they start with a system of differential equations, typically non-linear, and they want to know something about its solution curves, asymptotic behavior, or the existence and nature of any equilibrium points.

Since this is such a universal feature of all the sciences, we have made it a central theme of this course. This topic is introduced on page 1, in a model for the progression of an epidemic, and is developed in many of the later chapters.

### Modelling

While we do not view this as a modelling course, it is important for students to develop a reasonably sophisticated appreciation for the interplay between real-world problems and the mathematical models we construct to help us think about these problems. A first difficulty in this process for many of our students is simply one of translation—from descriptions expressed in English to mathematical equations, and vice versa. Once students begin to be comfortable with this translation process it is possible to go on to discuss issues like what makes for a good model, the value and place of both quantitative and qualitative prediction, and the like. In particular, it is useful for students to begin to develop a good feel for the role parameters play in constructing models.

### Successive Approximations

Up to this point in their mathematical education, every problem our students have encountered has had one correct answer. In this course, though, this rarely happens. Solutions can be approximated to high degrees of accuracy, but the solution itself can not be written down in closed form. Thus the approximations are not just useful clues leading up to the "real" answer like 2 or $\pi$ or $\sin x$—often the approximations are all we have. This is a startling shift for many students to make, and an important one. Moreover, students need to develop a strong appreciation of the tradeoff in time (and perhaps money) in getting the next decimal place of accuracy in an approximation.

### Geometric Visualization

A computer's ability to produce and manipulate graphical images introduces a conceptual element that is very helpful in thinking about mathematical problems. We have tried to incorporate this into our course wherever we could, encouraging our students to cultivate their geometric intuitions and to see calculus as more than a collection of algebraic rules for manipulating strings of symbols. When students can actually see the graph of a function becoming linear as they zoom in on it, or when they can see a series of piecewise-linear curves approaching a smooth curve as a limit, these concepts become very real and concrete in ways that are difficult to achieve through more formal arguments.

### Numerical Methods

Traditionally, numerical methods have been the last recourse, to be used when one could think of no clever technique for producing a closed-form solution to a problem. Students were implicitly taught to expect that most problems would be tractable, that only if they were really unlucky would they have to resort to Simpson's rule or Runge-Kutte techniques. With readily available computers, though, the position is reversed—students can be taught to approach every problem of integration, for instance, knowing in advance that it is solvable by numerical methods at least, and that if they are really lucky it might even yield to a clever analytical shortcut like integration by parts. We feel that this shift in attitude is an important one, making our students more effective users of calculus since the concepts are seen in a more universal light.

At the same time, though, it is important to stress that this is not a course in numerical techniques, and we often stay with a particular approach—Euler's method, for instance—because of its conceptual simplicity, even though there may be other techniques that give more rapid convergence.

## Pedagogical Aspects

In addition to these curricular themes, we have designed this course to encourage our students to think about what it means to do mathematics in several ways which are new to many of them.

### Collaboration

A great deal has been written about the role of collaborative learning, which we won't go into here. In our calculus classes the students are strongly encouraged to work in groups of two or three on the homework problems, and we have found this to be very effective. The students encourage one another and work productively to make suggestions and try out possibilities that they would not have had the confidence or energy for if working individually. In fact, many of the problems are so involved that it would be discouraging and difficult for a typical student to work on them alone.

This immediately raises a number of questions for teachers: Do we assign students to groups or let them choose their own? How do we assign credit for the work done? Do the groups turn in one solution set per group? How do we get shyer students into groups? While each teacher develops his or her own response to such questions, we have found none of them to be a major hurdle. Some of us ask students to submit a single, joint homework paper representing their group work. We try to be sure that the responsibility for writing up joint solutions is shared evenly (for example, by asking students not only to list all members of the group but also to indicate who wrote up each portion). Having regularly scheduled problem sessions outside of class is another good way to encourage students to work together. Having a student assistant on hand at these sessions is helpful, too, so long as the assistant has been trained not to fall into the trap of being too helpful.

## Calculus as a Language

For many people—both teachers and students—the most striking feature of the text is the number of words. Students used to a largely algebraic approach to mathematics will be wondering where the formulas and equations are. Moreover, the lack of "template" examples in the text that students can turn to and readily adapt for doing their homework forces many of them to revise the successful strategies they have evolved for dealing with mathematics courses.

The text is designed to remind students that mathematics problems arise out of real world contexts and to give them ample practice in the art of translating such problems into mathematics. We believe that this translation process from words to mathematics is an important part of being an effective user of mathematics. The real problems that our students will encounter outside our classes rarely come labeled and broken down into tidy parts. While they need to be proficient in the routine manipulations, students need to realize that there is more to being good at mathematics than proficiency in manipulating symbols. We have found in previous courses that the ability to successfully perform mathematical manipulations does not always coincide with the ability to assign meaning to such manipulations—to think mathematically. We want our students to be fluent in moving back and forth between English statements and mathematical ones, and have structured the text to reinforce this.

## Tackling Large, Messy, Ill-Defined Problems

Once students leave their mathematics classes, they will often encounter problems that don't immediately suggest a specific technique for their solution. There may be incomplete or irrelevant information, there will be too many complexities to deal with all at once, and the like. We feel it is important for our students to begin getting some practice with how they can make a start in such a situation. We have therefore included a number of problems that require simplification or are first explored by generating data and looking for patterns. We also ask students to think and write about what they are doing, articulating the pros and cons of various approaches.

### Experimentation

An important part of thinking about hard problems is trying things out and experimenting with different possibilities to see what happens. As others have pointed out, computers can provide an experimental flavor to mathematics for the average student. One of the most striking features of teaching this material in a computer classroom is how quickly the students escape the control of the teacher. They try different models of a problem, vary parameters to see what patterns emerge, and exchange discoveries with one another. Having a setting where students can discover some of the truth for themselves rather than simply having it handed on by the teacher is very powerful.

### Approximation

Thinking in terms of initial, approximate answers to a problem rather than leaping immediately to *the* right answer is very difficult for many students, yet it is an attitude they must develop if they are ever to be able to approach large, messy problems with confidence. Making approximations also forces students to think about the structure of a problem in ways they can often avoid if all they have to do is perform certain manipulations to get the right answer. The notion of approximation is central to most of the topics in this text, and by the end students have a much more sophisticated conception of the role of approximations and how to use them intelligently.

### The Importance of Problems

Most students come to this course with the general notion that mathematics is about learning concepts and tools, and that doing problems—often exercises, really—is where they get good at using the tools, which will then be tested on the exams. This course is structured to make wrestling with problems much more central. We want the students to feel that they are learning the tools so they can think about interesting problems, rather than that they are doing problems so they can learn the tools. While there is a place for exercises that are moderately repetitive variations on a common theme to help develop facility with a certain tool or concept, the skill our students will need in the long run is to feel comfortable jumping into a problem when it is not obvious what the appropriate approach is. Many of our problems are designed to develop this attitude.

Moreover, many problems anticipate ideas and directions that will not have been covered yet in class, to get the students thinking about the issues before they are raised by the teacher. This is a significant change in what the students are used to, and it helps if the teacher discusses the reasons for this with the students.

There are several mechanisms we use for getting the students to take the homework problems more seriously. One is simply to insist on more clarity in the written explanations accompanying their answers. Another is to assign more credit to the homework in computing the final grade—some of us count the homework for as much as 50% of the grade. One arrangement we use that helps the students' learning is to permit students to resubmit homework papers. It is helpful to have the original work attached to the revision so the reader can easily compare the two versions. A useful grading scheme is to assign one of three overall grades to each paper, for example 1, 2, or 3. We might tell students that "1" indicates they should get help and then re-do the assignment (or those portions presenting difficulties); "2" indicates that they would benefit from re-doing the assignment; and "3" that the paper is good enough to study from. In practice this tends to mean that almost all groups end up with the grade of "3" on nearly all assignments, but this seems fine to us.

## The Role of the Text

As working on the problems becomes more central, students will need to learn to use the text in ways that may be different from their previous uses of mathematics texts. *Calculus in Context* is more than a convenient summary of a set of techniques, theorems, and worked examples. There is a narrative flow to the ideas which reaches its culmination as the students grapple with the exercises. We make specific reading assignments before each class and then spend the bulk of the class time working on and discussing the problems and questions that arose from that reading. It is our experience that most students soon learn the value of doing the reading ahead of time, especially if during the first several weeks the teacher helps students draw answers to their questions out of the reading. We try to resist the temptation to present the text in lecture format.

### Intuition and Rigor

In mathematical learning, as in the rest of developmental biology, "ontogeny recapitulates phylogeny"—that is, the development of the individual's mathematical understanding can often proceed most productively if it follows the evolution of the discipline itself. Just as the 19th century's concern for definition and proof only came after more than a century of at times free-wheeling imaginative leaps, we should allow our own students time to develop substantial intuitions about the material before pushing them too hard to be rigorous. Thus, for instance, the word "limit" occurs quite early in this text, and is used fairly often thereafter, with increasing precision, but it is only halfway into the second semester that anything like a precise definition is offered.

The kinds of reasoning skills most often required in this course are somewhat different from the tightly-reasoned mathematical arguments some students (and many mathematics teachers!) enjoy. If you have such students in your class, you can steer them to points in the text that offer opportunities to explore this side of mathematics. These include the second treatment of the exponential function in 4.3, the proofs of the differentiation rules in 5.1, the proofs of periodicity in 7.3, the recursion relations in 10.4, the treatment of convergence in 10.5, and Stirling's formula in 12.1. By contrast, problems in this course are rarely "hard" in the usual sense of requiring a lot of clever algebraic manipulation. When problems in this text are perceived as hard, it is because they require a lot of common sense, together with a good feel for the underlying mathematical ideas.

### Rethinking Techniques

While much of calculus builds on broad, general insights, many of the techniques of calculus respond to the need to perform specific kinds of complex calculations rapidly. In many instances, though, computers potentially reduce the importance of these tools. For computational purposes, crude, brute-force algorithms can replace many of the elegant methods developed by generations of mathematicians. Many branches of math—statistics, linear algebra, calculus—currently seen as "advanced" to a greater or lesser degree are difficult because of the time and effort needed to develop their *techniques*. Many of the underlying *concepts* of those subjects, however, are straightforward and can be understood by students working at an elementary level. The impact of this potential shift and how to accommodate it in our classes will undoubtedly engage mathematics teachers for years to come. Let's look at a couple of examples from the calculus curriculum.

**Example 1: Maxima-minima**

Traditionally, a lot of time in calculus courses is spent on max-min problems, where the student sets up the function, takes the derivative, finds where it equals 0, and tries to determine which points are maxima and minima of various kinds. The concept is certainly important, and the techniques can be an excellent exercise in algebra and analytical thinking.

However, most students (and virtually all professionals) now have access to high-quality graphing software. Once they have set up the function, they can simply display it and zoom around the graph to locate the maxima and minima, using no calculus at all. While it is possible to create examples which will fool the naive user of this approach, by and large graphing software leads to answers more rapidly than hand analysis, with a lower probability of algebraic and arithmetic errors. More importantly, unlike the traditional calculus course where we have to be careful to choose functions where the students can actually solve the equation $f'(x) = 0$, the computer approach is general—all functions are dealt with the same way. The concept of extrema is simple—9th graders can grasp it easily. It is only the traditional techniques that are at all advanced, requiring the treatment of such problems to be deferred.

Moreover, graphing software liberates the student to tackle more complex and interesting problems than the traditional Norman window problem or the lighthouse keeper forever rowing his boat to a point on the shore and walking into town. Attention can be focused on the initial stage—that of setting up the function in the first place, or analyzing the appropriateness of the model—which is where many of our students have the greatest difficulty. Finally, while traditional analytical techniques are not eliminated, their use is shifted somewhat to areas where numerical methods are less useful. For instance, traditional methods of analysis are very powerful in the exploration of max-min problems involving parameters.

### Example 2: Differential equations

Traditionally, differential equations is an advanced topic, requiring two years of calculus as a prerequisite. It is divided into a number of subcases, and an array of techniques is developed to deal with different cases. Many of these techniques are very clever and elegant, and display the kind of intricate reasoning and analysis that attracted many of us to mathematics in the first place.

Differential equations are central to this course and are introduced on the first day. We treat *all* differential equations the same way, using a simple and intuitively clear numerical approach. The student thus spends no time worrying about about which technique to use, or whether the problem is solvable at all (in the sense of there being some transformation which will reduce it to a recognizable form). We can thus, at an elementary level, address problems and concepts which the overwhelming majority of our students would never get to see in the traditional curriculum.

## Computing

This course cannot be implemented without ready student access to good graphing and computing facilities. Here we discuss some of the issues that come up frequently in thinking about making these facilities work well. Although we set forth our vision of the ideal setup, each school will need to adapt the suggestions to its own realities. Even at our own schools we can't all provide the ideal arrangements.

### Classroom Layout

Ideally, this course should be taught in a space where each group of two or three students has access to a computer linked by a network, or in which each student has a good graphing calculator. If you are using computers, it is most effective if the computers are arranged around the edge of the room rather than being in rows in the middle of the room. When each group of students can look around and see what every other group is doing, very productive sharing of results and ideas occurs. Less effective, because it makes it much harder for the students to become actively involved and try out their own ideas, is to have a single computer with good projection facilities, with the students' computer work left to lab sessions. A class where each student has a good graphing calculator is somewhere in between—students are still able to generate their own data, but tend to work a little more in isolation than in the ideal layout.

### Computer Labs

An experimental flavor, with students collecting data and looking for patterns, is an important feature of this course, and one which distinguishes it from standard treatments. Some teachers emphasize this by explicitly designating this as a laboratory course with a specific lab period once a week. Students are given projects to investigate and are expected to write their work up in a laboratory notebook. Besides reinforcing the experimental aspect, this also has the benefit of getting the students to write more descriptively about what they are doing, a process which helps many of them think more clearly about how they are approaching problems.

Even in courses without a separate laboratory component, though, students will need to use computers or graphing calculators outside of the classroom. Ideally, there should be a room with tables and a number of reasonably fast computers with high-resolution color monitors served by a network. There should be a printer attached so students can get copies of their output, or so they can print out programs that don't seem to be working to show to the teacher or course assistant. Students should be encouraged to have their own disks on which their versions of the various programs can be stored and, for those following a laboratory course format, on which the lab notebooks can be maintained. The advantage of having the computers in a single room rather than scattered about is to encourage students to collaborate and share results and ideas. For this reason, even courses based on graphing calculators might think about providing a working space where students can gather to work on problems outside of class.

### Appropriate Use of Technology

Properly used, technology allows students to think about many interesting problems that would otherwise be inaccessible. Technology can introduce an experimental flavor to mathematics, making the student much more actively involved in the learning process. Technology can reduce the amount of time spent on tedious drill and on hand calculations so that the student can focus on the underlying conceptual frameworks. These are all major benefits with far-reaching implications.

There are at least two major traps to be avoided, though. The first is to make sure that our students don't become mindless button pushers, punching in the problem, waiting for the machine to produce its output, and transcribing the result to paper without ever engaging their higher cortical processes. It is essential that students pause to reflect upon the significance of what their computers or calculators are telling them. Even better, they should think about the problem to develop develop a qualitative expectation before doing any computer calculation. This tension between math teachers trying to get their students to think and students wanting to reduce everything to rote mechanical processes is not new, of course. While omputers and calculators add to the potential for this kind of abuse, though, they also offer wonderful possibilities for breaking out of it.

Second, our students need enough practice with hand calculations to develop a good understanding of the principles involved, even though there is no longer the need for them to become as adept as earlier generations were expected to be. Students should at all times view their computer or calculator as a labor-saving device rather than as a superior intelligence. They should always be able to at least contemplate the possibility, for sufficient remuneration, of doing any given problem by hand.

### Software vs. programming

In our courses, output from the computers is obtained in three different ways: 1) through the Basic-like programs which are scattered throughout the text; 2) through software packages we have developed for manipulating the graphs of functions, for solving differential equations, and for working with density plots and contours of functions of two variables; and 3) through commercial numerical and symbol-manipulation packages like Mathematica, Maple, or Derive.

We feel that it is important in the early stages that students use the Basic programs to make sure they realize how simple the underlying concepts really are. There is a lot of variation in how rapidly we move from this stage to using software packages. Some of us continue with the programs through much of the second semester, while others have already moved to differential equation-solving software for most of their work by the end of the first semester. While this is clearly a matter of the taste of the teacher, there are two goals we would urge on you: 1. The students should not use the computers in ways that cause them to view them as magical black boxes that can mysteriously do things they could never do (if they had the time). This means they should use the Basic programs long enough that the manipulations feel mundane. 2. On the other hand, this is not a computer course, and the teacher should at all times resist the temptation to make it one. Questions of programming style, algorithmic efficiency, and the learning of all sorts of clever computer commands should always be subservient to the mathematical ideas under consideration.

For those of you who would like to use Basic programs for all the topics, Appendix E to this *Handbook* offers some programs for use with some of the more advanced topics. These programs are written in TrueBasic and can be used on either Macintosh or PC platforms. It should be relatively easy to adapt these programs to whatever programming language your computers or calculators are using.

We maintain an annonymous ftp transfer site at emmy.smith.edu where we store copies of the graphing and numerical software we have developed, together with supplemental Quick Basic and True Basic programs. A README file gives more details of what's available on emmy.

## Calculators vs. computers

Our own experience has largely been with computers having VGA monitors with color graphics, although a few of us are using Macintosh labs. The computer applications have largely been developed with this kind of facility in mind. Nevertheless, some users have taught the course using graphing calculators and report little difficulty converting the material to that platform. With the increasing power of hand-held calculators and the improvement of their graphics, there should be even fewer problems. Note that Appendix A in the text includes translations of all the Basic programs for the main graphing calculators currently available: TI-81, TI-82, and TI-85; Casio $fx$-7700G and $fx$-9700GE; and Sharp EL-9200/9300. Translations for use on the H.P will be available soon on the program's ftp site (emmy.smith.edu).

The advantages of using a computer network are: 1) The speed and accuracy make it possible to pursue limits a bit further, which has pedagogical merit at times; 2) The networking capabilities make it possible to maintain the software and the Basic programs easily; 3) If the computers are connected to a printer of some sort, students find it very helpful to be able to print out graphic images and programs (particularly when they don't work and they want to ask us why!); 4) The high-resolution graphics support the development of sophisticated visualization on the students' part; 5) There are a number of sophisticated software packages for applications like solving differential equations, graphing vector fields, or dealing with large systems of equations which permit explorations that simply can't be done or done as well (yet!) on calculators.

The advantages of using hand-held calculators are: 1) Lower cost to the institution; 2) Commuting students can have access to the computational tool at their convenience; 3) Any classroom can be readily converted into a computational laboratory; 4) Students have a tool they can take with them to their other courses; 5) Technophobic students often find the calculators much less intimidating.

## Support

Over the past five years we have seen great changes in our students in terms of their familiarity and comfort with computers. Nevertheless, if the computational component of the course is to be successful, a lot of timely support along the way is essential. Here are some things we would strongly recommend you have in place.

### Handouts

Carefully written and well-indexed handouts dealing with topics like how to sign on to the network, how to access the software and programs, anticipating the most common problems and what to do about them, and so on are very important. Moreover, the material should be packaged so that students can absorb it in digestible portions—students should have the essentials in the first couple of pages, near-essentials in the next several pages, and so on, with the clever but optional topics put at the end, if at all.

### Prompt answers

It is important to have knowledgeable assistants available to lend a hand, particularly in the initial stages. Moreover, these assistants should be trained to answer only the question asked and resist demonstrating their own knowledge by being too free to show the inquirer the clever way to do things instead.

As was mentioned above, it is also very helpful to have an on-line printer available so that when students run into problems they can get a screen dump to bring in to you. These lead to much more fruitful discussions than what you get when the student is trying to tell you verbally what went on.

### Traps for the Unwary

Since not every calculus teacher is also an experienced computer teacher, we mention here a couple of features inherent in using computers which will almost certainly come up at some point during the course. You can wait until they intrude themselves before bringing them up with the students, but they should eventually become a part of their general education.

#### Roundoff error

Here is a simple program you should try writing on your machine:

```
delta = 1/100
S = 0
FOR k = 1 TO 10000
        S = S + delta
NEXT k
PRINT S
```

Apparently, this program simply adds .01 to itself 10000 times, so we would certainly expect to get 10.0000000000 as the printout. In some languages, though, you won't. The reason is that most computers keep track of numbers in binary form. Since .01 doesn't have an exact finite binary expression, the computer uses an approximation. The resulting error is small enough so that it doesn't matter most of the time, but in some of the applications in this course involving many iterations, the errors can accumulate in ways that become quite visible.

By contrast, you might try the program

```
delta = 1/128
S = 0
FOR k = 1 TO 12800
        S = S + delta
NEXT k
PRINT S
```

In this case, the answer is exact since the fraction involved will be carried exactly.

The moral is that you need a reasonable amount of precision in whatever language you use. We have discovered, for instance, that ordinary Basic rapidly produces results that are quite far off when you try for finer approximations, and that it is important to specify that the program run in extended precision mode. The built-in level of precision in TrueBasic, on the other hand, appears to be adequate for most purposes.

**Overflow**

Computers have a limit on the size of numbers they can deal with. If your program generates numbers that exceed these limits, you will get an error message. In this course, numerical solutions to differential equations can generate such messages in a couple of different ways. The most obvious way is when you try to solve a differential equation that grows very fast—say $P' = P^2$. Here it is not too surprising that if you start off, say, with $P(0) = 10$ and try to get $P(100)$ using small values of $\Delta t$ the values may exceed the machine's capacity.

Overflow errors can be generated more subtly, though, by failing to turn corners sharply enough using Euler's method. For instance, one place where this is almost certain to occur is in the May Model discussed in problem 6 of chapter 4.1. If the students try for an initial approximation using $\Delta t = 1$, they will generate an overflow message. What happens is that the piecewise-linear solution has crossed over an axis into either negative rabbits or negative foxes, with the result that the corresponding variables grow very rapidly (try it out!).

**Misleading results**

Computers do lie, in the sense that an uncritical acceptance of their output can lead to erroneous conclusions. Here are four examples to illustrate some of the kinds of things that can happen:

1. The first place where many students are likely to encounter this phenomenon is with the graphing software they use, where scaling factors may cause important features to be missed. For instance, a student who mindlessly graphs $y = x(x-1)(x-2)$ over the interval $[-10, 10]$ will often miss the humps in the graph. In fact, to naive students, all polynomials of degree $> 1$ tend to look either like $y = x^2$ or $y = x^3$.

2. During the course students are asked to calculate slopes of curves at various points by zooming in on the curves, finding the coordinates of a couple of nearby points, and getting the value of $\Delta y/\Delta x$. Typically the results will seem to be converging for a while, then will begin to wander off. This is because not enough significant figures have been

used for the values of $\Delta y$ and $\Delta x$, either due to the limitation in the number of digits their computers report, or on their own failure to use all the digits provided, thinking the ones "way at the end" are irrelevant.

3. Solutions to differential equations such as $y'' = -y$, using Euler's method will appear to fluctuate more and more as time goes on, even though the true solution is periodic. This is because in problems like this, Euler's method always overshoots the true solution in the same way, so that the accumulated errors will inevitably become noticeable if you continue long enough.

4. A classic problem is to have students calculate the value of the harmonic series. They will almost invariably come back with an answer, either because the results diverge so slowly that they decide after a while there will be no more change, or, if they are more patient, because the computer itself will begin treating $1/n$ as 0 for $n$ suitably large. The same problem crops up in a different form if one calculates the improper integral
$$\int_1^\infty \frac{1}{x}\,dx$$
by Riemann sums using midpoints—the computer will give a small finite answer no matter how small $\Delta x$ is.

While we don't want to give our students the message that computers can't be trusted at all, it is important that they not get in the habit of mindlessly writing down whatever the computer says—they should always be interpreting the results and trying to generate some intuition about what is going on.

## Time Demands

This course has the potential for taking much more time on the part of both the teacher and the student than a traditional calculus course. Here are some of the points at which this can happen, with suggestions for dealing with it.

### Time Demands on the Teacher

Under the best of circumstances, there are one-time startup costs in teaching this course. The teacher will need to spend more time thinking and learning about computer facilities, exploring the mathematical models he or she may not be familiar with, possibly doing some collateral reading to become more familiar with some of the contexts, and working through the problems to get a feel for how long they take and what some of the pitfalls are likely to be for the students. Ideally, you would get released time from your institution to make some of these preparations. It also helps considerably if there are at least two of you teaching the material, both so you have someone to talk with about the course and so you can divide some of the startup preparation (although, on the other hand, conferring regularly also is an extra time demand!).

Even after you have taught the course a couple of times, it can still be more time-demanding than a standard calculus course.

1. Maintaining the computer facilities takes time, and if your department doesn't have a staff person designated to do this, it might end up being you.

2. Getting students oriented to the computers and answering the questions that arise throughout the semester takes time. A good course assistant can be very helpful here.

3. To be able to provide interesting and current examples for your students, you may want to browse regularly through some of the scientific journals to see what can be adapted for your course. Your colleagues in the other disciplines can be a real help here—once they find out what you are trying to do, many of them can be very good at keeping their eyes open for articles you might find interesting. Your ex-students can also be a good source of examples, bringing you the uses of calculus they run into in their further work.

4. Certainly teachers thinking of running this as a laboratory course need to consider the increased time required to comment thoughtfully on the written work the students will be submitting. Some teachers have dealt with this by having students submit a single paragraph or two summarizing their laboratory work rather than submitting a full laboratory report or notebook.

5. Correcting the homework can take longer in this course than usual. More written responses are asked of the students, which typically take longer to read. Goals like training students to report only significant figures require written comments on their papers. Again, a well-trained student assistant can be a big help.

6. Training the assistants in the first place, though, requires some time—you can't simply give them the solution sheets and send them off. Moreover, it is good to meet with them regularly throughout the term to make sure they are clear on what the criteria are and to answer their questions. This is particularly true in the initial years when you will be using assistants who have not themselves been through the course.

### Time Demands on the Student

We have discovered that even if students are not actually spending more time on the homework than they would in a traditional course, they perceive it as taking more time. This seems to be due largely to the fact that they need to exercise more conscious thought at a number of points: they need to figure out how to use the computers or calculators and they need to work out the logistics of getting together with their partners. There is also more writing involved than many of the students are used to as they are asked to explain their reasoning and defend their answers. Most important, though, the problems require more thought—there are fewer of the template-type problems than they are used to where there are several worked examples in the text that only need to be modified to fit the assigned problem. In fact, some of the problems are designed to get students thinking about issues that won't be covered until the next class, and it takes explicit attention on the teacher's part to help the students appreciate the value of this kind of problem.

Some of the homework assignments ask the student to compute a certain quantity to a specified number of decimals. Such problems presume that the students have access to computers at least as fast as 386 machines. If your students are working on a system that is substantially slower than this, you may need to reduce the number of digits asked for. While it is important for the students to realize that each additional decimal of accuracy takes roughly ten times as long to obtain, the pedagogical point tends to be lost if the students have to wait two hours for the output! You should try some of these problems beforehand yourself to get a feel for how much time will be required using your system.

As with most mathematics classes, there is a great deal of variation in the amount of time students need to spend on the more drill-type exercises to become comfortable with the underlying idea. Some of the computer investigations in particular can begin to seem like busy work to your quicker students. You might want to give them the option of writing up solutions to the harder problems only, accompanied by some clear prose demonstrating that they really do understand what is going on. Some of us have been able to use this strategy quite effectively.

## Testing and Evaluation

This text represents a shift in what we expect of our students, and our mechanisms for evaluating student work necessarily reflect that shift. Most of us assess our students' progress in four ways: Through homework, lab reports, in-class exams, and take-home exams. Some of us also use weekly quizzes. In addition, all of us learn a great deal about our students from class discussions and from shameless eavesdropping as students work in groups.

Before saying more about the ways we handle these various mechanisms, we should outline some of our general views on evaluation. Since we value thinking over rote learning, all of us put the primary emphasis on process and explanation: we don't just want an answer, we want a clear indication of the method of solution. We also value clear, well-organized writing, whether we are reading a few sentences, a paragraph or an essay. This means that quizzes and fixed-time exams should probably not be the only information used in evaluating a student's work, since they are best suited for testing more routine matters, and computer use is less feasible.

We have already discussed the importance we place on the homework and some of the ways we help students to take it seriously for its own sake.

Some of us have students do projects in the second semester in which they are given a journal article using the ideas under discussion in class. They are expected to write up an analysis of the article and its techniques, and see if they can confirm (or better yet, expand) the mathematical results in the paper. Such exercises can be very exciting for the students, but they need to be carefully structured.

If you teach in a setting where a take-home exam is reasonable, this is an excellent vehicle for eliciting more thoughtful responses and for letting students demonstrate their ability to use technology appropriately. All of us set a higher standard in grading a take-home examination than one taken in class. A typical pattern is to give two mid-term exams and a final, each with an in-class and a take-home part. Weightings of the two parts of each exam vary from 50%-50% to 33%-66% in some order. Some sample in-class and take-home examinations are provided in Appendix B, to give you more of an idea of what we have tried. All of us expect students to work on examinations individually. Some of us have students "check out" a take-home examination for 24–48 hours to work on it, while others give everyone a week.

Asking students to master larger, more complicated ideas makes them very uneasy. They miss the familiar sign-posts of accomplishment from high school: the algorithm mastered, the technique learned, the end-of-chapter test. They easily become discouraged, even when they are making good progress, partly because they don't know how to recognize or value their own accomplishments. A weekly 15 minute quiz can be very reassuring to students. It is important to avoid having quizzes distort the course by putting too much emphasis on little discrete chunks, but several of us have found them very useful, both for helping students see their progress and for helping us see what the stumbling blocks are. Some sample quizzes are also provided in Appendix B.

Naturally, we, our institutions, and the National Science Foundation also want to evaluate how the outcomes of calculus courses like this one compare to those of the traditional course. This is a much harder task. We believe that asking the same questions of students in this course and in the traditional one is likely to be unfair to both groups, since the goals of the two courses differ so substantially. We have, however, collected information on student attitudes at the beginning and end of our courses, and we would be happy to respond to inquiries about our methods and/or our findings. In addition, we want to know what alumni of our courses bring to subsequent courses in mathematics and in the mathematics-using disciplines, so we discuss these questions with our colleagues. We pay attention both to the number of students who go on to take more mathematics and also to *who* the students who go on are. We also compare the sophistication of the kinds of questions we ask of our students now with those we asked in the past. Sharing both old exams and current ones with colleagues, in or out of the department, contributes to our conversations with them.

We would welcome hearing from you about your own efforts to evaluate the effects of this course on your students.

# Chapter-by-Chapter Commentary

## Chapter 1. A Context for Calculus

This chapter and the next introduce most of the major themes of the course: Modelling, differential equations (called rate equations initially), numerical calculations, and successive approximations and limit. The temptation is to see all this as merely the introduction, to be skimmed through quickly to get to the "real" start of the material in chapter 3. In fact, we typically spend two to three weeks on each of these opening chapters. The ideas are new to most of our students, and we have found that it pays to adopt a somewhat leisurely pace at the beginning to give the students time to immerse themselves in what are some very different ways of doing mathematics. Moreover, many students will be using computers or graphing calculators extensively for the first time, and this also calls for a certain deliberateness to ensure that they are sufficiently comfortable with the mechanics of interacting with a computer to be able to explore effectively the concepts being developed .

### 1.1 The Spread of Disease

It is startling to many mathematics teachers (but not to the students!) that we begin the course with a system of non-linear differential equations. For some, the response is to want to back off a bit and start out with something simpler and more tractable—say a single linear equation. Here's why we chose to begin this way:

1. We feel that a facility in understanding and working with such differential models is, in fact, one of the most important skills our students need, and we want to emphasize its centrality from the beginning.

2. Models with interacting variables are, in reality, typical of many of the problems students will encounter in their further work. This is perhaps especially the case for those students in disciplines—like the life sciences and economics—who are not particularly well-served by the standard calculus course. It has been our experience that such models are conceptually no more difficult for our students to work with than are single-variable problems. Differential equations courses traditionally begin with the study of the single-variable case for reasons of mathematical tractability, a criterion which is not of immediate concern to us as our approach is via numerical methods which apply equally well to single-variable and multi-variable systems.

3. We chose this particular model because it is accessible—the underlying problem is both clear and of general interest, and no time is needed to explain difficult technical concepts.

4. We wanted a model that was rich in terms of leading early to non-trivial implications (such as that of threshold in our example) and in terms of being readily modifiable to investigate related problems.

5. Since some of our students have had some calculus before, we wanted to begin with a problem of a kind that would be new to them as well, giving the sense at the outset that this was not going to be merely a review of what they had already done.

You will notice that the book treats both a variable $S$ and its rate of change $S'$ as intuitively clear concepts, without trying to define one formally in terms of the other. That comes later, and seeing how this is done is one of the main points of these first two chapters. By and large, the students seem to have little trouble with this and accept quite readily the introduction of a variable designating the rate at which something is changing.

Before leaping into the mathematical analysis you might find it helpful to solicit from the students their experiences of the course of epidemics. They can probably come up with general sketches of what the graphs of the numbers of susceptibles and infecteds will look like over the course of the infection, and will probably get into arguments over whether either graph will go all the way to 0 by the end. Before writing down and analyzing any model, it is good to have established some expectations like this so we can tell if the proposed model is behaving as we would want it to.

Invariably students challenge the model as being simplistic, as they should. It is important to acknowledge the truth of the charge—all mathematical models of physical systems are simplifications—while simultaneously discussing the value of the modelling process nevertheless. While it would be a mistake to expound at great length on the modelling process at this stage, some points you might want to make are:

1. Even simple models can lead to surprisingly useful insights about the dynamics of a system.

2. It is usually best to start with a simple model and then make it more sophisticated later if the core seems to capture the essence of the system. You might want (briefly!) to solicit suggestions for features the $S$-$I$-$R$ model lacks and how they might be expressed mathematically.

3. There is value in qualitative predictions as well as quantitative ones. While models in physics can provide very accurate numerical predictions, models in, say, ecology are typically more often used to capture the general dynamics of a system and predict the kinds of phenomena one might expect from such a system.

Ultimately, the proof of any model is the quality of the insights it provides. An elaborate, sophisticated model which doesn't tell us anything we didn't already know is of less value than an obviously simplistic model which leads us to think about the system being modelled in new ways.

Pages 9 and 10 make a point that is important for students to understand: Once we have a model that seems like it might capture some of the reality we are looking at, the subsequent manipulations of the model belong strictly to the world of mathematics, and it is mathematical criteria that determine the validity of what we do. It is only after we have obtained the results of these manipulations that we look again to the original problem, to see if the predictions of the model seem to be consistent with the system being modelled.

The problems at the end of this section take a lot of time, and you may want to assign some subset of them. Since problem 18 (*There and Back Again*) is referred to in the next chapter, you should be sure to deal with it in some fashion before then.

## 1.2 The Mathematical Ideas

How much time you spend on this section will depend in part on your sense of your students' need for review. You may want to supplement the exercises with drill sheets on some of the algebra involved—some samples you could duplicate are contained in Appendix C of this *Handbook*.

Your students should have access either to a reasonably sophisticated graphics package on a computer or to graphing calculators, and this section is a good place to introduce them to the workings of whichever device you choose to use.

This is a good point to begin weaning your students from an excessive dependence on the slope-intercept form of thinking about lines (which is not particularly insightful for many calculus applications), getting them to think more in the $\Delta y = m \cdot \Delta x$ form. It is essential for much of what follows that they come to think of linear relationships as being characterized by the fact that there is some fixed multiplier $m$ such that any change in the independent variable produce a change $m$ times as big in the dependent variable. This is a surprisingly difficult shift for some of them to make.

## 1.3 Using a Computer (or Graphing Calculator)

Care spent in making sure your students feel comfortable using computers and reading simple programs at the beginning can avoid many difficulties later on. Handouts explaining carefully how to use your system are important. Prompt feedback and readily accessible help are crucial. Be careful not to overload your students at this point—give them only the information they need to know to do the current problems. While there will be some who will be eager to do things in a sophisticated way, for most of your students the finer points of editing, saving, and elegant shortcuts can come later. It is important to remember that this is not a course in computer programming—crude but effective methods are fine.

## Some Historical Notes on the $S$-$I$-$R$ Model

The mathematical modelling of diseases began in the early part of this century. W. H. Hamer in 1906 published an article on "Epidemic disease in England" in the medical journal *The Lancet* (**i**, 733-9). His model was a discrete-time model, and was the first to postulate the so-called 'mass action principle' (the analogue of a fundamental principle in biochemistry) in which the rate of new infections is assumed to be proportional to the product of the number of susceptibles times the number of infecteds. In 1908 Ronald Ross (who also discovered that malaria is transmitted by mosquitoes) published a continuous-time version of the model in his *Report on the prevention of malaria in Mauritius.*

Mathematical epidemiology really got its start, though, in 1927 when W. O. Kermack and A. G. McKendrick published "A contribution to the mathematical theory of epidemics" in the *Proceedings of the Royal Society* (**A115**, 700-721). This was the first articulation of the $S$-$I$-$R$ model as we are seeing it. It was also the first to develop the concept of the threshold theory.

A number of efforts have been made to fit the $S$-$I$-$R$ model to actual epidemics. In their original 1927 paper, Kermack and McKendrick analyzed the Bombay plague epidemic of 1905-6. This was a severe disease in which almost everyone who became infected died. Using for $R'$ the number who died each week (so $R$ definitely stands for removed, rather than for recovered in this case), they found a very good fit.

As a second example, in his book *Mathematical Biology* , J. D. Murray analyzes the data on a flu epidemic in a boy's boarding school. Since the disease was severe, all infected boys were hospitalized, which made possible a precise determination of $I(t)$ each day. Out of 763 boys, 512 boys became sick. He found that he got excellent agreement of the data and the model if he assumed a threshold of 202 and a transmission coefficient of .00218. Murray's book contains the graphs of both this example and the Bombay plague example.

## Further Reading

1. Kermack, W.O. and A.G. McKendrick. 1927. "A contribution to the mathematical theory of epidemics", *Proceedings of the Royal Society* **A115**, pp. 700-721.

2. Kingsland, Sharon. 1985. *Modeling Nature*. University of Chicago Press. This is an excellent book for getting a sense of the history of modeling in biological systems.

3. Murray, J.D. 1989. *Mathematical Biology.* Springer-Verlag. An excellent resource with a good bibliography and lots of projects you could get your students working on by the time they are in the second or third semester of calculus.

## Chapter 2. Successive Approximations

The basic idea in this chapter—that when we can't solve a problem, being able to approximate the answer to an arbitrary degree of accuracy may be just as good—requires a radical shift for many students in the way they think about what it means to solve a mathematics problem. Until now, almost every mathematics problem they have encountered has had a single, clear answer. It is worth spending some time on this chapter to help them appreciate this shift in outlook.

### 2.1 Making Approximations

In chapter 1, we only considered values for $\Delta t$ that were an integral number of days. While this allowed us to predict future and past values, it also led to the disquieting phenomenon that when we used the model to go forward one day to get new values for $S$, $I$, and $R$ and then applied the same model to these new values to go back one day, we didn't end up at our starting point. This difficulty is used as the motivation to use values for $\Delta t$ of less than one day.

At this point many mathematicians get quite concerned about the legitimacy of this fairly casual transition from the discrete to the continuous. After all, wasn't the original model developed on an assumption of gathering data on a daily basis? Besides, what are we to make of all those fractional people (this issue even occurs in the previous chapter)? Both concerns can be addressed in part by reminding the students that we are now dealing with a mathematical object—the model—and that while all manipulations have to be mathematically defensible, it is only at the end, when we want to check the appropriateness of the model, that we check the results with the original system. A second observation is that it is often helpful to think of the numbers generated by the model as being average values, which can quite legitimately be non-integers, resulting from a number of trials of the original system. As for the first concern—about the tension between the discrete and the continuous—this does not seem to be an issue which troubles the students. Here as elsewhere, it is probably not helpful for the teacher to raise objections before the students have run into situations which make the objections real for them.

The term 'limit' is first introduced in this section. It is not defined, except by example, and its use is meant to be a convenient shorthand for a phenomenon which is already clear to the student. In this chapter students see numbers emerging as the limit of a sequence of other numbers, they see curves emerging as the limit of a sequence of other curves, and they see functions emerging as the limit of a sequence of other functions. In every case, though, the limit only emerges through the approximations—at no point is there an independent expression for it. The standard $\epsilon$–$\delta$ definition of limit thus is not appropriate, since we don't know the "answer" $L$ to see how close we are coming to it. Our definition (which is made formally only in chapter 10) is essentially that of Cauchy. The existence of a limit is inferred from the fact that as more and more detailed approximations are made, more and more digits of these approximations become fixed. The limit can thus, in principle, be expressed to any finite degree of accuracy, but in general it can never be known in its entirety. This is in fact a much more realistic view of limits in terms of the way they actually occur in many applications.

Teachers are strongly urged to be fairly casual in their use of the term at this point. Most students have a vague intuition of what limit means, and it is one of the chief goals of this course to increase their experiences with this concept to the point where the 'real' definition almost feels like a statement of the obvious. While we can all think of cases where a string of terms in a sequence of successive approximations appears to be fixed to a certain number of decimals, when in fact the approximations are still quite bad, this is not a concern that immediately arises for the students. If we can wait for our students (rather than the teacher) to ask the question "How do we really know that those 6 digits will remain fixed forever?", the question will probably receive a much more receptive hearing. We would recommend teacherly restraint on this crucial question until it arises from the class, perhaps not for a month or so.

At this point many students respond to a problem by running one approximation with what they perceive to be a small value of $\Delta t$ and assuming that the resulting answer will have to be close to the true answer. It is important to stress that **a single approximation gives no information about the answer**. It is only when one has a sequence of approximations, by seeing how much agreement there is between them, that one can begin to develop a sense of where the answer really lies. A related point is that students should be strongly discouraged from writing down meaningless digits in reporting their answers. Just because the computer or calculator gives them 8 decimals does not necessarily mean that all of these digits are significant. We have found that for many students developing a reasonable sense of the concept of significant figures requires a great deal of experience, but that most of them do develop a good feel for it after a month or so if the teacher is insistent on the subject from the beginning.

Problem 2, which focuses the student's attention on the near-linear relation between $\Delta t$ and the change in the approximation is useful to spend some time on. The fact that the changes in the approximation respond so predictably to changes in $\Delta t$ is very suggestive that a limit is indeed being approached. We have found it helpful to get the students to guess the value of the approximation for a new value of $\Delta t$ before running the program as a way of recognizing the pattern.

An important byproduct of this approach is that students develop a very real appreciation of the tradeoff between the degree of accuracy to which the limit can be known and the cost, in terms of time and equipment, to obtain such accuracy.

While most students find the notion of piecewise linearity to be fairly straightforward conceptually, many find dealing with it algebraically to be quite hard. Teachers should therefore assign problems 4–8 only if they are willing to spend a fair amount of time preparing their students to make the algebraic translations. Some teachers may see this as an excellent opportunity to work on students' algebraic skills, while others may feel it is too much of a diversion.

## 2.2 Euler's Method

Here is the fundamental tool for much of the rest of the course. By the time the students reach this point, Euler's method should largely feel like a summing-up of ideas they have been working with for some time. The key point to stress here is that simply finding one approximate solution, even with a very small value of $\Delta t$, gives little information. It is important to have a sequence of approximations that can be compared with each other before we can get some sense of how good they are.

## 2.3 Approximate Solutions

This section continues the discussion of successive approximations in a couple of new settings: finding arc length and finding square roots. The real point being made here, besides additional exposure to a new idea, is that the only way we know most numbers—even familiar numbers with names, like $\pi$ or $\sqrt{2}$—is by some process which allows us to determine as many decimals as we need (at a cost!). Even these familiar quantities aren't known in any different way from the way we now know the value of $S(3)$ in the $S$-$I$-$R$

model. This is also probably the first time most of the students will have seen how lots of digits of a number like $\pi$ can be determined (a problem which is revisited in the second semester, with more efficient techniques).

This section also gets the concept of an **algorithm** out on the table. Since this is a central theme of the course—that numbers and functions are computable from first principles using some sort of effective rule for determining the quantity to any predetermined level of accuracy—it is helpful to have this term available for future reference.

## Chapter 3. The Derivative

One of our central premises is that the idea of the derivative is separable from and more fundamental than the process of differentiation. Far too many students carry away from a first semester calculus course nothing more than the magical formula they are likely to recite as "$x^n = nx^{n-1}$." They have no geometric understanding and no appreciation for how the derivative relates changes in independent and dependent variables. In fact, though, in scientific contexts it is much more common to be given information about the rate at which a function is changing and from there try to determine the function—as in Newton's laws of motion or Maxwell's equations—than it is to be given a function and from there find its derivative. For that reason, we spend considerable time working with derivatives before we introduce the differentiation formulas.

The key element of our treatment is that students have use of a graphing utility which allows them to "zoom in" on a curve and see that it eventually seems indistinguishable from a straight line. We then define the derivative of a function at a point to be the slope *of the curve* at that point—i.e., the slope of that straight line the curve is clearly becoming. There are three things to emphasize here: the definition is fundamentally geometric, it requires no mention of a tangent line, and it only makes sense for a locally linear function.

Thanks to the computer graphics, this idea of the slope of a curve is completely natural to students. The existence of the limiting straight line (and its slope) as the student zooms in closer and closer to the point in question is clear, and the traditional definition of the derivative as $\lim_{\Delta x \to 0} \Delta y / \Delta x$ is merely the analytical expression of what is geometrically obvious. Thus this treatment is explicitly linked to the geometry (and to visualization), and it doesn't have to fight with fears about dividing by zero or contend with confusing arguments in which sometimes it's okay to set $\Delta x = 0$ and sometimes it's essential that $\Delta x$ is *not* zero.

There is nothing to be gained, we feel, to introduce another geometric object, the tangent line, at this time. If the curve itself is linear (if we only look closely enough!), why introduce another element? Tangent lines appear for the first time in chapter 5.5, in the discussion of Newton's method for approximating a root of an equation. There, in a neat reversal of the usual roles, the tangent line at a point is defined as the *extension* of the locally linear approximation at the point.

The restriction to locally linear functions is, in fact, a restriction. We know that there are functions which are differentiable at a point but not locally linear there. Students, however, do not know enough yet to generate examples like these. In fact, it was many years after Newton and Leibnitz before mathematicians discovered the pathologies that occur when functions have derivatives which are not themselves continuous functions. We thus follow our practice of not raising issues before students encounter the situations that prompt them.

We emphasize throughout that a derivative has three meanings: it is a rate, it is a slope, and it is a *multiplier*. By this last term we mean to evoke the fundamental relationship we call the *microscope equation*: $\Delta y \approx y' \cdot \Delta x$. The microscope equation, of course, underlies Euler's method, and the students' familiarity with Euler's method in turn prepares them for the microscope equation.

### 3.1 Rates of Change

In chapters 1 and 2 students should see that rates are important quantities because they provide a natural language for describing a changing world and because they permit us to make predictions. In this section, however, rates are introduced afresh and in a different setting: based on a table of data. We start with a naive rate, the change in the time of sunset in minutes per day, given by a difference quotient. The use of a data table rather than a formula in this first example both makes the point that functions can be specified by data and illustrates how rates give natural descriptions of the data. The fact that the rate itself varies with time is familiar from traditional treatments. Note, however, that we don't yet call attention to the fact that these rates are average rates.

This is a short, readable section. You can ask students to read it and do exercises 1-7 (on a falling object) and bring their solutions to class for discussion. If you have longer than a 50 minute period, you might start section 2 the same day.

### 3.2 Microscopes and Local Linearity

To study the slope of a curve, students need a tool, the graphing utility with a zoom capability. We speak of it as a *microscope* for studying the graph of a function. With the aid of the microscope, students see easily that most graphs are locally linear. They also understand quite readily that the rate of change of the function is the slope of the straight line they see under the microscope.

After these preliminaries, the successive approximations to the slope (or to the rate) appear, and the student is now on fairly familiar territory, thanks to chapter 2. Finally (page 99), the slope (and the rate) at a point is defined formally as the limit of these successive approximations as the magnification increases, i.e., as $\Delta x$ approaches zero.

Some students find the pictures of successive magnifications in this section of the text confusing, but the confusion clears up right away when they experiment with the "microscope." Beware of students trying to do the exercises algebraically only. They really need to *see* the graphs.

There are, however, three cautions for the teacher in this section.

**Roundoff error.** Since they will be dividing by small numbers, roundoff error becomes particularly problematic in this section. This often is manifested in the following form: the student is calculating a sequence of values for $\Delta y/\Delta x$ for smaller and smaller $\Delta x$. The quotients appear to be converging nicely, with more and more decimals becoming fixed. Suddenly, though, a point is reached where the results begin to diverge, bouncing about with no apparent pattern! This typically occurs because the student has begun to use values of $x$ so close together that the limitations of the computer or calculator display means that the digits reported for the $y$-values are so close that the corresponding $\Delta y$ is missing some crucial digits, and the computed ratio $\Delta y/\Delta x$ is less accurate than the ratio computed with a larger $\Delta x$. (This problem is much less of a danger in the approximations of chapter 2, which is one of the reasons why there is a real advantage in beginning with *other* successive approximations like these before considering the derivative.)

**Significant figures.** Students too often either unthinkingly write down all the digits the computer gives them or discard them capriciously. Since dividing a small $\Delta y$ by a small $\Delta x$ means that *most* of the information in the coordinates of points on a curve lies in the later decimal places, this is a good time to insist that students use significant figures appropriately.

**Tedium.** Doing lots of arithmetic is boring. At what point should the numerical computation of slopes be automated? You will have to decide, and the answer will be different for different students. You must balance the tension between the tedium of the repetitive calculations and the mindlessness that can take over when the student just presses a button to get the result.

Select from among parts of problems 1-5, but it is best to include at least one part from each of the first 3. These exercises reinforce the limit ideas of chapter 2 as well as the new ideas in this section. The graphical problems 6-10 are very important, but it is not necessary to do all of them. Problems 11-15 concerning the failure of local linearity are less essential.

### 3.3 The Derivative

This section has two important purposes. First, it codifies the ideas of the first two sections and introduces the standard language and notation for the derivative. Thus we formally define the derivative of the function $f(x)$ at $x = a$ (provided the function is locally linear there) to be the *rate of change* of the function at that point, which is the same as the *slope of the graph* at $(a, f(a))$. Our emphasis on arriving at the slope from the graph makes most natural the following analytic form of the definition:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}$$

although the standard form of the difference quotient appears on the next page (see pp. 107, 108).

As students can see for themselves in exercise 3, the "two-sided" form given first is actually more computationally efficient. Of course the two forms agree, provided the function is locally linear at $x = a$. Because the absolute value function is readily available, the better students will notice that the "two-sided" difference quotient can have a limit when the "one-sided" one does not; we advise not raising the issue until then (although you can stack the deck a little by being sure your students look at the exercises in 3.2 where local linearity fails).

The second important purpose is to give formal expression to the importance of local linearity for estimation (and Euler's method) via the "microscope equation." Because the equation $\Delta y \approx f'(a)\Delta x$ is so important, we found we had to give it a name so we could refer to it. Although the language is non-standard, students find it helpful and easy to use. They find the local coordinates difficult at first, but gradually this gets better.

Note that if the graph of a function is vertical at a point, we say the derivative is infinite there, so locally linear and differentiable are exactly synonymous.

You can profitably spend two days on this section, one on computing derivatives and one on the microscope equation. After substantial work with a graphing utility in section 2, the numerical exercises 2-5 are meaningful to students. Note that problem 2 is needed for problem 10. Problems 6-8 on the exponential functions are referred to later, in 4.3. Students find problem 11 very illuminating.

### 3.4 Estimation and Error Analysis

This section reinterprets the microscope equation in terms of error (and relative error). It is our experience that even very bright students do not carry away from a standard calculus course the important idea that the derivative gives them a way to gauge the effect of a change in $x$ on the dependent variable $y$. When given an explicit functional relationship between $x$ and $y$ and a question about how an error in measuring $x$ affects the accuracy of the determination of $y$, they had no idea that the derivative of the function could help them answer the question.

This section also shows students that it would be useful to have a convenient way to obtain the value of the derivative of some simple functions, thus paving the way for the formulas in section 5. Some of us revisit the exercises in section 4 after section 5. A selection of exercises is sufficient; many students find exercises 8 and 9 particularly engaging.

### 3.5 A Global View

The key idea here is that the derivative of a function is a *function*, and there are two approaches to finding the derivative function. There are the familiar formulas, but also considerable emphasis is placed on sketching a qualitative graph of the derivative of a function based only on a graph of the function. You can profitably spend two full days on this section, and we recommend that you spend the first of them concentrating on the qualitative, graphical approach. This graphical exercise is very hard for many students, but when they master it, their understanding of the derivative is substantially strengthened. It is important, though, to be sure that students really understand what the slope of a straight line is and can visually estimate slopes of lines. The exercises on lines in 1.2 and the appendix are valuable, and some students might profitably revisit them now.

Students who have had some calculus will try to guess formulas for the functions given graphically and then write down formulas for their derivatives. It is important to discourage this, since we are trying to develop students' geometric understanding of what the derivative means.

Some instructors also introduce the product and quotient rules (in chapter 5) at this point. Others have derived the formulas for a few derivatives (e.g., of $x^2$ and $x^3$) here (again, in chapter 5). There is considerable flexibility on this, as long as the ideas of the derivative as rate, slope and multiplier are well-established, so they don't get lost in the differentiation rules.

Student weaknesses in algebra will often show up here, especially in manipulating exponential notation. Because this text shifts the emphasis away from algebraic computations, this may be the first time you really confront these difficulties. Depending on your students' strength in algebra, you may want to devote a full day just to the mechanics of differentiation; additional drill is in Appendix C.

### 3.6 The Chain Rule

Although we defer the product and quotient rules to chapter 5, we treat the chain rule here because it grows so naturally out of the microscope equation and the idea of the derivative as multiplier. In fact, the argument for the chain rule that is given here can be made precise (but the precision is not appropriate in this course, in our view).

The expanding house problem provides the context for the question about combining rates of change, and the microscope equation provides the answer. The use of units reinforces the naturalness of the multiplication of the rates.

The chain rule is given first in Leibnitz notation, since most of us have found that easier for most students; the version in functional notation is given as well, and some students find that easier to use. We usually spend a single day on the chain rule, with additional problems reappearing in later homework assignments.

### 3.7 Partial Derivatives

This is a modest introduction to partial derivatives, emphasizing rate of change and the multivariable microscope equation. In our original vision of this course, we wanted to emphasize that multiple variables arise naturally and that the ideas of the calculus extend readily to the multivariable case. However, we found that we got so bogged down in 3-dimensional geometry that the shape of the first semester was too distorted. So we have deferred the geometry until chapter 9. We do make free use of multivariables, wherever they are natural, and we find students can master the idea of holding all independent variables but one constant and observing the effect of changes in the remaining one. The multivariable microscope equation also seems natural to them, although they have to think about it carefully. One day is enough for this section, and it can be omitted without harm. Partial derivatives appear again in 5.2, which is also skippable.

## Chapter 4. Differential Equations

Here we return to the central theme of the course, differential equations. But this is not simply a reprise. In fact, there are two fundamental (but somewhat subtle) new ideas at work here. The first is the idea that differential equations *define functions*, and the second is the notion of a *solution* to a differential equation.

### 4.1 Modelling with Differential Equations

In this section the students get a wealth of experience using differential equations to define functions, in constructing solutions to differential equations, and in studying the properties of those solutions. On the surface, the emphasis is on modelling issues using a variety of population models of increasing sophistication and complexity. In fact, there is an interesting "meta" version of successive approximation here in the succession of models, each capturing more of the complexity of the reality being described.

A second goal of this section, though, is to give the students sufficient experience using Euler's method to produce functions to drive home the point that differential equations really *define* functions. In many branches of science, functional relationships are naturally expressed through differential equations. While it is convenient when there are ways to obtain closed form solutions to such equations, this is often not possible, and it is important that our students realize that in such cases we can still go ahead and explore the nature of the functions involved. This theme is treated explicitly in section 3.

Exercises 4 and 5 are referred to in section 2. Exercises 12-14 on Newton's Law of Cooling are referred to in section 2 in exercises 10-13. This exercise set is a rich source of laboratory explorations: there is real meat to investigate here. Students particularly enjoy problems 8-11 on fermentation, for the questions are very real to them. A special caution is in order on problem 6, the May model. It is easy to create overflow problems by using too "coarse" a $\Delta t$.

## 4.2 Solutions of Differential Equations

Until this chapter we have used a differential equation as a *description* of a physical problem, and, via Euler's method, as a *procedure* for constructing a solution to the problem. However, a differential equation can also be viewed as posing a problem: find a function (or functions) which, when substituted into the equation, makes the equation true. That a function produced by Euler's method does satisfy the differential equation seems almost obvious by the very construction (although the actual proof of the fact is slightly tricky). The new question raised in this section is how to determine whether a function *not* given by Euler's method—for example given by a formula—is a solution, and to anticipate the question of how one goes about finding such solutions.

Having a formula for the solution has a number of benefits. The obvious one is that it is often much faster to obtain, and is usually much easier to write down and visualize than a numerical solution. Another important feature of closed-form solutions is that they make it much easier to explore the impact of various *parameters*—initial conditions and coefficients—used in the differential equation. We try to exploit these benefits.

Since few anti-differentiation methods are yet available, we do very little here about *finding* solutions to differential equations. In chapter 11, however, we introduce the technique of separation of variables and use it to find closed form solutions to a number of differential equations, including several arising in this chapter.

An important theoretical issue is articulated on page 181 – what we call the *existence and uniqueness principle* for the solution of an initial value problem. We don't call it a theorem for two reasons: the statement lacks the precision a theorem requires, and we give no proof. The lack of precision ("What do you mean by 'Under most conditions ...'?") will bother some teachers and a few students. In earlier versions of the text we actually tried to discuss the hypotheses that are necessary to guarantee existence and uniqueness, but we have become convinced that this goes too far for beginning students. At the same time, this principle is an expression of an intuition that the students have been developing experientially from using Euler's method to solve a number of initial value problems, and it is important to reassure them that this approach is generally effective.

If you have students who really want to know why the method doesn't always work, there is a handy counter-example embedded in exercises 14 – 16 *A Leaking Tank* at the end of this section (pp. 195 and 196). If you use the rate equation of problem 14

$$V'(t) = -k\sqrt{V(t)},$$

but change the initial condition to

$$V(C) = 0,$$

where $C$ has the value determined in problem 16(b), then the initial value problem has two solutions – the one they've been looking at all along, and the new solution $V(t) \equiv 0$ for all $t$.

Problems 6-9 foreshadow the development of the exponential function in section 3. The problem sequence on falling bodies is revisited in section 3. The example of the oscillating spring occurs again, in much greater detail, in chapter 7, although that treatment is independent of these problems.

## 4.3 The Exponential Function

In this section we make explicit the important point that differential equations can be used to define functions, using the exponential function as our chief example.

The exponential functions are solutions of the differential equation $y' = ky$. We give two independent developments of these functions.

In the first, we start with the functions $y = b^x$, which students have been studying since chapter 1 (see 1.2, problem 14). They already know that $y = b^x$ satisfies the differential equation $y' = k_b y$, where $k_b = y'(0)$ (see 3.3, problems 7 and 8). With this solution in hand, we can find out many interesting things. For example, $e$ is the special base for which the constant $k_b = 1$, and (on page 203) we obtain $e = \lim_{n\to\infty}(1+1/n)^n$ by using Euler's method to solve the the initial value problem $y' = y$ and $y(0) = 1$. Notice that (modulo the existence theorem for differential equations) we *know* this limit exists.

Some teachers may want to stop here: we have the exponential functions and their properties, we have the special base $e$, and we have the solution $y = C\,e^{kx}$ to the initial value problem $y' = ky$ and $y(0) = C$. Others will want to continue for two reasons. One reason is that the second development directly uses the initial value problem $y' = y$ and $y(0) = 1$ to *define* the exponential function, driving home the idea that differential equations define functions. This is a way of thinking that is particularly important for students continuing in physics and engineering to develop. There is another important reason. A small minority of our students particularly enjoy the sophisticated reasoning and the interplay of abstract ideas that so many of us love in mathematics. Such students delight in the second treatment.

A natural step in this chain of inferences requires that the function giving the solution be *continuous*. Continuity occurs again in 5.3 in the discussion of the existence of extremes. Even more importantly, the crux of the argument (see p. 207) is that the solution to the initial value problem $y' = y$ and $y(0) = C$ exists and is *unique*. The notational complexity alone is more than most students can handle at this level. Following this argument is challenging for many students.

You should note that we give a third treatment in 10.3, finding a series solution to $y' = y$ and recognizing the Taylor series for the exponential function.

Exercises 8–10 revisit models first seen in 1.2. Problem 13 leads students to a formula for the solution to the differential equation modelling Newton's law of cooling; problem 15 does the same for the model for a falling body with air resistance. Additional drill problems on exponentials are in the appendix.

## 4.4 The Logarithm Function

The problem of finding the doubling time of a population leads naturally to the definition of the natural logarithm as the inverse of the (base $e$) exponential function. In the era of numerical calculators, many of our students are quite used to the idea of an inverse function, because they have learned to use the key marked INV on their calculators. A geometric argument is used to find the derivative of the logarithm, and the argument is repeated in a discussion of the general relationship between the derivatives of a function and of its inverse.

Exercises 1 and 4 offer practice in manipulating and differentiating logarithms and exponents; more drill is in the appendix. Problem 24 compares growth rates of exponential functions with different bases, revisiting ideas from section 3.

## 4.5 The Equation $y' = f(t)$

Section 5 defines the antiderivative as a function and gives a more formal introduction to antidifferentiation. But the crucial element here is Euler's method for solving differential equations of the form $y' = f(t)$, leading to programs for finding the *accumulated change* in a solution $y$. Here we have the essential foundation for our treatment of the fundamental theorem in 6.4.

We have two methods for solving the differential equation $y' = f(t)$. First we can solve it by finding an antiderivative for $f(t)$. Second we can use Euler's method to find a solution. We call the numerical program adapting Euler's method to this situation TABLE (see p. 236). The identical program reappears in 6.4 as the program for computing a left-endpoint Riemann sum; the programs TABLE and RIEMANN appear side by side on p. 357. It is enormously satisfying to see the delighted "aha!" of our students as we prove the fundamental theorem of calculus by observing that the limit of left-endpoint Riemann sums is the same as the solution of an initial value problem by Euler's method.

## Further Reading

1. There are a number of interesting books available on mathematical population biology which can be referred to at this point. Murray's book mentioned in the discussion of chapter 1 is a good source, as are some of the other books mentioned in the Further Reading section of chapter 8.

2. Clark, Colin W. 1990. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, 2nd edition. John Wiley & Sons, Inc. This is an interesting book with a different set of applications, particularly in the first few chapters, which can be explored by students at this level.

## Chapter 5. Techniques of Differentiation

In this chapter are all of the standard differentiation rules and (often in the exercises) their proofs. You can incorporate some of this material into chapter 3 at your discretion (although exercises often include the exponential and logarithm functions, which are not introduced until chapter 4). Most of us find, however, that it works well to treat the material on graphing, optimization and Newton's method *after* chapter 4, to emphasize the way chapter 4 draws together the ideas from the first three chapters.

### 5.1 The Differentiation Rules

The only formulas in the section which are new are the product and quotient rules; all except the derivatives of the exponential and logarithm functions appear in 3.5. Many of us consider calculations like those on page 243 for the function $f(x) = x^3$ when we treat the formulas in 3.5. Otherwise, we think proofs of the formulas are best deferred to this point of the course, since too much algebra can drive out the geometric intuition we are trying to build at the start.

Note that heuristic arguments, as well as formal proofs, are provided in a number of cases. Especially at this stage of our students' development, we think that the most important function of a proof is to strengthen understanding, rather than merely to validate a statement, so we concentrate on arguments, formal and informal, that reinforce the ideas and give students a way to think intuitively about them. Also important is the text example showing that the rules for the derivatives of sums, products and quotients (and the chain rule) can be useful even when formulas are not available. Additional drill problems are provided in the appendix.

Problems 13–21 lead students through proofs of a number of the differentiation formulas; most of us assign only a few of these. Problems 22–25 introduce the *second* derivative and *second order* differential equations.

### 5.2 Finding Partial Derivatives

The only thing new in this brief section is the use of the product and quotient rules in finding formulas for partial derivatives. The extended example on eradication of disease reinforces the meaning of the partial derivative.

## 5.3 The Shape of the Graph of a Function

This section prepares the student for the optimization in the next section by codifying the observations already made about the relationship between the derivative of a function and the shape of the graph of the function. Note that the definition of critical point includes a place where *the derivative is infinite* (recall that for us a locally vertical function is differentiable). It also explicitly introduces the idea of a continuous function, and makes the distinction between open and closed intervals. Because we reserve the term "theorem" for something we actually prove, we speak in this section of the "principles" governing the existence of extremes. By this time in the semester, students have had considerable experience informally finding extremes, and most find these principles natural and convincing. Because it is less general, and because students find deducing it very satisfying, we have reserved the "second derivative test" for the exercises.

Throughout this section and the next we encourage students to confirm the results of their calculations by using a graphing utility to produce the graph of the function they are studying. Students with weak algebra skills will have trouble in this section, but we advise against getting bogged down in algebra.

## 5.4 Optimal Shapes

This brief section provides an extended example of a geometric optimization problem, and all of the exercises are geometric. More of the traditional optimization problems are in supplement (1.2c) in Appendix C. Many of us put students to work on these supplementary problems (christened "one-a-days") early in the semester. Algebraicizing the verbal descriptions is a useful exercise, and they can get practice with the graphing utility by graphing the functions they find and determining extremes by inspection. Those of us who do this return to these supplementary exercises in this section.

## 5.5 Newton's Method

Newton's method is introduced as a tool to find critical points by solving $f'(x) = 0$, but it carries more pedagogical baggage than that. In this section we introduce the idea of the *tangent line* at a point as the extension of the local linear approximation at the point. The algorithm reminds students about successive approximation and the use of the computer loop, which prepares them well for the return of computing in chapter 6. A single laboratory period on Newton's method can achieve these goals.

Exercises 2-5 contain considerable history, and many of us encourage our students to read all of them, even though we usually do not assign them all. Problems 10-15 return to the $S$-$I$-$R$ model and provide an extended exploration.

## Chapter 6. The Integral

This is a long chapter with many ideas and an approach which we have found very effective. First time users of these materials will find the approach nonstandard and *we strongly urge that you read the chapter in its entirety before beginning to teach it.*

If you are using the chapter as the end of the first semester, you will encounter far fewer difficulties than if you are using the chapter as a first chapter of the second semester in which some of the students have had a standard first semester course. In the latter case, we recommend that you spend a couple of class meetings going over Euler's method for solving differential equations—this material will be new to students coming from the standard calculus course and is essential for understanding the main point of the chapter.

All the examples developed in the first several sections take the same form: we want to calculate the accumulated value of some quantity—human effort, work, energy consumed—that is the product of a rate and a time interval, where the rate is changing. This gives a concrete way of visualizing the process and leads almost trivially to the connection between antidifferentiation and accumulation.

### 6.1 Measuring Work

We recommend that you stress the notions of work and energy and take care to emphasize that the process of interest is that of *accumulating* work and energy. It is important to underline for the students that they are dealing with a product in which one of the factors is varying.

### 6.2 Riemann Sums

We begin by expanding the examples of §1 to approximating distance travelled, areas and lengths. Of course, the goal is for students to see the similarities between the different types of accumulation, so this is what you should stress. It is worthwhile to go over the steps in the program RIEMANN and have the students work some of the problems 4-12 in class. The point of all this is to have the students come to see a Riemann sum as a basic object of interest in its own right. This is especially important psychologically for students who have had a previous calculus class and who have carried away the notion that the Riemann sum is something to be muddled through before getting to the real stuff.

### 6.3 The Integral

This section consolidates the material in section 2, with an emphasis on visualizing the integral. The truly new element here is the error bound. This is the first time we provide an honest-to-goodness proof of convergence (pp. 339 - 345). Section 3 also includes the integrations rules for sums, differences and constant multiples of functions. Exercises 9 and 12 are well worth having the students do in class.

### 6.4 The Fundamental Theorem

Most students find this section a revelation, and we have had many wax eloquent on course evaluations. In class you should actually display, side by side, the programs RIEMANN and TABLE, as on p. 357. This is the proof of the fundamental theorem of calculus, and it never fails to create a deep impression. The point, of course, is that two different looking processes, computing left-endpoint Riemann sums for $f$ and Euler's method for solving the differential equation $y' = f(t)$, are actually the same. After this, the section on antidifferentiation comes as an anti-climax, but can be fun to cover. The short section on parameters should not be overlooked. Avoid the temptation to spend more than one class on this material—it is easy to undo all one's careful work and leave the student with the impression that integration is really just antidifferentiation!

## Chapter 7. Periodicity

After a brief overview of some of the contexts which naturally exhibit periodic behavior, the chapter describes the sine and cosine functions. The bulk of the chapter is devoted to some simple differential equations whose solutions are periodic functions. The key ideas that the student should carry away are the notion of periodicity, the properties of the sine and cosine as the simplest periodic functions, and the key role of differential equations in modelling periodic behavior. Many of us cover chapter 12.3 (The Power Spectrum) as part of this chapter. Here are some specific comments on each section. See the comments about 12.3 for detailed remarks on the power spectrum.

If you are still using Basic programs rather than commercial software to solve the differential equations that arise, we strongly urge that you move to a Runge-Kutta algorithm rather than continue with Euler's method. By now your students should have a very good feeling for how Euler's method works, and the increased accuracy of Runge-Kutta will make many of the points of this and the following chapters much more clearly and rapidly. Appendix E contains a TrueBasic Runge-Kutta algorithm which you can adapt to your system.

## 7.1 Periodic Behavior

This section is straightforward and needs little comment. We underscore the importance of presenting lots of examples of periodic behavior. We feel that it is important to point out explicitly that real world data are frequently noisy and that it is sometimes difficult to tell if some behavior has a periodic component. (Pursuing this question leads naturally to the methods detailed in chapter 12.3.)

## 7.2 Period, Frequency, and the Circular Functions

The sine and cosine should be presented as the simplest nonconstant periodic functions. Although students will have seen these functions in high school, they frequently have very hazy ideas about what frequency, period and cycle mean in connection with these functions. We advise you not to skip the table of physical interpretations of amplitude and frequency of sine and cosine functions in various contexts. You may want to give further examples.

Exercises 7 and 8 are useful for linking the material to the previous chapter on integration. Students will be hesitant about 7e) and 8b)—encourage them to consult one another. If you choose to do §3 of Chapter 12, then students should do exercise 20 (and possibly 21, 22, and 23). They should not attempt these exercises without doing 7 first.

## 7.3 Differential Equations with Periodic Solutions

This section is the heart of this chapter. Remember that many students will not have had a physics course, and those that have will frequently only half remember some poorly understood formulas. Make sure that they understand what pendulums, vibrating strings and springs are. Being scrupulous in the use of units is a useful way to keep the context in mind at all times, and to make sure that students can check equations with reality.

This is the first time students find out how to obtain values of sine and cosine to high levels of accuracy, and they are usually delighted by this use of Euler's method. It is worth taking time to make sure that they understand this—you may find that you have to review Euler's method (especially if you did not do so when discussing the integral, or if you begin the second semester with Chapter 7).

Likewise, we recommend that you do not skip the discussion of May's predator-prey model. The fact that the frequency and amplitude do not depend on the initial conditions is striking (and is also important as an example of the behavior of periodic behavior in generic solutions). If your students are still using Euler's method (as opposed to more sophisticated software) to solve these equations, remember that too large a value for $\Delta t$ leads to overflow problems.

The section on proving that a solution is periodic is perhaps best handled as a reading exercise, with a brief review and some of exercises 18-23 done in class. (Exercise 24 is best left for homework or extra credit.) Make sure, however, that students grasp the key notion of a *first integral*—it pervades physics.

## Further Reading

A good discussion of the question of whether or not animal populations really exhibit periodic behavior is James Patrick Finerty's *The Population Ecology of Cycles in Small Mammals* (1980 Yale University Press). It also has a very useful bibliography.

For other references, consult the Further Reading section at the end of chapter 12.

## Chapter 8. Dynamical Systems

In this chapter we resume our systematic investigation of systems of differential equations, introducing some of the basic concepts for thinking about such systems. One exercise that some teachers have used with considerable success is to assign journal articles using these ideas for the students to read and interpret. Many students find it very exciting to see that the tools they are acquiring are actually used by working researchers in a variety of fields. The further readings at the end of this chapter list some articles which have worked well in this way.

### 8.1 State Spaces and Vector Fields

Until now, when we have solved a dynamical system we have expressed the solution by plotting the different variables against time. In this chapter we introduce the concept of **state space**, where we suppress the time axis and use as coordinates the values of the dependent variables. The solutions are called **trajectories**, and the state space can be decomposed into a disjoint union of all possible trajectories. We review the concept of **first integrals** to see that where they exist, they give us the equations of the trajectories.

For those still using Basic programs, it is easy to emphasize the simplicity of the ideas involved since the students will only need to change the `PLOT` command in the programs they've been using all along to solve differential equations. Many of you, though, will by now have moved to more sophisticated software packages of one sort or another which can plot vector fields and trajectories. It is important that you point out to your students that the underlying concepts are still essentially those of Euler's method. Beware, though, that if one strictly uses Euler's method with systems—like the Lotka-Volterra predator-prey model or the undamped pendulum—that have first integrals, the trajectories may look like spirals rather than the closed loops they ought to be. While for any given time interval, the appearance of such spirals can be made to disappear by using small enough values of $\Delta t$, it is an inherent problem with Euler's method. Appendix E contains a simple TrueBasic program to draw vector fields.

Note that we are continuing to assume implicitly that there is a unique solution for any given set of initial conditions. If you have a fairly sophisticated class you might want to point this out to them and look at cases where this breaks down. They even looked at such a case back in problems 14 and 15—A Leaking Tank—of chapter 4.2, although this point was not made at the time.

The other important concept introduced in this section is that of **equilibrium point**, which can be considered as a point trajectory. At this point we introduce the idea that equilibrium points can be classified, and look at the types we get in the plane.

The section concludes with a consideration of a model with two attracting equilibria, talks about the idea of **basin of attraction**, and shows how such models can capture switching behavior, in which a system changes from one steady state to another.

## 8.2 Local Behavior of Dynamical Systems

Here we are dealing with a topic which should, arguably, only be dealt with after students know a little linear algebra and have a bit of complex number notation. Since the concepts are useful to a wider audience than those who will take such courses, though, we have developed an analysis using only the tools they have already acquired which gets at most of the ideas they need to think intelligently about local behavior. Since it puts these earlier tools together in new ways, it is also a good review and helps them think about these tools in a broader context

## 8.3 A Taxonomy of Equilibrium Points

This section should be viewed as an extended exercise using ideas the students have learned up to this point to explore the local geometry of equilibrium points. The goal is not so much to provide the kind of complete and elegant classification that more advanced methods can provide, but to show students that they can combine the calculus tools they've developed with common sense geometrical intuition to think about fairly sophisticated questions.

Note that the discussion of fixed lines doesn't explicitly address the question of vertical fixed lines. You might ask your students whether or not this case needs to be treated separately.

## 8.4 Limit Cycles

The section on limit cycles is quite brief (one example). There is obviously room here for considerable expansion on the teacher's part if desired. This is a good place to steer students to examples in some of the scientific journals where such examples occur.

### 8.5 Beyond the Plane:Three-Dimensional Systems

This is also a very brief introduction to a very large topic, providing considerable scope for elaboration for any teacher who is so inclined. It probably should not be attempted without good computing software which can help the students visualize the three-dimensional graphics.

## Further Reading

This is a good point in the course to give the students projects to see how some of these ideas are used in the literature. You can either give the students a single article to explore, explain, and elaborate on, or you can help them find articles which appeal to their own particular interests. Journals like *Science* or *The American Naturalist* are good sources of such articles. Here are a few specific sources which have interesting examples that can be adapted and most of which also have good bibliographies.

1. Abraham, Ralph H. and C.D. Shaw. 1984 , 1988. *Dynamics—The Geometry of Behavior.* In four parts (now available in one volume). *part 1: Periodic Behavior; part 2: Chaotic Behavior; part 3: Global Behavior; part 4: Bifurcation Behavior.* Santa Cruz, CA: Aerial Press, Inc. This is a superb exploration of the geometry of dynamical systems, helpful for both teachers and interested students. There are virtually no equations anywhere in the book.

2. R. M. Anderson and R. M. May, “Population Biology of infectious diseases: Part I”, *Nature* **280**, pp. 361–367 (2 August 1979).

3. —. “Infectious Diseases and Population Cycles of Forest Insects”, *Science* **210**, pp. 658–661 (7 November 1980). Some of us have used this quite effectively with our classes—it is at about the right level of difficulty for students at this point.

4. —. 1992. *Infectious Diseases of Humans.* Oxford University Press. Contains many examples and references which could be adapted.

5. Edelstein-Keshet, Leah. 1987. *Mathematical Models in Biology.* NY: Random House.

6. Segel, Lee. 1980. *Mathematical models in molecular and cellular Biology.* Cambridge U. Press.

7. — (ed). 1991. *Biological Kinetics*. Cambridge U. Press. For instance, ch. 1 on the Michaelis-Menten equation makes a very nice investigation.

8. Tuchinsky, Philip M. 1981. *Man in Competition with the Spruce Budworm*. Birkhäuser. Part of the UMAP Expository Monograph Series. An interesting development, presented as an investigation for undergraduates. It even throws in a little catastrophe theory.

## Chapter 9. Functions of Several Variables

This is a long chapter which can be covered after Chapter 10 and/or 11, if you desire. Although the chapter could be covered before Chapter 8, we recommend that, on the first time through, you cover the chapter after Chapter 8 (otherwise you will need to spend some time getting the students accustomed to the idea of a vector field).

We recommend encouraging students to devote *at least* as much time to studying the pictures as to reading the text. Our intention is to develop their visual intuition. Once they have a good visual intuition, students find it much easier to use more algebraic concepts like vectors in an appropriate fashion.

### 9.1 Graphs and Level Sets

The first section introduces the various visual tools for representing functions of two variables: graphs, sections, level sets, contour plots, density plots, terraced density plots. We recommend covering this material in a leisurely fashion, with much attention to the types of phenomena that can occur, without spending too much time on trying to systematize what occurs in what sorts of functions. Do stress the different representations and attempt to elicit from students the advantages and disadvantages of the different representations with respect to different examples.

Take time to make sure that students understand what is being represented. We have found that time spent on different representations is time well spent, and accord it much more class time than is typical.

To do this, you will need to settle on some graphing utility. We all have our own preferences here, but find that almost any well-designed program works. Home grown stuff is available by ftp—we especially recommend the program `Tint`, which produces colorful density plots. We have also used

Mathematica, Maple, Derive, and Beverly West and John Hubbard's package with good results.

As in the case of graphing utilities for functions of one variable, you will need to spend a little class time discussing the mechanics of running the utility. When you discuss level sets of functions of two and three variables, we recommend that you explicitly draw the connection to first integrals and dynamical systems. As usual, you will want to stress that the level set of a function of two variables (resp. three variables) is generically a curve (resp. surface). We recommend having the students work on one of problems 29 - 32 in class and do another one for homework.

## 9.2 Local Linearity

This section extends the ideas of linearity to functions of several variables and, in particular, to functions of two variables. The material in this section will take some time for students to digest. It falls into three main pieces.

The first important point is that graphs of functions of two variables generically approach a flat plane if you magnify repeatedly. You will then need to spend some time with formulas and graphs of linear functions to make precise what one means by a flat plane. You should emphasize that the contour lines of a linear function are parallel straight lines and carefully cover the notion that a plane has different slopes.

Next come the key concepts of the gradient of a linear function and the gradient vector field of an arbitrary function of two or three variables. We have found that the notion of *trade-off* appeals to some students—if it appeals to you, you may want to consider having students do exercise 25 and one or two of exercises 26 - 30. These exercises can again be done in class.

Third, we have the essential ideas of local linearity, the microscope equation, and linear approximation. We have found that students have difficulty using the microscope equation to estimate error—care must be exercised to ensure that students do not resort to blindly plugging numbers into the microscope equation. (You may find it useful to refer the students back to the discussion of error propagation in chapter 3.4.) Incidentally, time spent on the microscope equation will vastly ease the students' understanding of the machinery of differential forms in later courses.

### 9.3 Optimization

If the students have understood sections 1 and 2, section 3 can be covered very quickly. We have found that students have no difficulty with the method of steepest ascent—here you should emphasize that a simple program does the job (point out that one does not even need a graphing utility, a programmable calculator works fine).

We find that students have little difficulty with the idea of finding extrema by setting partials equal to zero. Similarly, they have little problem grasping the idea behind the use of Lagrange multipliers. We stress the geometric aspects, and spend relatively little time on solving the resulting systems of equations. Students will, of course, have difficulties solving simultaneous systems of equations—we deliberately choose to spend very little time on this topic: in practice, very few systems of equations are soluble by hand and it is, in our view, a mistake to spend large amounts of time on a topic which in practice is usually handled by a machine, if at all.

## Chapter 10. Series and Approximations

In the light of the general building-objects-from-first-principles approach we've used throughout this course, this topic plays a somewhat more important role than it does in other approaches to calculus. Moreover, in this chapter students will see that the mathematical sophistication they've been developing now leads to some major efficiencies in approximating quantities and expressions they've looked at before—integrals, the value of $\pi$, solutions to differential equations.

### 10.1 Approximation at a Point and Over an Interval

This section makes the observation that there is more than one possible criterion we can use when we want to approximate a function. Which we use depends both on the nature of the problem we are trying to solve and on the tools we have at hand. For many purposes (like designing a calculator), we need a good approximation over some entire interval. For other purposes (like studying the local behavior of a function), we only need a good approximation at a point. In traditional calculus courses the first problem is seldom addressed, not because it is unimportant, but because it is computationally messy.

### 10.2 Taylor Polynomials

The material in the first part of this section is fairly straightforward. Note that all approximating polynomials are really polynomials, of finite degree. We defer the concept of Taylor series to the next section.

The second half of this section, though, explores the concept of goodness of fit and introduces the "big oh" notation. This concept allows a precise formulation of Taylor's Theorem (in fact, three versions are given). The material is fairly technical, though, and will be difficult for some of your students. You should think carefully about how much time you want to spend on this topic, since you can't lightly skim over it—you should either skip it entirely, defer it, or spend the time it will require.

Problem 9 raises some interesting conceptual issues about the nature of mathematical definitions which will be referred to in the next section.

### 10.3 Taylor Series

In this section we introduce Taylor series—polynomials of "infinite degree". We interpret such objects in terms of limits and raise the question of whether or not the implied limit exists or not. This leads to the concept of **interval of convergence**.

In addition to elaborating on some of these ideas, the exercises also introduce some new themes. Problems 6 raises the important point of how the domain of mathematical functions like exponentiation can be extended to larger sets. This is usually done (as was the case even in defining things like $2^{-3}$ or $2^{\frac{1}{3}}$), not by looking at the original definition, but by looking at the key properties of the function under consideration and seeing how they could be preserved. This point was explored in problem 9 of at the end of section 10.2.

Problems 8 through 13 look at some technical questions around computing values of polynomials rapidly.

### 10.4 Power Series and Differential Equations

This section is again a fairly standard introduction to the ways polynomial approximations to the solutions of differential equations can be obtained. It makes the point that one reason for obtaining such approximations is for data storage—it is much easier to store the coefficients of a polynomial than to store several thousand numerical values of a function. In fact, many numerical packages, such as those in *Mathematica*, do just this.

Since the arithmetic is messy, you should check through the solution to the *S-I-R* model before deciding to cover it with your students. The results are a nice addition to their understanding of this problem, but the teacher had better be prepared to help lead the students through the maze!

### 10.5 Convergence

Here some of the basic tests—alternating series test and the ratio test—for series convergence are developed. **Geometric Series** and the **Harmonic Series** are presented as prime motivating examples for some of the issues involved. Teachers so inclined can easily use problem 8 in the exercises to motivate the integral test.

### 10.6 Approximation Over Intervals

This section introduces the major concept of **least squares criterion** for approximations and applies it to find polynomial approximations to functions over intervals. Since the idea of the least squares fit is so important in the field of statistics, it would be useful for many of your students to see the same idea in this very different context. It makes sense to investigate these polynomial approximations only if you have good software for solving systems of simultaneous equations, since the arithmetic is otherwise excruciating.

It is natural at the end of this section to jump to chapter 12.4, where the same ideas are used to develop Fourier Series. Students find the simplicity of the Fourier Series approximations, compared to the messiness of the polynomial approximations, to be very elegant and appealing.

### Further Reading

Campbell, P.J. 1991. "Computer and Calculator Computation of Elementary Functions", *The UMAP Journal* **12.4** (Winter 1991).

## Chapter 11. Techniques of Integration

There are few surprises here. Most of the material is familiar to teachers (and students) of traditional calculus courses. What is unusual is the placement of the chapter near the end of the book. This is to emphasize the distinction between the *concepts* of integration (chapter 6), which are general, and the *techniques* of integration, which apply to a variety of special (and not complete) subsets of functions. We intend this chapter as a resource for the teacher and the student, with sections to be used as they seem appropriate. Some of us don't cover the material in this chapter as a single unit, but intersperse one or two sections at a time between coverage of other topics.

Most teachers will want to cover the first three sections (through integration by parts) and perhaps part of section 4 (separation of variables) and the last two (Simpson's rule and improper integrals). Section 5 is useful primarily for students going on in physics or engineering. Even for these students, the widespread availability of good integrating software means there is less justification for spending large amounts of class time becoming proficient in techniques that will be rarely used.

The traditional second semester course often begins with techniques of integration. That does make a natural connection with the preceding semester, which often ends with the fundamental theorem of calculus. However, many of us have chafed at this organization, especially when we teach students who are beginning their college level study of calculus with the second semester. Techniques are techniques. While they have their importance, they seldom engage a student's imagination. Further, techniques are a poor vehicle for weaning a student from attitudes too common in high school: mathematics consists *only* of computational algorithms, and learning mathematics means imitating examples in the text.

Those of us who teach an entry level course for students with high school calculus have found that chapter 7, on periodic functions, makes an excellent introduction both to the more conceptual and contextual thinking we want our students to do and also to Euler's method and systems of differential equations. Most of us then treat all or part of chapter 8 and then turn to studying integration techniques. The treatment of separation of variables in section 4 makes the link back to differential equations. On the other hand, for students who began their study of calculus with this text, chapter 11 can naturally and effectively follow chapter 6. One consideration that may affect your decision about when to teach techniques of integration is compatibility with your institution's mathematics prerequisites for introductory physics. We recommend consultation with your colleagues in physics before making a final decision.

Whenever you decide to use the material in this chapter, you will see that it assumes that a student will have access at least to a table of integrals, if not to software capable of carrying out antidifferentiation. (Many integration formulas appear at the end of the Quick Reference section at the back of the text.)

## 11.1 Antiderivatives

The treatment of antidifferentiation of basic functions is straightforward. We also pick up the consideration of inverse functions which was begun in chapter 4 and find formulas for the derivatives of the arcsine and arctangent functions. For those of you who haven't thought about it before, you might want to look at problem 18 ahead of time, where there is an exploration of why the standard formula

$$\int \frac{1}{x} dx = \ln|x| + C$$

is, in fact, incorrect.

## 11.2 Integration by Substitution

The technique of integration by substitution is explained using the notation of differentials because most of us find that easiest for our students. While the connection to the chain rule is made explicit later in the section, several of us have had the interesting experience of our students—used to understanding ideas before doing computations—rebelling against being shown the mechanics of the differential notation before they are convinced that the result makes sense. We find this very satisfying!

Note that exercise 7 foreshadows the improper integral. Additional drill exercises are in the appendix.

## 11.3 Integration by Parts

Integration by parts is presented using functional notation, because our experience has been that our students make an easier connection to the product rule that way. You may wish to show students the differential notation as well, if you think they may encounter that notation elsewhere, but most of us have not done this.

Exercises 5 and 7 again foreshadow the improper integrals. The later exercises provide the formulas used in 12.3, in the case study on the power spectrum, and should, we feel, only be done in that context.

### 11.4 Separation of Variables and Partial Fractions

This section approaches solving a differential equation by separation of variables via the relationship between the derivatives of inverse functions. We use separation of variables to obtain the formula for "supergrowth" in 4.2, and we also consider the differential equation for logistic growth from 4.1. The logistic leads to the technique of partial fractions, which some of us like to include because it reappears in later algebra courses, as well as for its usefulness in this context.

### 11.5 Trigonometric Integrals

This section pulls together a variety of techniques involving trigonometric integrals and trigonometric substitutions. A principle thread that runs through many of the techniques is that of a **reduction formula**. This material is largely needed by students going on in physics or engineering, and you should give thought ahead of time to how much of this material your students will actually need, and how adept they need to be at it, given the widespread availability of integrating software.

### 11.6 Simpson's Rule

In this section we return to numerical methods and introduce the notion of the *efficiency* of an algorithm. The trapezoidal approximation is shown to be the average of the left and right endpoint Riemann sums, and the accuracy of the left, right, and midpoint Riemann sums and the trapezoidal approximation are compared. Students enjoy the surprising discovery that the midpoint Riemann sum is the best. We then present Simpson's rule as a weighted average of the midpoint and trapezoidal approximations.

Exercise 4 asks how many subdivisions are needed to obtain a specified accuracy using Simpson's rule, foreshadowing the formal study of limits in more advanced courses. We have found that our students' extensive numerical experience with sequential convergence in first year calculus serves them very well in the junior analysis course when we work with the formal definition of the limit.

### 11.7 Improper Integrals

We introduce the improper integral with an example involving the lifetime of light bulbs that is very easy for students to understand, but then we turn to the more important example of the *normal density function* of probability. Exercises 5–10 form a nice sequence on the *gamma function*.

## Chapter 12. Case Studies

This chapter consists of four extended explorations of topics connected to one or more of the previous chapters. They are quite different in tone, and each appeared in some previous draft of the book as part of the body of the text and each has its enthusiastic adherents who use the material each time they teach the course. As the course evolved, though, it was felt that they were best seen as optional exercises which teachers could use when it seemed appropriate.

### 12.1 Stirling's Formula

This section is a straightforward derivation of Stirling's formula for approximating $n!$ using only the technique of integration by parts, together with some tight mathematical reasoning bringing together a number of threads from elementary calculus. This is an excellent section to assign to students who like to work with the logical structure of mathematics and are looking for some challenging exercises of this sort.

It makes sense to assign this study in conjunction with studying chapter 11.3 on integration by parts, or with the material of chapter 10, where factorials are used extensively. Another good place to use the material is in conjunction with, and a class or two before, section 12.2 as part of a unit on probability and statistics.

The section is written rather straightforwardly—good students can follow the explanation in the text by themselves. We have found that having students work through the derivation in small groups is an effective way of covering the material that is written. (In fact, this is essentially exercise 1.) We urge that you resist the temptation to give a polished lecture recounting the material in the book. Rather, the chief pedagogical challenge is to convince students that Stirling's formula is useful (and, in fact, almost indispensable) for working out the probabilities that arise when the number of events is large. This usually requires an explanation of how to compute simple probabilities for situations that can be modelled by urn models. This does not take a great deal of time, since most students have had some probability in high school.

Running an example with large numbers will convince students that even a computer has difficulty handling $n!$ for large $n$—indeed, writing a brief program to compute $n!$ for large $n$ will, with many older languages, cause horrendous output errors.

You can also observe that computing $n!$ requires $n$ multiplications (so runs in linear time in $n$), while computing $\sqrt{2\pi}\sqrt{n}(\frac{n}{e})^n$ requires on the order of $\log_2(n)$ operations, a huge difference.

## 12.2 The Poisson Distribution

This section can be covered profitably once the power series formula for $e^x$ is in the students' possession. Those of us who are fond of the section use it to introduce the notion of a *probability model*. Here, the chief points to make are that naive linear models often do not work and that the final arbiter of whether or not a mathematical technique is applicable to a phenomenon under investigation is how well the predictions of the model agree with the data.

The central puzzle is that of modelling $\alpha$-ray emissions. Having students work through the details of the derivation in small groups is again a good strategy for covering the derivation of the Poisson distribution. Exercises 6, 7, or 8 are particularly good for class discussion. If you are assigning homework individually, we recommend that you do not assign one of these exercises without a previous class discussion of one of the other ones.

## 12.3 The Power Spectrum

Fourier transforms are the basis of a number of engineering applications of mathematics and have become a common analytical technique for many scientists. Moreover, the basic ideas underlying the sine and cosine transforms are quite straightforward and can readily be followed by students in their second semester of calculus. At one point this topic was a central part of our treatment of periodicity, but we have (temporarily?) backed off a bit and placed it in this optional category, for two main reasons. The first reason was not that the topic was too hard, but, in some ways, that it was too simple. The students could understand the idea, they could take a data set, find its power spectrum and interpret the spikes, and this was all very interesting. But there were few active investigations they could get into. Unless you are willing to push the topic further—getting into inverse transforms, discussing the phenomenon of aliasing in greater detail, etc.—the topic is probably best left as a case study. A second reason is that for this topic to go well, the teacher needs to have on hand a number of interesting data sets and know how to import data into the programs for finding their spectra. In some ways, the best data sets to use are those generated by your colleagues in the other disciplines—ask around and see what they have.

This section can profitably be covered by the entire class immediately after finishing chapter 7. It also makes a good independent study project for some of your more eager students. Here are some specific comments:

1. Use the material as a chance to emphasize that real world data are noisy. It is frequently very difficult to tell whether or not some behavior has a periodic component.

2. A main theme should be that noise is random, so averaging will take care of it. The integral is frequently used as a tool for averaging.

3. The notion that the transform (more generally, that an integral depending on a parameter) defines a function will confuse some students. Resist the temptation to work out a carefully chosen example in closed form—this will benefit only those who already understand. Rather, stress the mechanical analogies of probe or detector to make sure the idea gets across. Assign the closed-form example as a homework problem if you must.

4. We have a number of data sets—such as the number of measles cases in New York City, $CO_2$ levels in Hawaii over a 20-year period, and the number of lynx and hare pelts purchased by the Hudson Bay Company over a period of 120—available by annonymous ftp transfer from emmy.smith.edu. An excellent in-class exercise is to have the students run the programs on the data sets to see if, in fact, there appear to be periodicities in the data. We advise doing the exercise in class because the exercise is more in the nature of a demonstration. For students who appear especially interested in the topic, we recommend exercises 13-22. Since these exercises are fairly technical, we recommend that they not be assigned to the class as a whole (unless you are willing to spend time going into the topic of spurious information and phenomena which are artifacts of the numerical methods one uses). Fourier transforms also crop up frequently in science journal articles—see the readings below for some examples—and you are strongly urged to have students read and discuss some of these to see how the transforms are used by working scientists.

## Further Reading

1. Broadhurst, T. J. et al. 1990. “Large-scale distribution of galaxies at

the Galactic poles", *Nature* **343**, pp. 726–728 (22 February 1990).

2. Chappelaz, J. et al. 1990. "Ice-core record of atmospheric methane over the past 160,000 years", *Nature* **345**, pp. 127–131 (10 May 1990).

3. Herbert, T. D. and A. G. Fischer. 1986. "Milankovitch climatic origin of mid-Cretaceous black shale rhythms in central Italy", *Nature* **321**, pp. 739–743 (19 June 1986).

4. Körner, T.W. 1988. *Fourier Analysis*. Cambridge University Press.

5. Ruddiman, W. F. and A. McIntyre. 1981. "Oceanic Mechanisms for Amplification of the 23,000-Year Ice-Volume Cycle", *Science* **212**, pp. 617–627 (8 May 1981).

6. Scuderi, L. A. 1993. "A 2000-Year Tree Ring Record of Annual Temperatures in the Sierra Nevada Mountains", *Science* **259**, pp. 1433–1436 (5 March 1993).

## 12.4 Fourier Series

This material was originally the final section of chapter 10 and obviously continues the ideas in 10.6. It was shifted to being a case study to keep down the length of chapter 10. It is a beautiful topic, though, and one which many students find fascinating. They especially appreciate the beauties of an orthogonal basis after having waded through the simultaneous equations which arose in 10.6!

In addition to its inherent mathematical interest, this topic is also an excellent exercise in circular functions and in integration by parts.

# Appendix A: Sample Syllabi

Here are sample syllabi, drawn from the way some of us have structured the year. They are, of course, only samples, but we hope they will be useful guides as you steer your own path through the material. The syllabi are based on a 13-week semester, with the equivalent of three 70-75 minute class periods each week. Obvious modifications allow them to be adapted to four 50 minute classes per week. As explained below, the syllabi do assume that students have access to some technology in the classroom. More substantial rearranging will be needed if that is not the case.

Classes vary widely in terms of the students' preparation and the interests of the teacher. Your classes may need to move more or less rapidly than the pace suggested by these syllabi, and you should certainly feel free to adjust them to your circumstances. There are many points at which supplementary topics can be inserted, and there are also topics that can be dealt with more summarily if you find yourself pressed for time, especially in chapters 7–12.

For the second semester syllabus, we assume that students have had a first semester Calculus in Context course which included Chapter 6 on integration in its entirety. Even so, the syllabus is an ambitious one, and you may well want to omit some topics to allow for a more leisurely development of the others. In particular, fall offerings of Calculus II often need more time for review of Calculus I topics than in the Calculus II syllabus offered here, even when all students have used this text for Calculus I. When the student population is heterogeneous, even more time may be needed.

It is important to stress that we expect students to have read the material to be covered in class *before* class. (In the early weeks we have to give careful attention to *teaching* them to read the text.) Reading assignments are part of most homework assignments. Note that the assignments are listed on the day the assignment is *made*, and normally the assignment is *due* at the following class. Some teachers find frequent, brief quizzes to be a useful way either to give students a sense of confidence on some of the more basic ideas or to encourage them to study the material in advance. Samples of such quizzes are also included in Appendix B. Each syllabus indicates the timing of the in-class portion of two examinations. We assume the final exam occurs after the end of classes.

**Explanation of syllabus layout.** For each class there is a set of problems you might work on during that class. Thus on Class 6 you see the entry "Work on 1.3: 7-15; 1.4:5" meaning that during the 6th class you and the students will work on (some of) problems 7-15 of chapter 1.3, and problem 5 of chapter 1.4. The problems are, in our view, the heart of the course. We find that class time is profitably spent actually working on some of the problems, as well as discussing strategies for attacking them and what is learned from solving them. Problems begun in class might comprise part of the homework assignment for that day. The final column gives their assignment for the next class. In this case they are expected to read chapter 2.1 and write up problems 6 and 7 from chapter 1.4, to be turned in next class.

Some of the suggested classroom exercises require the use of technology. If you have sufficient computers or graphing calculators in the classroom, students can work on them in small groups. We use this format quite often, and spend much of our time roaming around talking with and listening to the various groups as they work. If you have only a single computer with an overhead display, you can still generate active, useful discussion as the class collectively works on an exercise. If no technology at all is available in the classroom, only the preliminary and follow-up discussions of these problems can occur in class, and you will have to adjust accordingly.

We begin each class meeting by inviting questions and discussing issues raised as seems appropriate, but we try to resist lecturing on the reading. Most of our time in class is spent working on problems or discussing problems and their implications. We frequently move back and forth between problem-solving in small groups and larger class discussions informed by the problem-solving efforts. Again, we would emphasize that this way of running the class works only if the students have studied the material ahead of time.

# Calculus I Syllabus

| | In Class | Homework |
|---|---|---|
| Class 1:<br>Modelling the spread of a disease | Discussion of model;<br>work on 1.1: 1-5 | Read 1.1 |
| Class 2:<br>Analyzing the model | Work on 1.1: 15, 16, 17 | 1.1: 18, 19;<br>read 1.2 |
| Class 3:<br>Computer graphing | Work on 1.2: 8, 10, 12 | 1.2: 11, 14;<br>(Appendix C) 1.2a: 1 |
| Class 4:<br>Linear functions | Work on 1.2: 16, 17, 18, 21 | 1.2: 19, 23, 24, 25, 27; |
| Class 5:<br>Using a computer program | Work on 1.3: 1-6 | 1.3: 16, 21<br>(AppC) 1.2a: 3 |
| Class 6<br>Using a computer program, cont. | Work on 1.3: 7-15; 1.4: 5 | 1.4: 6, 7;<br>read 2.1 |
| Class 7:<br>Successive approximations | Work on 2.1: 1-3, 10 | 1.4: 1-4;<br>read 2.2 |
| Class 8:<br>Euler's method | Work on 2.2: 1-4 | 2.2: 5, 7<br>(AppC) 1.2a: 3 |
| Class 9:<br>Euler's method, cont. | Work on 2.2: 8-10 | 2.2: 13;<br>read 2.3 |
| Class 10:<br>Approximating lengths | Work on 2.3: 2-10 | 2.3: 11, 12 |
| Class 11:<br>Approximating roots | Work on 2.3: 1;<br>1.2: 13 (select) | (AppC) 1.2a : 4, 5 |
| Class 12:<br>Review of chapters 1 and 2 | Questions and discussion | None; prepare for<br>midterm #1 |
| Class 13:<br>Midterm #1 | In class portion of midterm #1 | Read 3.1 and 3.2;<br>do 3.1: 1-6 for<br>discussion next class |
| Class 14<br>Rates of change | Discuss 3.1: 1-6;<br>work on 3.2: 1c, 2c, 3b, 6 | 3.1: 7, 8; 3.2: 1a, 2ab, 3a 7;<br>read 3.3 through p. 110 |
| Class 15:<br>The derivative | Work on 3.3: 2, 3, 6 | Finish class problems<br>and read the rest of 3.3 |
| Class 16:<br>The microscope equation | Work on 3.3: 9a, 10, 14, 17 | 3.3: 10bcd, 13, 17;<br>read 3.4 |
| Class 17:<br>Estimation and error analysis | Work on 3.4: 3, 7, 9 | 3.4: 6, 8;<br>read first 2 pages of 3.5 |
| Class 18:<br>The derivative as function; graphs | Work on 3.5: 5 (parts), 2 | 3.5: 1, 5 (parts);<br>read the rest of 3.5 |
| Class 19:<br>Differentiation formulas | Work on 3.5: 6 (parts), 7, 9, 17 | 3.5: 6 (parts),<br>8ad,10, 13, 18, 19; read 3.6 |
| Class 20:<br>Chain rule | Work on 3.6: 1, 2 (parts), 8 | 3.6: 2 (parts),<br>7, 11; read 3.7 |
| Class 21:<br>Partial derivatives | Work on 3.7: 1, 7, 9 | 3.7: 3, 10;<br>read 4.1 |

| | In Class | Homework |
|---|---|---|
| Class 22: Modelling with differential equations | Work on 4.1: 6 | 4.1: 6e; read 4.2 through p. 188 |
| Class 23: Solving differential equations | Work on 4.2: 3, 4 | 4.2: 1, 2, 5; read the rest of 4.2 |
| Class 24: Effect of parameters | Work on 4.2: 6, 7, 17-20 | 4.2: 24, 25; read 4.3 through p. 204 |
| Class 25: Review | Questions and discussion | None; prepare for midterm #2 |
| Class 26: Midterm #2 | In class portion of midterm #2 | Read 5.1 and 5.2 |
| Class 27: The exponential function | Work on 4.3: 1, 2, 3 (parts), 4 (parts), 9 | 4.3: 3 (parts), 4 (parts), 8; read the rest of 4.3 |
| Class 28: The exponential function, cont. | Work on 4.3: 5, 6 | 4.3: 7, 10; read 4.4 |
| Class 29: The natural logarithm | Work on 4.4: 1 (parts), 3, 4 (parts), 7 | 4.4: 1 (parts), 5, 8; read 4.5 |
| Class 30: Solving $y' = f(t)$ | Work on 4.5: 3, 5, 6 | 4.3: 13ab, 4.4: 15; 4.5: 1abc, 2, 4 |
| Class 31: Differentiation | Work on 5.1: 1 (parts), 2 (parts), 8, 13, 14, 15 | 5.1: 1 (parts), 2 (parts) 4, 7, 20 a-d; 5.2: 3; read 5.3 and 5.4 |
| Class 32: Shape of a graph; optimization | Work on 5.3: 1, 2, 3, 7 (parts), 6; 5.4: 1, 2 | 5.3: 7 (parts); (AppC) 1.2a: 1-5; read 5.5 |
| Class 33: Newton's method | Work on 5.5: 7, 9 | Read 5.5: 1-5; 5.5: 6; read 6.1 |
| Class 34: Work | Work on 6.1: 1, 2, 6, 7 | 6.1: 9, 10; read 6.2 and do 6.2: 1, 2 for discussion |
| Class 35: Riemann sums | Discuss 6.2: 1, 2; work on 6.2: 3, 4, 16b, 19b, 28 | 6.2: 9, 16ac, 19ac; read 6.3 through p. 339 |
| Class 36: The integral. | Work on 6.3: 1, 2, 3, 16 | 6.3: 5ac, 17; read the rest of 6.3 |
| Class 37: Error bounds | Work on 6.3: 9, 12 | 6.3: 10, 11, 13, 14; read 6.4 |
| Class 38: The fundamental theorem | Work on 6.4: 2, 3, 4c, 8a, 10 (parts) | 6.4: 4g, 8d, 10 (parts) |
| Class 39: Review | Questions and discussion | |

## Calculus II Syllabus

| | In Class | Homework |
|---|---|---|
| Class 1:<br>Review Euler's method | Work on 1.4: 5-7 and 4.1: 6c, 7<br>by hand and by computer | Finish 1.4: 5-7, 4.1: 6c, 7;<br>read 7.1, 7.2 |
| Class 2:<br>Sines and Cosines | Work on 7.2: 1-6, 9-13 | Finish 7.2:1-6, 9-13;<br>read 7.3 to p. 394 |
| Class 3:<br>Spring and pendulum | Discuss behavior of springs<br>and pendula; work on 7.3: 1-3 | 7.3: 5, 6, 10, 18-20;<br>read rest of 7.3 |
| Class 4:<br>Predator-prey | Discuss predator-prey models;<br>work on 7.3: 25-26 | 7.3: 27-29;<br>re-read 7.3 |
| Class 5:<br>Periodic solutions, first integrals | Work on 7.3: 13-17, 22-24 | Finish 7.3: 13-17, 22-24;<br>(re)read 6.3, 6.4 |
| Class 6:<br>Review Riemann sums,<br>integral, fundamental theorem | Work on 6.3: 19, 20;<br>6.4: 1-2, AppD: 6.4 | Finish problems; read 11.1<br>(omit inverse functions) |
| Class 7:<br>Anti-differentiation | Work on 11.1: 9 (parts), 10 (parts) | 11.1: 9 (parts), 10 (parts),<br>12, 17; read 11.2 |
| Class 8:<br>Integration by substitution | Work on 11.2: 1 (parts), 2 (parts) | 11.2: 1 (parts), 2 (parts), 3, 8;<br>read 11.3 |
| Class 9:<br>Integration by parts | Work on 11.3: 1 (parts), 13, 14 | 11.3: 1 (parts), 15, 16, 17, 18;<br>read 12.3 up to power spectrum |
| Class 10:<br>Noise and periodicity,<br>the power spectrum | Discuss detection, introduce<br>power spectrum,<br>examine noisy data | 12.3: 9, 10 |
| Class 11:<br>Review | Questions and discussion | None;<br>prepare for midterm #1 |
| Class 12<br>Midterm #1 | In class portion of midterm #1 | Read 8.1 |
| Class 13:<br>Dynamical systems<br>and state spaces I | Discuss trajectories, vector<br>fields, first integrals and<br>equilibria; work on 8.1: 1 | 8.1: 2, 3, 6 |
| Class 14:<br>Dynamical systems and<br>state spaces II | Discuss Anderson-May model;<br>work on 8.1: 7 | 8.1: 5; read 8.2 |
| Class 15:<br>Local behavior | Discuss localization and<br>linearization; work on 8.2: 1 | 8.2: 1 (finish), 2, 4;<br>read 8.3 |
| Class 16:<br>Taxonomy | Work on 8.3: 1 (parts), 7 | 8.3: 1 (finish), 10, 11;<br>read 8.4 |
| Class 17:<br>Limit cycles | Work on 8.4: 1;<br>discuss why systems with limit cycles<br>and attractors can't have first integrals | 8.4: 2, 7.3: 35, 36<br>from phase plane point of view;<br>read 8.5 |
| Class 18:<br>Beyond the plane | Linearization and localization; Hopf<br>bifurcation and strange attractors | 8.5: 1 and/or 2; read 11.1<br>(inverse functions) |
| Class 19:<br>Inverse functions | Work on 11.1: 5 | 11.1: 1-4, 6, 7;<br>read 11.4 |
| Class 20:<br>Separation of variables | Work on 11.4: 1, 2 (parts) | 11.4: 2 (parts), 3, 4, 6 |
| Class 21:<br>Partial fractions | Work on 11.4: 13 | 11.4: 9-11 (parts), finish 13;<br>read 12.1 |

| | In Class | Homework |
|---|---|---|
| Class 22: Stirling's formula | Discuss reading, work on 12.1: 1 | 12.1: write up part of 1 based on class discussion |
| Class 23: Improper integrals | Work on 11.7: 1 (parts), and begin 3 | 11.7: finish 1 and 3, 4-7; (opt: read section on gamma function ) |
| Class 24: Review | Questions and discussion | None; prepare for midterm #2 |
| Class 25: Midterm #2 | In class portion of midterm #2 | Read 9.1 |
| Class 26: Graphs and Level Sets I | Introduce graphics utility; work on 9.1: 1, 5, 6, 11, 12 | 9.1: 2, 3, 14, 18 |
| Class 27: Graphs and Level Sets II | Slices, terraced density plots; work on some of 9.1: 29-32 | 9.1: finish 29-32, 24, 27; read 9.2 to p. 480 |
| Class 28: Local linearity | Magnify surfaces to get planes; equations of planes; work on 9.2: 1-3, 5, 6 | 9.2: 4, 14, 18, 19; read pp. 480-483, 489-491 |
| Class 29: Gradients | Work on 9.2: 20, 21, 25 partial rates of change | 9.2: 22-24, 63, 66, 67; read rest of 9.2 |
| Class 30: Microscope equation and linear approximation | Work on 9.2: 42, and begin 49-54 | 9.2: 45-47, 57-59; read 9.3 to p. 517, omitting pp. 506-510 |
| Class 31: Method of steepest ascent | Maxima, minima and saddles; work on 9.3: 7, 15, 16 using inspection and steepest ascent | 9.3: 13, 14 and those not done in class; read rest of 9.3 |
| Class 32: Constraints and Lagrange multipliers | Work on 9.3: 6, 11; geometric idea behind Lagrange multipliers | 9.3: 3, 4, 10, 12; read 10.1, 10.2 to p. 538 |
| Class 33: Taylor polynomials | Discuss reading; work on 10.2: 10-13 | 10.2: 1, 4, 7, finish 10-13, 16-18; read rest of 10.2 |
| Class 34: Goodness of fit, Taylor's theorem and Taylor's theorem | Work on 10.2: 19-21 | 10.2: 22, 23, 30; read 10.3 |
| Class 35: Taylor series | Graphs of functions and their Taylor polynomials (c.f. pp. 534, 556); work on 10.3: 2,3 | 10.3: 6-8; read 10.4 |
| Class 36: Differential equations | Work on 10.4: 5 | 10.4: 1 (parts), 2, 9; read 10.5 to p. 583 |
| Class 37: Convergence; geometric, harmonic, alternating series | Work on 10.5: 3, 4 (parts), start 7 | 10.5: 1 (parts), 2 (parts), 4 (parts), 7 (finish), 8 |
| Class 38: Radius of convergence, ratio test | Work on 10.5: 11 (parts), 12 | 10.5: 10 (parts), 11 (parts), 13, 14 |
| Class 39: Review | Review | |

# Appendix B: Sample Exams and Quizzes


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# CALCULUS I
## SAMPLE MIDTERM 1, take-home portion
## (week 5, on chapters 1 and 2)

This is an "open-book" test; you may consult freely your notes, homeworks, text, and any other books you wish; you may use a calculator or a computer, and any programs available on a computer. However, you must not receive help, in any form, from anyone else. Make your responses brief but complete; explain your reasoning, and write clearly. *Whenever you have a numerical answer deduced from a rate equation, you should indicate how many digits you know to be exact and why you can guarantee they are exact.*

1. Find $x$ so that $x^5 = 4 - x$. Give the value of $x$ accurate to 3 decimal places.

2. Make a sketch of the graph of the function

$$y = \frac{x^3 - 7x}{3x^2 + 2x + 5}$$

on the interval $-3 \le x \le 3$. In your sketch indicate clearly where the highest and lowest values of the function appear, and indicate where the graph crosses the $x$-axis.

3. **Hooke's law.** If you hang a weight from a steel spring, the spring stretches. If the weight is not too large, then the distance stretched is proportional to the weight. (This is called Hooke's law.) Suppose a particular spring is 12.3 inches long when there is no weight on it, but it becomes 12.68 inches long when a 10 pound weight hangs from it.
a) How long is the spring when a 5 pound weight hangs from it?
b) Let $L$ denote the length of the spring in inches and $W$ the weight in pounds. Produce a formula that describes how $L$ depends on $W$.

4. **The vanishing elephants.** According to one environmental group, the population of wild elephants declines by 5% per year. If $E$ denotes the population $t$ years from now, then we can express the decline by a rate equation of the form

$$E' = -kE \quad \text{elephants per year.}$$

a) What value would you give $k$?
b) Suppose there are 200,000 elephants now. Using the rate equations with your value of $k$, determine how many elephants there will be in 10 years. Your answer should be accurate to the nearest 100 elephants. (Remember, make it clear why you know your answer has that much accuracy!)
c) How many years will it take for the population to be cut in half—to 100,000?
d) Consider this argument: "Since 1/20-th of the population disappears each year, in 20 years the population will vanish completely." Does your rate equation predict that the elephant population will vanish in 20 years? How many elephants *does* your rate equation predict there will be in 20 years? What, if anything, is wrong with the argument quoted in the first sentence?

5. **Predicting the human population.** One model for the growth of the world's human population uses the rate equation

$$P' = .015P^{1.2} \quad \text{billions of persons per year.}$$

Here $P$ is the population, *measured in billions*. Right now, $P = 5$ billion people.
a) At what rate is the population growing now, in billions of persons per year?
b) According to this model, what will the population be in 25 years? Make the value of $P$ exact to 4 decimal places. (At that accuracy, you predict the size of the population to the nearest million people.)
c) At what rate will the population be growing in 25 years?

6. The effects of scale. When I bake cookies and cut in half all the ingredients in the recipe, I get about half the number of cookies I would by using the full recipe. The is called "scaling down." The purpose of this question is to see if the $S$-$I$-$R$ model scales down the same way. Use the standard $S$-$I$-$R$ model found in the text:

$$\begin{aligned} S' &= -.00001SI \\ I' &= .00001SI - I/14 \\ R' &= I/14 \end{aligned}$$

Take the initial values used in the text and cut them down by a factor of 10. Thus, the *new* initial values are

$$S = 4540, \quad I = 210, \quad R = 250$$

a) What is the value of $S$ after 15 days? Compare this with the value of $S$ after 15 days when we use the original initial values (which were 10 times larger). In particular, is the new value of $S$ equal to 1/10-th the original value of $S$ after 15 days? (In other words, does the value of $S$ "scale down" the way a cookie recipe would?)
b) Using the new initial values, sketch the graphs of $S(t)$, $I(t)$, and $R(t)$ for $0 \leq t \leq 30$ days. Compare these to the graphs of $S$, $I$ and $R$ from the original problem (in which the initial values were 10 times larger). In particular, do the new graphs have the same shape as the original graphs ("scaled down" by a factor of 10)?
c) If the same epidemic (i.e., we'll use the same model to describe it) strikes a large city and a small town, will the same effects be observed? Write a brief essay summarizing your conclusions about the effects of scaling on the $S$-$I$-$R$ model. In particular, you know that the $S$-$I$-$R$ model has a threshold. What is the conneciton between the scaling and the threshold?

CALCULUS I
SAMPLE MIDTERM 1, in-class portion (75 minutes)
(week 5, on chapters 1 and 2)

This is a "closed-book" test; no books or notes are allowed. You may use a calculator if you wish.

1. Find a formula for the linear function $y = f(x)$ that satisfies the conditions $f(3) = 0$ and $f'(0) = -2$.

2. The growth of the population $P$ of a small town is modelled by the rate equation $P' = .02P$ persons per year. Assume that $P = 1000$ when $t = 0$. In each of (a) and (b), organize your work in a table.
a) Use a single calculation with $\Delta t$=4 to estimate $P(4)$.
b) Use two rounds of calculations with $\Delta t$=2 to estimate $P(4)$.
c) Illustrate your calculation in (b) with a graph of a piecewise linear function that approximates the graph of $P$ on the interval $0 \leq t \leq 4$. Label points and slopes.

3. [Provide a figure for this problem showing graphs of two positive functions $y = F(t)$ and $y = M(t)$ in the first quadrant, each with a single maximum, and with the peaks at different values of $t$.]
Consider the functions $F(t)$ and $M(t)$ of time $t$ whose graphs are sketched above. Let $t_M$ be the time when $M$ is at its maximum, and let $t_F$ be the time when $F$ is at its maximum.
a) Which is larger: $F(t_M)$ or $M(t_F)$?
b) At time $t_M$, is $F$ increasing or decreasing?
c) At time $t_F$ is $M$ increasing or decreasing?

4. Here are a succession of Euler approximations for the population $P$ of a city after 3 years. [Provide a table of estimates of $P(3)$ for values of $\Delta t$ = 1.0, 0.1, 0.01, 0.001, 0.0001, 0.00001.] What is the most precise value you can give for the exact value of $P(3)$? How many digits do you know exactly? Why are you sure they are exact?

5. In 1970 the Science Library had 81,000 volumes, and by 1975 the number had grown to 88,500.
a) During 1970-75, the Science Library grew at a constant annual rate. What was that rate, in volumes per year?

b) If the 1970-75 rate continued without change, how many volumes would the Science Library have in 1990?
c) The old building housing the science Library had a capacity of 105,000 volumes. At the 1970-75 rate, when would it reach capacity?

6. A beaker contains three kinds of molecules called *dimers*, *monomers* and *trimers*. The variables $D$, $M$ and $T$ keep track of the number of molecules of each type. The following model describes how the numbers of each kind of molecule change over time.

$$\begin{aligned} D' &= .1M^2 - .2MD \\ M' &= -.1M^2 - .2MD \\ T' &= .2MD \end{aligned}$$

Below are graphs of $D(t)$, $M(t)$ and $T(t)$ (in some order). Label the graphs with $D$, $M$ and $T$ in a way that is consistent with this model.
[Provide graphs like those on page 17 of the text.]

7. Around 1920 L.F. Richardson constructed a simple model to describe an "arms race" between two countries. If $x$ and $y$ are the annual military budgets of the two countries (in billions of dollars), then the model expresses the rates at which $x$ and $y$ change (in billions of dollars per year) in terms of the values of $x$ and $y$. Consider two countries for which the model says

$$\begin{aligned} x' &= -4x + 2y \\ y' &= 5x - 4y + 12 \end{aligned}$$

and suppose this year x = 5 and y = 6.
a) According to the model, will $x$ increase or decrease next year? Will $y$ increase or decrease next year?
b) Assuming the rates given in the model stay fixed for an entire year, estimate the values of $x$ and $y$ one year from now.
c) According to the model, there are budgets $x$ and $y$ which will *not change* from one year to the next. What are those values of $x$ and $y$?

CALCULUS I
SAMPLE MIDTERM 2, take-home portion
(week 9, on chapter 3 and 4.1, 4.2)

This is an "open-book" test; you may consult freely your notes, homeworks, text, and any other books you wish; you may use a calculator or a computer, and any programs available on a computer. However, you must not receive help, in any form, from anyone else. Make your responses brief but complete; explain your reasoning, and write clearly.

1. a) Let $H(x) = x^x$. Find $H'(2)$ to three decimal places accuracy.
b) You know $H(2) = 2^2 = 4$. Use your answer to (a) to estimate the value of $2.015^{2.015}$.
c) Now use a calculator to determine $2.015^{2.015}$, and compare this with your estimate. According to the calulator, how many digits of your estimate are accuarate.

2. [Sketch a graph of $y = f'(x)$.] Sketched above is the graph of the *derivative* of an unknown function $f(x)$. Make an accurate copy of this graph on your answer paper.
There are many different functions $f(x)$ whose derivative could be the graph shown above. On a separate coordinate plane just above your copy of $f'(x)$ sketch the graphs of *two* such functions $f(x)$. Call them $f_1(x)$ and $f_2(x)$. Choose $f_2$ so that it satisfies the additional condition $f_2(0) = 0$.

3. **The distance to the stars.** When we look at an object, our two eyes have slightly different views of it. This difference is called **parallax**. If the object is close, the angle between the views is large (and we look "cross-eyed"). As the object moves farther away, the angle gets smaller. Our brain senses the parallax angle and uses it to judge the distance to the object.

Astronomers use parallax to judge the distance to the nearby stars. They take two views of the star six months apart, and measure the angle $2\theta$ between the views. They call $\theta$ itself the **parallax angle**. Even though the viewpoints are on opposite sides of the earth's orbit (they are about 186 million miles apart), $\theta$ is still very small. It is always less than 1" (one *second*, or 1/3600-th of one degree). If the parallax angle of a star is $\theta$ seconds, then the distance $S$ to that star is

$$S = \frac{3.26}{\theta} \quad \text{light-years.}$$

(Note: A **light-year** is the distance light travels in one year; it is about 6 trillion miles! Also, there is no need to measure $\theta$ in radians because no circular functions are involved.)
a) The parallax angle $\theta$ of the nearest star is 0.762". How many light years away is that star?
b) Since $\theta$ is obtained by measurement, its value is never known precisely. Any error $\Delta\theta$ in measuring $\theta$ will propagate to an error $\Delta S$ in the calculated value of $S$. Write the error propagation equation for $\Delta S$ in terms of $\Delta\theta$.
c) Suppose you measure the parallax of a star and then calculate that it is 8.35 light-years away. If you want your calculation to be accurate to within .05 light-years, how precisely do you have to measure the parallax angle $\theta$?
d) Write the propagation equation for *relative* errors.
e) If you want to calculate the distance to a star to within 1 %, what percentage error can you tolerate in the measurement of the parallax angle $\theta$?
[Provide a figure showing the star as a point at the apex of an isosceles triangle whose base is a diameter of the earth's orbit around the sun. The angle at the apex is $2\theta$, and the altitude is $S$.]

4. **Fermentation.** Do problems 8-11 on pages 172-174 of the text.

5. **Falling bodies—with gravity and air resistance.** Do problems 21-23 on pages 197-198 of the text.

Calculus I
SAMPLE MIDTERM 2, in-class portion (75 minutes)
(week 9, on chapter 3 and 4.1, 4.2)

This is a "closed-book" test; no books or notes are allowed. You may use a calculator if you wish.

1. The graph below shows a runner's distance $D$ in meters from the starting line after $t$ seconds.
[Provide a graph of distance versus time on a grid so coordinates can be read.]
a) When is the runner speeding up? How can you tell?
b) What is your best estimate of the runner's velocity at $t = 3$? Be sure to include units.

2. [Provide a sketch of the graph of a function, as in exercise 1 of 3.5 on page 134.] Copy the graph of the function $f(x)$ above onto your answer paper. In a coordinate plane just below your graph of $f$, sketch the graph of the *derivative* of $f$.

3. Find formulas for the the derivatives of each of the following functions.
[Choose a reasonable selection of four or five based on 3.5 and 3.6.]

4. a) Find the microscope equation for the function $y = \sqrt{x}$ at $x = 100$.
b) Use (a) to estimate the value of $\sqrt{99.3}$.

5. A block of ice is melting, and its volume shrinks at the steady rate of 15 cubic inches per minute. Assume that the block of ice is a perfect cube. At what rate is the length of the edge of the block decreasing when the edge is 10 inches long? Be sure to include units.

6. Verify that

$$y(t) = \frac{1}{\sqrt{6 - 2t}}$$

is a solution to the initial value problem $y' = y^3$, $y(1) = 1/2$.

CALCULUS I
FINAL EXAMINATION, take-home portion
(on chapters 1 - 6)

This is an "open-book" test; you may consult freely your notes, homeworks, text, and any other books you wish; you may use a calculaator or a computer, and any programs available on a computer. However, you must not receive help, in any form, from anyone else. Make your responses brief but complete; explain your reasoning, and write clearly.

1. a) Determine the value of each of the following integrals to four digits accuracy.

$$\int_0^1 \frac{1}{1+x^2}\,dx \qquad \int_{-2}^2 e^{-x^2}\,dx$$

b) Demonstrate how you know that the first four digits of your answers to part (a) are correct.

2. a) Until now, our only solutions to the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{C}\right)$$

have been provided by Euler's method. In fact, though, the formula

$$P(t) = \frac{Ce^{kt}}{1+e^{kt}}$$

*also* provides a solution. Use algebra and the rules for differentiation to verify that this formula does indeed give a solution to the logistic equation.
b) Suppose we extend the formula in part (a) to

$$P(t) = \frac{CAe^{kt}}{1+Ae^{kt}}$$

where $A$ is any number whatsoever. Verify that this extended formula is also a solution to the logistic equation, for any value of $A$.
c) In the formula for $P(t)$, let $k = 1$, $C = 12$, and $A = 2$. Sketch the graph of $y = P(t)$ on the interval $0 \le t \le 20$. What is the value of $P(0)$?
d) Find the formula $P(t)$ for the solution to the initial value problem

$$\frac{dP}{dt} = .2P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 200.$$

Sketch the graph of this solution over $0 \le t \le 20$.

3. **Growth in a seasonally fluctuating environment.** You have already considered the growth of a rabbit population $R(t)$ that is governed by a logistic model:

$$\frac{dR}{dt} = .1R\left(1 - \frac{R}{C}\right) \quad \text{rabbits per month.}$$

In this model, the carrying capacity of the environment was assumed to have the constant value $C = 25000$ rabbits. However, it is reasonable to think that the environment can support fewer rabbits in the winter than in the summer. We can therefore make a more realistic model by having $C$ fluctuate periodically with the seasons. This question investigates what happens to the population $R(t)$ when the carrying capacity depends on time according to this formula:

$$C(t) = 25000 - 5000\cos(\pi t/6) \quad \text{rabbits per month.}$$

Here $t$ measures the time in months since January 1990.
a) Sketch the graph of $C(t)$ for an interval of 60 months. Indicate on your graph the lowest value that $C$ achieves, and the months when that occurs. Also indicate the peak value and the months when *that* occurs. How many months are there between one peak and the next? (This is called the **period** of $C$, and $C$ is said to be **periodic**.)
b) Sketch the solution of the new logistic equation

$$\frac{dR}{dt} = .1R\left(1 - \frac{R}{25000 - 5000\cos(\pi t/6)}\right)$$

for which $R(0 = 2000$ rabbits. Show at least the first 120 months. [Note: part (c) below asks you to make a *second* sketch of $R(t)$ after completing parts (c) and (d).]
c) After about 60 months, $R(t)$ settles in to a fluctuating pattern that is similar to the pattern of $C(t)$—that is, $R(t)$ becomes periodic. Determine the period of $R(t)$, and compare it to the period of $C(t)$. (This means: decide whether the two periods are the same, or whether one is larger than the other.)
d) What are the peak and lowest values of $R(t)$, and in what months do they occur? Compare the peaks of $R$ and $C$. Are they the same size, or is one larger? Which one? Do the peaks happen in the same month? If not, which peaks first, $R$ or $C$? Compare the lowest values of $R$ and $C$ the same way.

e) Summarize your findings in parts (c) and (d) by sketching on the same coordinate plane the graphs of $R(t)$ and $C(t)$ over the interval $60 \le t \le 120$. Your graphs should show how the peaks of $R$ and $C$ relate to each other.

4. **Prices, demand and profit.** Do problems 1 and 2 on pages 294-295.

5. [Draw a graph of $f$ on a grid and choose values of $A < B < C < D$ appropriately for your graph and the questions below.]
This question concerns the function $f(x)$ whose graph appears above.
a) Give a simple argument that shows

$$A \le \int_0^5 f(x)\,dx \le D.$$

b) Give a more detailed argument that shows

$$B \le \int_0^5 f(x)\,dx \le C.$$

Calculus I
Final Exam, in-class portion (time limit 2.5 hours)
(on chapters 1-6)

This is a “closed-book” test; no books or notes are allowed. You may use a calculator if you wish.

1. Find the indicated derivatives. [Choose a sample of four or five from 5.1, and one accumulation function from 6.4.]

2. Times are tough in the decaying mill town of Detroit, Oregon. Its population $P$ is modelled by the differential equation $P' = -.01P$. The population of Detroit was 2000 in 1990.
a) Write a formula for $P$ as a function of $t$, the time in years after 1990.
b) Using this model, when will the population of Detroit fall to 1000?

3. Use Euler’s method with a step size of $\Delta t = .5$ to estimate $y(1.5)$ given that $y(0) = 1$ and $y' = y + 1$.

4. Find the critical points and global extreme values (that is, the global maximum and minimum) of $y = \frac{1}{3}x^3 - 2x^2 + 2$ on the interval $-1 \le x \le 3$. (Be sure to find both $x$ and $y$ coordinates.)

5. Write out and compute the Riemann sum for $\int_0^6 2^x \, dx$ obtained by taking 3 subintervals of equal length and sampling at the midpoints of the subintervals.

6. Here is some information about the second derivative of a function $f$:

$$f''(x) > 0 \quad \text{for} \quad x < 2 \text{ and } \quad x > 1$$

$$f''(x) < 0 \quad \text{for} \quad -2 < x < 1.$$

For each of the following functions $f$, indicate whether or not the graph is consistent with the given information. Briefly explain each of your answers. [Provide three or four suitable graphs.]

7. Which of the following are solutions to the indicated initial value problem? Be sure to *explain* your answer in each instance.
a) Is $y = \frac{1}{3}t^3 - 2$ a solution to $y' = t^2$, $y(3) = 7$?

b) Is $y = 2e^{3t}$ a solution to $y' = 2y$, $y(0) = 2$?
c) Is $y = \int_2^t \ln(x)\,dx$ a solution to $y' = \ln(t)$, $y(2) = 0$?

8. Use the Fundamental Theorem to find the exact value of the following definite integrals. [Choose 2 from 6.4.]

9. The following is the graph of the velocity (in units of 10000 km/sec) of an electron in a particle accelerator against time. Approximately how far does the particle travel from $t = 0$ to $t = 6$? Explain clearly how you are calculating your estimate. [Provide a suitable graph of velocity versus time, with a grid so that coordinates can be read.]

10. Suppose that you know $f(1000) = 375$ and $f'(1000) = -25$. Use this information to estimate $f(998)$.

11. After sketching graphs of a function $f(x)$ and its derivative $f'(x)$, a tired student spilled her coffee on the graph of $f(x)$, and part of the graph was obliterated, as shown.
[Provide a partial graph of $f$ on one coordinate plane, and a complete graph of $f'$ on a plane just below it.]
Please redraw the graph of $f(x)$ for the student, using her graph of $f'(x)$ as a guide.

Calculus I
Sample Quizzes (15-20 minutes each)

QUIZ 1 (on 1.1)
Consider an epidemic modelled by the rate equations

$$\begin{aligned} S' &= -.00002SI \\ I' &= .00002SI - I/5 \\ R' &= I/5 \end{aligned}$$

with initial values (i.e., values for "today" $t = 0$)

$$S = 20,000 \quad I = 100 \quad R = 100$$

1. Organize your work and your answers for this problem in a single table for $t, S, I, R, S', I', R'$.
a) Use the model to find values of $S$, $I$, and $R$ *tomorrow*.
b) Use your results in (1) to find values of $S$, $I$, and $R$ the *day after tomorrow*.

2. *Redo* your calculation of the values of $S$, $I$, and $R$ the day after tomorrow using a *single* time step of *two* days. Again, organize your work and your answers in a table.

3. For this model with these initial values, the graph of $I$ versus $t$ appears below. On a certain day—call it day $T$—the model says the number of susceptibles is $S(T) = 6,000$. Is day $T$ *before* or *after* the infection peaks? How can you tell?
[Provide a graph like the graph of $I$ on page 3 of the text.]

QUIZ 2 (on 1.2)
1. Find the equation of the straight line passing through the points $(-1, 5)$ and $(1, 9)$.

2. The volume $V$ of a quantity of gas at atmospheric pressure is a linear function of its temperature $T$ in degrees (centigrade). The gas occupies 500 cubic centimeters when the temperature is 10 degrees, and it fills 1300 cubic centimeters when the temperature is 12 degrees.
a) Find the slope = rate of change = multiplier of this linear function (include units).

b) If the temperature goes up by 10 degrees, by how much will the volume increase?
c) What temperature chanage would shrink the volume by 100 cubic centimeters?

QUIZ 3 (on 1.3)
1. The amount $R$ of radium (measured in grams) in a sample changes over time (measured in years). The rate $R'$ at which the radium changes into lead is proportional to the amount of radium present. Measurements show that $R' = -(1/2337)R$. Assume that you begin with a sample containing 10 grams of radium. Modify the attached copy of the program SIR to estimate the amount of radium in the sample after 5 years, using a time step of .5 year. Cross out lines of the program you don't need. Next to each line of the program that you need to change, write the appropriate variation.
[Provide a copy of SIR as on page 44 of the text.]

2. The variables $S, I, R, S', I', R'$ have their usual meanings in a model of the spread of an epidemic. Use the data sheet provided showing the estimated values of these variables over a 20 day period to answer the following questions. For each question, specify *which variable* you looked at to answer the question.
[Provide a printout of the values of the variables; label the columns with the variable names.]
a) On which day did the epidemic peak?
b) On which day did the largest number of persons fall ill?
c) On which day did the largest number of persons recover?

QUIZ 4 (on 3.1 and 3.2)
1. A question estimating a rate from a table of values, like problem 5 on page 94.

2. A question estimating a rate from a graph, like problem 9 on page 105.

QUIZ 5 (on 3.3 and 3.5)
1. A microscope equation problem, like one of those on page 120.

2. Given the graph of a function, sketch a graph of its derivative (like problem 5 on page 134).

QUIZ 6 (on 3.5)

1. Find formulas for the derivatives of the following functions [choose two like those in problem 6, page 134].

2. Write the microscope equation for $y = f(x)$ at $x = a$ and use it to estimate the value of the function at $x = a + h$ [choose one like problem 12 or 15 on page 136].

3. A motorized toy car is moving along a straight track. Its distance (in inches) from the starting point is given by $D = 3t^2 + 18\sqrt{t} + 5t$, where $t$ is the number of seconds the car has been moving. What is the velocity of the car after 9 seconds have passed? (include units)

QUIZ 7 (on 4.3)
1. Differentiate the following functions [choose two like those in problem 3 on page 210].

2. Check that $y = 300e^{.1t}$ is a solution to the initial value problem $y' = .1y$ and $y(0) = 300$.

3. The per capita growth rate of Afghanistan was .0216 in 1985, and the poulation then was 15 million people. The initial value problem $P' = .0216P$ and $P(0) = 15$ summarizes this information (assuming $t = 0$ in 1985). Write a formula for $P$ that is a solution to this initial value problem. (You don't need to check your solution.)

QUIZ 8 (on 4.4)
1. Determine the numerical value of each of the following [two like problem 1 on page 227].

2. Solve for $x$ in the following equation: $4e^{3x} = 15$.

3. Find $dy/dx$ for $y = \ln(2x^3 + 7x)$.

QUIZ 9 (on 5.3)
1. On the following graph of $f(x)$ on the interval $[a, b]$, mark with an $\times$ (directly on the graph) all critical points of $f$. [Provide a suitable graph.]

2. Find the critical points of f(x) [one or two like problem 7 on page 274, but without sketching the graph or analyzing the critical points].

# CALCULUS II
## SAMPLE MIDTERM 1, take-home portion
## (week 4 or 5, on chapters 7 and 11.1-11.3, 12.3)

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1. Do problem 12 on page 402 (soft spring).

2. Do problem 16 on page 403 (first integral for soft spring).

3. Do problem 27 on page 405 (predator prey—May model).

4. Do problem 23 on page 625 (finding the value of an accumulation function with a differential equation solver).

5. a) Obtain a formula for the Fourier sine transform of $f(x) = x$ on the interval $0 \le x \le 1$. That is, determine

$$F_s(\omega) = \int_0^1 x \sin(2\pi\omega x)\, dx\,.$$

b) Sketch the graph of $y = F_s(\omega)$ on the interval $0 \le \omega \le 5$. At what point $\omega$ on this interval does $F_s(\omega)$ attain its maximum? What is the maximum?

Calculus II
SAMPLE MIDTERM 1, in-class portion (time limit 75 minutes)
(week 4 or 5, on chapters 7 and 11.1-11.3, 12.3)

This is a "closed-book" test; no books or notes are allowed. You may use a calculator if you wish.

1. Questions on period and/or amplitude (like problems 1 or 4 on page 379).

2. a) Show that $E = \frac{1}{2}v^2 + \frac{1}{2}b^2x^2$ is a first integral for the linear spring

$$\begin{aligned} x' &= v \quad x(0) = a \\ v' &= -b^2 x \quad v(0) = p \end{aligned}$$

b) If $v = 0$, what is the value of $x$? (Your answer will be in terms of the parameters $a$, $b$ and $p$.)
c) Use the first integral to show the solution to this system of differential equations is periodic.

3. Find a formula for each of the following integrals.
[Include a selection of 5 or 6 indefinite and definite integrals drawn from 11.1-11.3.]

CALCULUS II
SAMPLE MIDTERM 2, take-home portion
(week 9, on chapters 8 and 11.1, 11.4, 12.1)

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1. The value of the integral

$$J = \int_1^\infty x^p \, dx$$

depends on the value of $p$. For example, if $p = -2$ then $J = 1$, while if $p = 1$ then $J = \infty$.
a) Determine all values of $p$ for which $J = \infty$.
b) Determine all values of $p$ for which $J$ is finite, and determine the value of $J$.

2. **Predator-prey interactions with harvesting.** The basic models of predator-prey interactions take no outside environmental factors into account. This question adds one such factor—the effects of certain human intervention, called **harvesting with equal effort**.

Consider a population of insects, such as moths, that damage an agricultural crop. Suppose the moths are kept under control naturally by a predator—in this case, spiders. But suppose the crop is sprayed repeatedly with an insecticide like DDT, to reduce the moth population even more. Does the strategy work? The problem is that both spiders and moths die from the poison, so the DDT may do more harm than good.

Here is another example, connected with Volterra's original studies of fish catches in the Adriatic sea. One species preys upon the other, and both are caught in nets. Continual fishing reduces the breeding population of each species. How does this affect the sizes of the two populations?

Proceed in the following way. Assume that the orignal predator-prey interaction is modelled by the standard equations in which the prey population grows logistically in the absence of predators:

$$\begin{aligned} x' &= ax\left(1 - \frac{x}{K}\right) - bxy \\ y' &= cxy - ey \end{aligned}$$

Here $x$ and $y$ are the sizes of the prey and the predator populations, respectively. Spraying or fishing (or, more generally, "harvesting") removes a fraction of the breeding populations of both species. In the absence of any more precise information, we assume it is the *same* fraction—$h$. (This is what we mean by harvesting with "equal effort.") Thus we must decrease $x'$ by $hx$ and $y'$ by $hy$. Making these modifications to the basic equations, we get

$$\begin{aligned} x' &= ax\left(1 - \frac{x}{K}\right) - bxy - hx \\ y' &= cxy - ey - hy. \end{aligned}$$

Now carry out an analysis of the following concrete problem:

$$\begin{aligned} x' &= .1x\left(1 - \frac{x}{2500}\right) - .005xy - hx \\ y' &= .00004xy - .04y - hy. \end{aligned}$$

a) Assume first that $h = 0$. That is, examine the model before any harvesting occurs. Make a sketch in the $(x, y)$-plane that shows where $x' = 0$ and where $y' = 0$. Find the equilibrium points of this system, and mark them clearly on your sketch.
b) Suppose we begin with $x = 2000$ and $y = 10$. What happens to $x$ and $y$ over time?
c) Now harvest the populations by setting $h = .02$. Make a new sketch in the $(x, y)$-plane that shows the new places where $x' = 0$ and $y' = 0$, and find the new equilibrium values of $x$ and $y$.
d) Suppose we again start with $x = 2000$ and $y = 10$ but have a harvesting effort of $h = .02$. What happens to $x$ and $y$ now?
e) The basic model (with $h = 0$) has an equilibrium with both predators and prey present (that is, $x > 0$ and $y > 0$). What happens to this equilibrium when harvesting occurs? At the new equilibrium, are there more or fewer predators, and are there more or fewer prey?
f) Does the model sugggest that fishing increases the proportion of predator species in relation to the prey, or is it the other way around?
g) Does the model support the use of DDT to reduce the population of moths? Explain your position clearly.

3. **The Richardson arms race model**. Between the two World Wars, the British physicist Lewis Fry Richardson devised a simple model to describe

the "arms races" carried on by various nations at various times. It concerns two aggressively hostile countries or alliances of countries; call them $X$ and $Y$. Let $x(t)$ represent the level of hostile activity of $X$ at any time $t$, and let $y(t)$ represent the same for $Y$. For example, take $x$ to be the annual armaments budget of $X$ in billions of dollars, and measure $t$ in years. There are diverse political pressures in $X$, some tending to make $x$ increase, some tending to make it decrease. The same is true in $Y$. Richardson concentrates on three sources of pressure:

1. The larger $y$ is, the greater the pressure to increase $x$. [An example: in the 1980's, the U.S. navy grew in size, partly because the Soviet navy grew.]
2. The larger $x$ is, the greater the pressure to reduce $x$. [An example: it is costly for the U.S. to maintain large garrisons around the world; some in Congress argue the money would be better spent supporting social programs at home.]
3. Citizens of $X$ may have a grievance against $Y$, independent of the size of either's armaments expenditures. [An example: Until quite recently, communism was widely considered to be repugnant in the U.S., and the Soviet Union was labelled "an evil empire."]

Richardson assumes that $x$ changes in response to each of these pressures, and he expresses the *rate* at which $x$ achanges by the following differential equation (representing, in order, each of the three sources of pressure listed above):

$$x' = ay - mx + g$$

A similar equation describes how $y$ changes:

$$y' = bx - ny + h$$

The values of the coefficients $a$, $b$, $m$, $n$, $g$ and $h$ are to be determined by the circumstances of a particular case.

Now, consider two possibilities: in the first, the type 2 pressures are larger than the type 1 pressures for both $X$ and $Y$. In other words, each prefers to spend its money on domestic needs rather than armaments. Here is a specific example:

$$\begin{aligned} x' &= .1y - .3x + 4 \\ y' &= .2x - .5y + 3 \end{aligned}$$

For the second possibility, suppose that the pressures for $X$ are reversed, while those for $Y$ are unchanged:

$$\begin{aligned} x' &= .3y - .1x + 4 \\ y' &= .2x - .5y + 3 \end{aligned}$$

a) For each of the two specific possibilities, determine whether there are hostility levels $x$ and $y$ for which the various pressures exactly balance, so that $x$ and $y$ do not change over time. If so, what are those levels?
b) Suppose the current hostility levels are $x = 10$ and $y = 20$. For each of the two specific possibilities, what are the hostility levels after one year and after two years? What happens in the long run?
c) What is the essential difference between the two possiblities presented above. Explain *how* the outcomes are different, and explain *why* they are different.

4. Problem 7(a)-(c) on pages 698-699 (one-dimensional random walk, using Stirling's formula).

Calculus II
SAMPLE MIDTERM 2, in-class portion (time limit 75 minutes)
(week 9, on chapters 8 and 11.1, 11.4, 12.1)

This is a "closed-book" test; no books or notes are allowed. You may use a calculator if you wish.

1. Consider the function $f(x) = x^5 - 4x^2 + 7$. Suppose that $g(x)$ is the inverse function for $f$. a) What is $f(1)$? What is $g(4)$?
b) What is $g'(4)$?

2. Use the method of separation of variable to find formulas for the solutions of the following differential equations. [Choose two or three like problem 2, page 653.]

3. **A warming liquid.** [Use a variation of problem 3 on page 653.]

4. **Logistic growth.** [Use a variation of problem 13 on page 656.]

5. Assume we have three initial value problems, each defining solutions $x = x(t)$ and $y = y(t)$. Below are graphs of their solutions $x(t)$ and $y(t)$ versus $t$, labelled (a), (b) and (c). There are three more graphs which show the trajectories corresponding to each of these solutions in the state space which is the $x, y$-plane. These are labelled (i), (ii) and (iii). Match each of the graphs (a)-(c) to the corresponding graph (i)-(iii), and explain your reasons for matching them as you do.
[Provide the six graphs; include, for example, a periodic solution, a solution that stabilizes, and a solution with damped oscillation.]

CALCULUS II
FINAL EXAMINATION, take-home portion
(on chapters 7-12, emphasizing 9 and 10)

This is an "open-book" test; you may consult freely your notes, homeworks, text, and any other books you wish; you may use a calculaator or a computer, and any programs available on a computer. However, you must not receive help, in any form, from anyone else. Make your responses brief but complete; explain your reasoning, and write clearly.

1. The function $E$ defined by

$$E(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is called the **error function** (and is important in mathematical statistics). Find its Taylor series centered at $a = 0$.

2. Use Taylor's theorem to estimate $\sqrt[3]{e}$ to 4 decimal places. Carefully justify your choice of the degree of the approximating Taylor polynomial.

3. A ball is dropped from a height of 1 meter onto a smooth surface. On each bounce, the ball rises to 60 percent of the height it reached on the previous bounce. Find the total distnace the ball travels. (Hint: be careful—draw a picture.)

4. Consider the function $f(x, y) = x^3 + y^3 - 12(x + y)$.
a) Find all critical points of $f$ and determine the type of each.
b) Sketch representative level curves of $f$ in the $(x, y)$-plane on the domain $-7 \le x \le 7$, $-7 \le y \le 7$. Be sure to include the zero level. Mark each level that you draw with the value of $f$ on that level.
c) Mark the location of each critical point of $f$ on your sketch. Indicate how the pattern of level curves around a critical point confirms its type, as you determined in part (a).

5. Use the function $f$ from problem 4 to construct the vector field grad $f$. Recall that the gradient field defines a dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial f}{\partial x}(x, y) \\ \frac{dy}{dt} &= \frac{\partial f}{\partial y}(x, y) \end{aligned}$$

a) Sketch representative vectors and trajectories for this vector field of the domain $-7 \leq x \leq 7$, $-7 \leq y \leq 7$.
Comment: Vectors and trajectories of grad $f$ should be perpendicular to the level curves of $f$ that you found in question 4. Is this so?
b) Find all the equilibrium points of grad $f$
c) Some trajectories of grad $f$ go to the local maximum of $f$. Mark on your sketch all starting points for trajectories that go to this local maximum.

Calculus II
Final Exam, in-class portion (time limit 2.5 hours)
(on chapters 7-12)

This is a "closed-book" test; no books or notes are allowed. You may use a calculator if you wish.

1. Evaluate each of the following integrals. [Choose an assortment of 5 or 6 from 11.1-11.4 and 11.7.]

2. Find a formula for the solution to the initial value problem

$$y' = 3t^2 y \quad y(0) = 5$$

3. a) Let $f(x) = e^{3x}$. Write a polynomial $P(x)$ of degree 2 satisfying $P(0) = f(0)$, $P'(0) = f'(0)$, and $P''(0) = f''(0)$.
b) Find a Taylor polyomial of degree 7 for

$$\int e^{x^2}\, dx$$

4. Determine the value of each of the following infinite sums.
[Choose two like parts of problems 1 or 2 on pages 588-589.]

5. Find the radius of convergence of each of the following.
[Choose two like parts of problems 10 or 11 on page 593.]

6. Write a sentence or two explaining why we studied the harmonic series.

7. Use Taylor polynomials to find an estimate for cos(.1) that is accurate to 3 decimal places. Explain how you know your answer has this accuracy (*other* than by comparing to what your calculator gives you for cos(.1)).

8. The variables $x$ and $y$ represent the sizes of two populations, one of which preys on the other. The way these two populations change over time is modelled by the differential equations

$$\begin{aligned} x' &= ax + bxy \\ y' &= cy - dxy \end{aligned}$$

where $a$, $b$, $c$ and $d$ are positive parameters. Which variable represents the predator and which the prey? How can you tell?

9. [A microscope equation problem like one of problems 34-56 on pages 496-499.]

10. The following questions refer to the contour plot of $z = f(x, y)$ below.
[Draw a contour plot with a local maximum and a saddle like the one on page 512; mark a point $P$ on the top edge.]
a) Mark the critical points of $f$.
b) Assuming that $f$ has a local maximum, draw several gradient vectors of $f$.
c) Draw on the contour map the *quickest* path down from the local maximum to the point $P$. What principle guides this path?

Calculus II
Sample Quizzes (15-20 minutes each)

QUIZ 1 (on 7.2)
1. What are the amplitude, period and frequency of $f(x) = 6\sin(5x)$?

2. Use the definition of the circular functions to explain the following.
a) $\sin(3\pi/2) = -1$
b) $\cos(-t) = \cos(t)$

QUIZ 2 (on 11.1)
1. Find a formula for each of the following indefinite integrals.
[Choose two like parts of problem 10 on page 621.]

2. Find the area under the curve $y = f(x)$ for $x$ between $a$ and $b$.
[Choose one like problem 24 or 25 on page 625.]

QUIZ 3 (on 11.2)
[Choose two like parts of problems 1 or 2 in on pages 632-633.]

QUIZ 4 (on 11.3)
[Choose two like parts of problem 1 on pages 637-638.]

QUIZ 5 (on 8.1)
[Choose a simple system of differential equations like problem 1a on page 424.]
a) Draw (in red) the set of points where $R' = 0$, and mark the regions where $R' > 0$ and $R' < 0$.
b) Draw (in blue) the set of points where $F' = 0$, and mark the regions where $F' > 0$ and $F' < 0$.
c) Mark any equilibrium points.
d) Sketch representative vectors of the vector field.

QUIZ 6 (on 11.1)
1. Sketch the graphs of $y = 2x + 5$ and its inverse on the same axes.

2. a) Sketch the graphs of $y = f(x) = x^2$ for $x \geq 0$ and its inverse $y = g(x)$ on the same axes.
b) What is the slope of $f$ at $x = 3$?
c) What is the slope of $g$ at $x = 9$? Answer this question *without* differentiating $g$.

QUIZ 7 (on 11.4)
1. Separation of variables—choose one like one part of problem 2 on page 653.

2. Partial fractions—choose one like one part of problem 10 on page 656.

QUIZ 8 (on 10.2)
1. Find a Taylor polynomial from the definition—choose one like problem 6 on page 548.

2. Find a Taylor polynomial for an anti-derivative—choose one like part of problem 1 on page 548.

QUIZ 9 (on 10.4)
Find a power series solution for a differential equation—choose one like part of problem 1 on page 571.

# Appendix C: Drill Sheets

Here is a list of the supplementary drill sheets, with an indication in parentheses of the section in the book they are appropriate for. You should feel free to photocopy them for your own use.


page

(1.2) Equations of Lines . . . . . . . . . . 107*

(1.2) Solution sheet for the preceding . . . . . . . . . . 108*

(1.2) A Drill Sheet on Lines (2 pages) . . . . . . . . . . 109

(1.2) One-a-Day Functions (2 pages) . . . . . . . . . . 111

(3.2) The Microscope Equation . . . . . . . . . . 113

(3.5) Differentiation Practice for 3.5 . . . . . . . . . . 114

(3.5 and 3.6) Differentiation Practice for 3.5 and 3.6 . . . . . . . . . . 115

(4.3) Thinking Exponentially . . . . . . . . . . 116

(5.1) Differentiation Practice for 5.1 (2 pages) . . . . . . . . . . 117

(6.4) Integration Practice for 6.4 . . . . . . . . . . 119

(11.1–11.3) Integration Practice for 11.1–11.3 . . . . . . . . . . 120


* *not in this document version*

**A Drill Sheet on Lines**

1. Circle the points that lie on the line $y = 5x + 2$:

$(0, 2)$ $(-\frac{2}{5}, 0)$ $(3, 17)$ $(17, 3)$ $(-1, -3)$ $(-\frac{1}{2}, -\frac{1}{2})$

2. The line which passes through the points $(1, 2)$ and $(7, 8)$ has slope ______ ; its equation is $y =$ ______ $x +$ ______ ; and it also passes through the points (16, ______) , (______ , .6), and (0, ______).

3. a) Specify a couple of other points on the line which passes through $(1, 1)$ and has slope 3: (______ , ______ ) , (______ , ______ ) .

b) What is the equation of this line? ____________________

4.
T F Lines are parallel if they have the same slope.
T F There is only one straight line passing through the points $(2, 17)$ and $(-3, 5)$.

5. At what point do the lines $y = 2x - 1$ and $y = x + 3$ intersect?

6. What is the equation of the line passing through the points $(5, 3)$ and $(10, 23)$ ?

7. a) A line with slope 3 passes through the point $(2, 5)$ ; it also passes through the points (3, ______) , (2.1, ______) , (1.8, ______) , and ($2 + a$, ______) . (You should do these **without** first finding the equation of the line.)

b) If, as it passes through the point $(1, 17)$ a curve is approximately a straight line with slope 3, approximately what will the second coordinate $y$ be if $(1.001, y)$ is to lie on the curve?

8. Knowing that the freezing temperature of water is 0° Celsius and 32° Fahrenheit, and that water boils at 100° Celsius and 212° Fahrenheit, see if you can figure out the formula which converts from one scale to another. (Hint: If you think of graphing Celsius temperatures along one axis and the corresponding Fahrenheit temperatures along the other axis, what sort of figure would you get? Do you know any points on this figure? Why is this exactly the same kind of problem as problem 6?)

9. Which of the following functions would you expect to have straight line graphs?

a. the cost of a taxi ride as a function of its length in miles
b. the height of a tree as a function of its age
c. the number of liters as a function of the number of pints
d. the circumference of a circle as a function of its radius
e. the area of a circle as a function of its radius
f. the population of the earth as a function of time
g. the logarithm of the population of the earth as a function of time
h. the value of a car as a function of its age

**One-a-Day Functions**

Here are some problems which have appeared in virtually every calculus book written over the last thirty years. For each problem you should

- (Except for problems 4 and 6.) Draw a simple picture labelling the different variables you are using and their relation to each other.
- Find the general functional relationship between the variables.
- Identify the domain of the function that makes physical sense.
- Sketch what you think the graph of the function looks like—looking at the edges of the domain is often helpful in this. Then use the computer to graph the function, and compare this with your sketch. With the computer, locate interesting points, such as maxima and minima and explain why these points might be of interest in the context of the problem.

1. A farmer has 90' of fencing out of which she plans to make a rectangular sheep pen up against the side of her barn, using the barn itself for one side of the pen and the fencing for the other three sides. (The barn is 100' long.) The total area $A$ of the pen will thus vary with the shape of the rectangle formed. Let $x$ be the length of the pen along the barn.

(Can you think of a simple way to find the value of $x$ giving the maximum area *without* using the computer (or calculus)?)

2. Suppose we want to make a topless box with a square bottom and rectangular sides, holding 80 cubic inches. The amount of material needed to make the box is essentially the same as the surface area of the box. How does the surface area of the box depend on the length of the bottom edge?

3. A lighthouse is 8 miles off shore, the coast line being straight. Fifteen miles up the beach is a town. Suppose the lighthouse keeper can row at the rate of 2 mph and can walk at the rate of 4 mph. If he rows his boat to some point on shore and then walks to town, how does the total time of the trip depend on where he beaches his boat?

4. A truck is to be driven 300 miles at a constant speed of $x$ mph. Speed laws require $30 \leq x \leq 65$ (on a rural highway). Assume diesel fuel costs 90 cents/gallon and is consumed at the rate of $2 + (x^2/200)$ gallons/hour. List some of the features of this model for the rate at which fuel is consumed and explain why the model is or is not reasonable. If the driver's wages are \$16/hour, how does the total cost of the trip vary with the speed $x$?

5. Suppose we want to make a poster containing 50 square inches of printed matter surrounded by a 3 inch border at the top and bottom, and a 2 inch border along each side. How does the total area of the poster depend on the width of the printed matter?

6. Suppose a manufacturer of plastic wombats figures out that the number $N$ of wombats she can sell at a price of $p$ cents each is given by the rule

$$N(p) = \begin{cases} 400(1-(p/50)) & \text{if } 0 < p < 50 \\ 0 & \text{if } 50 \leq p \end{cases}$$

and that the cost per wombat is given by $C(N) = (100N + 200)/(5N + 2)$ cents. List some of the key features of these models for $N(p)$ and $C(N)$ and explain why each model is or is not reasonable. Express her total costs and income in terms of the unit price $p$. How does her profit depend on $p$?

7. A piece of wire 30" long is cut into two pieces. One piece is bent into a square, the other into a circle. Express the sum of the areas of the two figures as a function of the length of the piece that is bent into a square.

8. We want to make a nice Roman window in the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is to be fifty feet, how does the area of the window depend on the diameter of the semicircle?

9. A 6 foot fence stands 8 feet away from a building wall. A ladder leaning on the top of the fence just reaches the building wall. Express the length of the ladder as a function of the distance from the foot of the ladder to the bottom of the fence.

**The Microscope Equation**

1. Suppose the volume of a right circular cylinder is:

$$V = h\pi r^2$$

a) Find $V$ when $h = 6$ cm and $r = 2$ cm.

b) If the height of the cylinder stays at 6 cm then $V(r) = 6\pi r^2$. Use $V(2)$ and $V'(2)$ to approximate $V(2.1)$.

c) What is the microscope equation that you used in b)? (i.e., what is the approximate change in $V$?)

d) What is the exact value for $V(2.1)$? What is the exact value for $\Delta V$?

2. The edge of a cube is measured to be 10 cm with a possible error of .02 cm. Use the microscope equation to find an upper bound on the error involved in calculating the volume of the cube to be $10^3 = 1000$ cubic cm.

3. Use the microscope equation to approximate the following expressions, using a nearby point where the functions is known.

a) $(9.06)^{\frac{1}{2}}$

b) $(3.07)^3$

c) $(48.8)^{\frac{1}{2}}$

d) $\dfrac{1}{1.98}$

e) $\dfrac{1}{31^{\frac{1}{5}}}$

f) $(.000063)^{\frac{1}{3}}$

g) $x^2 + 2x - 3$ at $x = 1.07$

h) $(15)^{.25}$

**Differentiation Practice for 3.5**

In problems 1–14, find formulas for the derivatives of the functions:

1. $x^5$

2. $\sqrt{x}$

3. $\dfrac{1}{x^3}$

4. $u^7$

5. $\sqrt[3]{w}$

6. $\dfrac{1}{z}$

7. $3x^3 - 5x^2 + 2$

8. $5u^3 - \pi u + 17$

9. $1.7w - \sqrt{w} + \frac{3}{w^2} - \pi$

10. $mx + b$ ($m$ and $b$ are constants)

11. $-\frac{g}{2}t^2 + v_0 t + d_0$ ($g$, $v_0$, and $d_0$ are constants)

12. $3\sin(x) + 2x^3 - 5$

13. $-\frac{1}{2}\cos u + \pi^2$

14. $\tan z - 3\sin z + 2z^5$

15. The formula $d = -16t^2 + 10t + 100$ gives distance travelled as a function of $t$. Find a formula for the velocity.

16. Use the derivative of $y = x^2$ to explain why the graph of $y$ has the shape it has: falling for $x < 0$, a minimum at $x = 0$, and increasing for $x > 0$.

17. (See the supplement to 1.2, problem 1) The formula for the area $A$ of the sheep pen as a function of the length $x$ along the barn is $A = 45x - \frac{1}{2}x^2$.

a) Find a formula for $dA/dx$

b) Use your formula from (a) to explain why $x = .45$ gives the maximum area.

18. $dy/dx = 6x^2$. What is a formula for $y$?

**Differentiation Practice for 3.5 and 3.6**

Differentiate the following functions:

1. $\cos 4w$

2. $\sin 6t$

3. $x^3 + 2\tan 5x$

4. $\sin(e^x)$

5. $5\cos(1-2u)$

6. $\tan(3z+2)$

7. $\sin(x^2+x+1)$

8. $e^{\tan x}$

9. $2\sin^6(3w)$

10. $\sqrt{\sin(2x)}$

11. $7\cos^3(4w)$

12. $\sqrt[3]{3w-5w^5}$

13. $\cos^2(\sqrt{1+2x})$

14. $1/\sqrt{2x+5}$

15. $8z^2\sqrt{z} - 5\sqrt[3]{z}$

16. $\dfrac{5}{(4x^2+25)^{2/3}}$

17. $\sqrt[4]{\dfrac{1}{t}}$

**Thinking Exponentially**

In the problems below, show all your work and your reasoning process.

1. Virtually all living things take up carbon as they grow. This carbon comes in two principal forms: normal, stable carbon—$C^{12}$—and radioactive carbon—$C^{14}$. $C^{14}$ decays into $C^{12}$ at a rate proportional to the amount of $C^{14}$ remaining. While the organism is alive, this lost $C^{14}$ is continually replenished. After the organism dies, though, the $C^{14}$ is no longer replaced, so the percentage of $C^{14}$ decreases exponentially over time. It is found that after 5730 years, half the original $C^{14}$ remains. If an archaeologist finds a bone with only 20% of the original $C^{14}$ present, how old is it?

2. The human population of the world appears to be growing exponentially. If there were 2.5 billion people in 1960, and 3.5 billion in 1980, how many will there be in 2010?

3. A cup of coffee at 80° C is placed in a room whose temperature is 20° C. After 10 minutes the temperature of the coffee is found to be 60°. If we assume that the rate at which the temperature difference changes is proportional to the difference, How long does it take for the temperature to reach 25°? (Thus we are assuming that if the temperature of the coffee at time $t$ is $T(t)$, then the difference $D(t) = T(t) - 20$ satisfies the condition $D'(t) = -kD(t)$ for some constant k.)

4. If bacteria increase at a rate proportional to the current number, how long will it take 1000 bacteria to increase to 10,000 if it takes them 17 minutes to increase to 2000?

5. Suppose sugar in water dissolves at a rate proportional to the amount left undissolved. If 40 lb. of sugar reduces to 12 lb. in 4 hours, how long will you have to wait until 99% of the sugar is dissolved?

6. Atmospheric pressure is a function of altitude. Assume that at any given altitude the rate of change of pressure with altitude is proportional to the pressure there. If the barometer reads 30 psi (pounds per square inch) at sea level and 24 psi at 6000 feet above sea level, how high are you when the barometer reads 20 psi?

**Differentiation Practice for 5.1**

Differentiate the following functions

7. $t^2 \cdot \cos t$

8. $\dfrac{x^3 - 2x}{\cos x}$

9. $\dfrac{\sin x}{x^3 - 5}$

10. $e^x(3x^5 + 7x^2)$

11. $3e^{-z} \cdot \sin(2\pi z)$

12. $\dfrac{(3t+5)^3}{(4-7t)^4}$

13. $(x^5 - 1)^3(7x + 5)^2$

14. $\sqrt{\dfrac{4t+7}{2t-3}}$

15. $\dfrac{\tan w}{w^2 + 1}$

16. $(3x^5 + 7)(2x - 1)^5$

17. $(15s + 3)(16s^2 + s^3)$

18. $\dfrac{x^2}{\sin(x)}$

Differentiation Practice for 5.1—page 2

19. $\sin(x^2)$

20. $\sin(x) + x^2$

21. $\dfrac{5x^2 + \ln(x)}{7\sqrt{x} + 5}$

22. $\dfrac{6e^{\cos(t)}}{5\sqrt[3]{t}}$

23. $3e^{7t} + t - 17$

24. Find partial derivatives of the following functions

a) $x^2 \cdot \sin(y)$

b) $\sin(x^2 y)$

c) $\ln(x + 2y)$

d) $e^{xy}$

e) $\dfrac{xy}{x^2y^3 + 1}$

25. Write the microscope equation for

a) $f(x) = 3x^2 + 5x - \sin(x)$ at $x = 0$

b) $f(x, y) = \dfrac{y + \cos(x)}{3xy + 5}$ at $(0, 2)$

26. Differentiate and *SIMPLIFY*:

a) $(\sqrt{x} - 1)e^{\sqrt{x}}$

b) $\ln(x + \sqrt{1 + x^2})$

**Integration Practice for 6.4**

1. Do each of the following two ways: (i) by hand (using antidifferention) and (ii) using RIEMANN

a) $\int_1^3 2x\, dx$

b) $\int_{-1}^2 -3x^2\, dx$

c) $\int_0^2 e^{3x}\, dx$

d) $\int_{\pi/2}^{3\pi/4} \cos x\, dx$

e) $\int_{\pi/8}^{\pi/4} \sin 2x\, dx$

f) $\int_2^3 (x^3 - \sqrt{x}) dx$

g) $\int_{-1}^2 (x^2 + 3e^x) dx$

2. Let $f(x) = \int_1^x \frac{1}{\sqrt{1+t^2}}\, dt$

a) What is $f(1)$? What is the *sign* of $f(x)$ for $x > 1$? What is the *sign* of $f(x)$ for $x < 1$?

b) Find a formula for $f'(x)$. Sketch a graph of $y = f'(x)$.

c) Find a formula for $f''(x)$. Sketch a graph of $y = f''(x)$.

d) Use the information in (a) - (c) to make a rough sketch of the graph of $y = f(x)$ on $[-4, 4]$.

**Integration Practice for 11.1-11.3**

1. $\int_2^4 \sqrt{x} + \frac{3}{x}\, dx$

2. $F(x) = \int_0^x t\sqrt{1+t^2}\, dt$. Find a formula for $F(x)$. Check that $F'(x) = x\sqrt{1+x^2}$.

3. $\int \frac{1}{t^2}\sqrt{1 - \frac{1}{t}}\, dx$

4. $\int \sin(2w)\, dw$

5. $\int \sqrt{\cos(x)}\, \sin(x)\, dx$

6. $\int 2e^{3x+2}\, dx$

7. $\int (e^x + e^{-x})^2\, dx$

8. $\int e^s \cos(e^s)\, ds$

9. $\int \frac{6}{3x+5}\, dx$

10. $\int \frac{1}{(2x+3)^2}\, dx$

11. $\int \frac{3x^2 - 2x}{x^3 - x^2}\, dx$

12. $\int \sqrt{\ln(x)}\, dx$

13. $\int_0^1 (e - e^y) y\, dx$

14. Find the *area* of the region under $y = xe^{-x}$ between 0 and 3.

15. Find the *volume* of the solid formed by rotating around the $x$-axis the region under $y = x^3$ between 0 and 2.

16. Find the *distance* travelled by a particle travelling with velocity $v = 2\cos(3t)$ for $0 \leq t \leq 1$.

17. Solve the initial value problem $y' = 3e^{2t}$, $y(0) = 5$.

18. Write a Riemann sum approximating the area in #15, using 3 subintervals of equal length and sampling midpoints.

19. Do the same for #17, using 2 subintervals.

# Appendix D: Supplementary Laboratory Exercises


page

(2.1) Exact Values of $S(1)$ and $S(10)$ using `SIRVALUE` (2 pages) .. 123

(4.4) Logs and Exponentials (2 pages) ........................... 125

(4.3) Graphing Exponential Functions ........................... 127

(4.5) Euler's Method for $y' = f(t)$ ............................... 128

(6.4) Accumulation Functions (3 pages) ......................... 129


*This chapter is not in this document version*

# Appendix E: Supplementary Programs

Teachers vary considerably in the point at which they switch from computer programs to commercial software. For those of you who like to continue the use of programming in your course here are some additional programs which are a bit more sophisticated than those in the text. They are written in TrueBasic, but can be readily adapted to other dialects. In addition to performing useful mathematical operations, these programs also contain features like drawing gridlines, using dialog boxes, and mouse input which make them more useful to students. These features can be left out if you want a bare-bones utility.

**Runge-Kutta Differential Equation Solvers** If you choose not to move to fancier software for solving differential equations, one of the most useful programs you will need is a numerical method that converges more rapidly than Euler's method. The most common improvement is the 4th order Runge-Kutta method. The method essentially involves a more sophisticated weighting of the slopes in the neighborhood of the current point to determine the direction and distance of the next step. Details can be found in most numerical methods books. We give two versions. The first (`RUNGE1.TRU`) solves a simple first order differential equation and plots the result in the $t$-$x$ plane. The second (`RUNGE2.TRU`) solves a pair of linked differential equations and plots the results in the $x$-$y$ phase plane.

The example used in this program is the logistic equation

$$\frac{dx}{dt} = .1\,x\,(1 - x/10000) \qquad \text{with } x(0) = 500.$$

While $x'$ only depends on the value of $x$ in this example, the method can be used unchanged for cases where $x'$ depends only on $t$, or on both $x$ and $t$. For this reason, the defining function is written as `func(x,t)`, even though only $x$ is involved in this case

RUNGE1.TRU

```
!!!!!!!!!!!!!  Specify initial values and functions
LET tinitial=0
LET tfinal=100
LET xmin = 0
LET xmax = 15000
LET numberofsteps = 1000
LET deltat = (tfinal-tinitial)/numberofsteps
DEF func(x,t) = .1 * x * (1 - x / 10000) !the rate of change function

!!!!!!!!!!!!!!  Subroutine to draw gridlines
SUB makegrid
SET COLOR "red"
    LET numbhlines = 10 !number of horizontal grid lines
    LET numbvlines = 20 !number of vertical grid lines
    LET hspacing = (tfinal - tinitial)/numbvlines
    LET vspacing = (xmax - xmin)/numbhlines
    SET TEXT justify "left","half"
    FOR k = 0 to numbhlines
       LET x = xmin + k*vspacing
       PLOT tinitial,x;tfinal,x
       PLOT TEXT, AT tfinal-.8*hspacing, x:str$(truncate(x,3))
    NEXT k
    SET TEXT justify "center", "bottom"
    FOR k = 0 to numbvlines
       LET t = tinitial + k*hspacing
       PLOT t,xmin;t,xmax
       PLOT TEXT, AT t, xmin+.2*vspacing:str$(truncate(t,1))
    NEXT k
END SUB

!!!!!!!!!!!!!!!  Set up the screen
SET WINDOW tinitial, tfinal, xmin, xmax
```

```
SET BACKGROUND COLOR "yellow"
CALL makegrid
SET COLOR "red"
PLOT tinitial,0; tfinal,0 !  Draws the t-axis (in red)
SET COLOR "blue" !Specify color for graph

!!!!!!!!!!!!!!!!  Run Runge-Kutta
LET t = tinitial
LET x = 500
PLOT t,x;
FOR K= 1 TO numberofsteps
    LET k1 = func(x,t)*deltat
    LET k2 = func(x + .5*k1,t + .5*deltat)*deltat
    LET k3 = func(x + .5*k2,t + .5*deltat)*deltat
    LET k4 = func(x + k3,t + deltat)*deltat
    LET deltax = (k1 + 2*k2 + 2*k3 + k4)/6
    LET t = t + deltat
    LET x = x + deltax
    PLOT t,x;
NEXT k
PRINT using "####.##### #####.#############":t,x
END
```

The example used solves the initial value problem

$$\frac{dx}{dt} = .15x(1 - .005x - .010y) \text{ and } \frac{dy}{dt} = .03y(1 - .004x - .005y),$$

with initial values

$$x(0) = 30 \text{ and } y(0) = 50.$$

Again, the defining rate equations could be specified to involve $t$ also.

RUNGE2.TRU

```
!!!!!!!!!!!!!!  Specify initial values, ranges, and functions
LET xmin = 0
LET xmax = 400
LET ymin = 0
LET ymax = 400
LET xinitial = 30
LET yinitial = 50
LET deltat = .1
LET numberofsteps =3000
!Enter the functions in the rate equations
DEF xprime(x,y) = .15*x*(1-.005*x-.010*y)
DEF yprime(x,y) = .03*y*(1-.004*x-.005*y)

!!!!!!!!!!!!!!!  Subroutine to draw gridlines
SUB makegrid
```

(Here you would type in the grid-making subroutine similar to that in RUNGE1)

```
END SUB

!!!!!!!!!!!!!!!!  Set up the screen and run program
LET hwidth = xmax - xmin
LET vwidth = ymax - ymin
SET WINDOW xmin-.05*hwidth, xmax+.05*hwidth,ymin-.05*vwidth,ymax+.05*vwidth
SET BACKGROUND COLOR "black"
CALL makegrid
!!!  Put an !  in front of this line if you don't want a grid
LET x = xinitial
LET y = yinitial
PLOT x,y;
FOR K= 1 TO numberofsteps
   LET k1 = xprime(x,y)*deltat
```

```
    LET h1 = yprime(x,y)*deltat
    LET k2 = xprime(x+.5*k1,y+.5*h1)*deltat
    LET h2 = yprime(x+.5*k1,y+.5*h1)*deltat
    LET k3 = xprime(x+.5*k2,y+.5*h2)*deltat
    LET h3 = yprime(x+.5*k2,y+.5*h2)*deltat
    LET k4 = xprime(x+k3,y+h3)*deltat
    LET h4 = yprime(x+k3,y+h3)*deltat
    LET deltax = (k1 + 2*k2 + 2*k3 + k4)/6
    LET deltay = (h1 + 2*h2 + 2*h3 + h4)/6
    LET y = y + deltay
    LET x = x + deltax
    PLOT x,y;
NEXT k
END
```

**Remark:** Note that `RUNGE2` can be immediately used to solve a second order differential equation in $x$ by defining $y = x'$.

The previous program—`Runge2`—had the disadvantage that you had to reenter the program each time you wanted to plot a new trajectory to type in the coordinates of the starting point. Here is a program that allows the student to point and click with a mouse to specify the starting point for a trajectory in the phase plane. There is also a dialog box that prints the coordinates of the starting point chosen and gives the student the option of plotting a new trajectory after each run. Instead of using mouse input, a dialog box could also be used to get the coordinates of the next trajectory.

PHASE.TRU

```
!!!!!!!!!!!!!!  Specify initial values, ranges, and functions
LET xmin = 0
LET xmax = 400
LET ymin = 0
LET ymax = 400
LET deltat = .1
LET numberofsteps =2000
DEF xprime(x,y) = .15*x*(1-.005*x-.010*y)
DEF yprime(x,y) = .03*y*(1-.004*x-.005*y)


!!!!!!!!!!!!!!!  Subroutine to draw gridlines
SUB makegrid
```

(Here again you would type in the grid-making subroutine similar to that in `RUNGE1`)

```
END SUB

!!!!!!!!!!!!!!!!  Runge-Kutta approximation
SUB runge
     PLOT x,y;
     FOR K= 1 TO numberofsteps
        LET k1 = xprime(x,y)*deltat !This would just be deltay in Euler's
method
        LET h1 = yprime(x,y)*deltat !This would just be deltayprime in
Euler's method
        LET k2 = xprime(x+.5*k1,y+.5*h1)*deltat
        LET h2 = yprime(x+.5*k1,y+.5*h1)*deltat
        LET k3 = xprime(x+.5*k2,y+.5*h2)*deltat
        LET h3 = yprime(x+.5*k2,y+.5*h2)*deltat
        LET k4 = xprime(x+k3,y+h3)*deltat
        LET h4 = yprime(x+k3,y+h3)*deltat
        LET deltax = (k1 + 2*k2 + 2*k3 + k4)/6
```

```
        LET deltay = (h1 + 2*h2 + 2*h3 + h4)/6
        LET y = y + deltay
        LET x = x + deltax
        PLOT x,y;
    NEXT k
END SUB

!!!!!!!!!!!!!!!!  Set up the screen
LET hwidth = xmax - xmin
LET vwidth = ymax - ymin
OPEN #1:screen 0,1,.88,1
OPEN #2:screen 0,1,0,.87
SET WINDOW xmin-.05*hwidth, xmax+.05*hwidth,ymin-.05*vwidth,ymax+.05*vwidth
SET BACKGROUND COLOR "black"
CALL makegrid
WINDOW #1
RANDOMIZE
LET ans$="y"
DO while ans$="y"
    WINDOW #2
    SET COLOR 1+int(15*rnd)
    GET POINT x,y
    WINDOW #1
    SET COLOR "white"
    PRINT "Initial values:  x= ";x; " y= ";y
    WINDOW #2
    CALL Runge
    PLOT
    WINDOW #1
    PRINT "Final values:  x= ";x;" y= ";y
    INPUT prompt "Another?  (y or n) ":ans$
    CLEAR
LOOP
PRINT "done"
END
```

Here's a program for quickly plotting a vector field (without arrowheads!), with the additional possibility of zooming in by a factor of 10 on the center point. You could combine this program with features of the previous program–PHASE–to get a program which would plot trajectories in a vector field.

VECFIELD.TRU

```
!!!!!!!!!!!!!  Specify initial values, ranges, and functions
LET xcenter = 50 !coordinates of center of screen
LET ycenter = 50
LET screenhalfwidth = 50
LET deltat = 1 !you may need to fiddle with this to get it started right
DEF xprime(x,y) = .15*x*(1-.005*x-.010*y) !the functions in the differential
equation
DEF yprime(x,y) = .03*y*(1-.004*x-.005*y)

!!!!!!!!!!!!!!  Subroutine to draw gridlines
SUB makegrid
```

(Here again you would type in the grid-making subroutine similar to that in RUNGE1)

```
END SUB

!!!!!  Subroutine to draw vector field
SUB arrows
    FOR j = 0 to 20
       FOR k = 0 to 20
          LET x = xmin + j*hwidth/20
          LET y = ymin + k*vwidth/20
          PLOT x,y;x+xprime(x,y)*deltat,y + yprime(x,y) * deltat
       NEXT k
    NEXT j
END SUB

!!!!!!!!!!!!!!!!  Set up the screen
OPEN #1:screen 0,1,.88,1
OPEN #2:screen 0,1,0,.87
LET ans$="y"
DO while ans$="y"
    WINDOW #2
    CLEAR
    LET xmin = xcenter - screenhalfwidth
    LET xmax = xcenter + screenhalfwidth
    LET ymin = ycenter - screenhalfwidth
```

```
    LET ymax = ycenter + screenhalfwidth
    LET hwidth = 2 * screenhalfwidth
    LET vwidth = 2 * screenhalfwidth
    SET WINDOW xmin-.05*hwidth, xmax+.15*hwidth,ymin-.05*vwidth,ymax+.05*vwidth
    SET BACKGROUND COLOR "black"
    IF screenhalfwidth > .001 then CALL makegrid
    SET COLOR "yellow"
    CALL arrows
    WINDOW #1
    SET COLOR "white"
    INPUT prompt "Another?  (y or n) ":ans$
    CLEAR
    LET deltat = deltat/10
    LET screenhalfwidth = screenhalfwidth/10
LOOP
PRINT "done"
END
```

# Appendix F: Solutions

*This chapter is not in this document version*