Lecture Notes on

C^*-Algebras and Quantum Mechanics

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1 Historical notes

1.1 Origins in functional analysis and quantum mechanics

The emergence of the theory of operator algebras may be traced back to (at least) three developments.

- The work of Hilbert and his pupils in Göttingen on integral equations, spectral theory, and infinite-dimensional quadratic forms (1904-);
- The discovery of quantum mechanics by Heisenberg (1925) in Göttingen and (independently) by Schrödinger in Zürich (1926);
- The arrival of John von Neumann in Göttingen (1926) to become Hilbert's assistant.

Hilbert's memoirs on integral equations appeared between 1904 and 1906. In 1908 his student E. Schmidt defined the space ℓ^2 in the modern sense. F. Riesz studied the space of all continuous linear maps on ℓ^2 (1912), and various examples of L^2-spaces emerged around the same time. However, the abstract concept of a Hilbert space was still missing.

Heisenberg discovered a form of quantum mechanics, which at the time was called 'matrix mechanics'. Schrödinger was led to a different formulation of the theory, which he called 'wave mechanics'. The relationship and possible equivalence between these alternative formulations of quantum mechanics, which at first sight looked completely different, was much discussed at the time. It was clear from either approach that the body of work mentioned in the previous paragraph was relevant to quantum mechanics.

Heisenberg's paper initiating matrix mechanics was followed by the 'Dreimännerarbeit' of Born, Heisenberg, and Jordan (1926); all three were in Göttingen at that time. Born was one of the few physicists of his time to be familiar with the concept of a matrix; in previous research he had even used infinite matrices (Heisenberg's fundamental equations could only be satisfied by infinite-dimensional matrices). Born turned to his former teacher Hilbert for mathematical advice. Hilbert had been interested in the mathematical structure of physical theories for a long time; his Sixth Problem (1900) called for the mathematical axiomatization of physics. Aided by his assistants Nordheim and von Neumann, Hilbert thus ran a seminar on the mathematical structure of quantum mechanics, and the three wrote a joint paper on the subject (now obsolete).

It was von Neumann alone who, at the age of 23, saw his way through all structures and mathematical difficulties. In a series of papers written between 1927-1932, culminating in his book *Mathematische Grundlagen der Quantenmechanik* (1932), he formulated the abstract concept of a Hilbert space, developed the spectral theory of bounded as well as unbounded normal operators on a Hilbert space, and proved the mathematical equivalence between matrix mechanics and wave mechanics. Initiating and largely completing the theory of self-adjoint operators on a Hilbert space, and introducing notions such as density matrices and quantum entropy, this book remains the definitive account of the mathematical structure of elementary quantum mechanics. (von Neumann's book was preceded by Dirac's *The Principles of Quantum Mechanics* (1930), which contains a heuristic and mathematically unsatisfactory account of quantum mechanics in terms of linear spaces and operators.)

1.2 Rings of operators (von Neumann algebras)

In one of his papers on Hilbert space theory (1929), von Neumann defines a **ring of operators** $\mathfrak{M}$ (nowadays called a **von Neumann algebra**) as a *-subalgebra of the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ (i.e, a subalgebra which is closed under the involution $A \rightarrow A^*$) that is closed (i.e., sequentially complete) in the weak operator topology. The latter may be defined by its notion of convergence: a sequence $\{A_n\}$ of bounded operators weakly converges to A when $(\Psi, A_n\Psi) \rightarrow (\Psi, A\Psi)$ for all $\Psi \in \mathcal{H}$. This type of convergence is partly motivated by quantum mechanics, in which $(\Psi, A\Psi)$ is the expectation value of the observable A in the state Ψ, provided that A is self-adjoint and Ψ has unit norm.

For example, $\mathfrak{B}(\mathcal{H})$ is itself a von Neumann algebra. (Since the weak topology is weaker than the uniform (or norm) topology on $\mathfrak{B}(\mathcal{H})$, a von Neumann algebra is automatically norm-closed as well, so that, in terminology to be introduced later on, a von Neumann algebra becomes a C^*-algebra when one changes the topology from the weak to the uniform one. However, the natural topology on a von Neumann algebra is neither the weak nor the uniform one.)

In the same paper, von Neumann proves what is still the basic theorem of the subject: a *-subalgebra $\mathfrak{M}$ of $\mathfrak{B}(\mathcal{H})$, containing the unit operator $\mathbb{I}$, is weakly closed iff $\mathfrak{M}'' = \mathfrak{M}$. Here the **commutant** $\mathfrak{M}'$ of a collection $\mathfrak{M}$ of bounded operators consists of all bounded operators which commute with all elements of $\mathfrak{M}$, and the bicommutant $\mathfrak{M}''$ is simply $(\mathfrak{M}')'$. This theorem is remarkable, in relating a topological condition to an algebraic one; one is reminded of the much simpler fact that a linear subspace $\mathcal{K}$ of $\mathcal{H}$ is closed iff $\mathcal{K}^{\perp\perp}$, where $\mathcal{K}^{\perp}$ is the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$.

Von Neumann's motivation in studying rings of operators was plurifold. His primary motivation probably came from quantum mechanics; unlike many physicists then and even now, he knew that all Hilbert spaces of a given dimension are isomorphic, so that one cannot characterize a physical system by saying that 'its Hilbert space of (pure) states is $L^2(\mathbb{R}^3)$'. Instead, von Neumann hoped to characterize quantum-mechanical systems by algebraic conditions on the observables. This programme has, to some extent been realized in algebraic quantum field theory (Haag and followers).

Among von Neumann's interest in quantum mechanics was the notion of entropy; he wished to define states of minimal information. When $\mathcal{H} = \mathbb{C}^n$ for $n < \infty$, such a state is given by the density matrix $\rho = \mathbb{I}/n$, but for infinite-dimensional Hilbert spaces this state may no longer be defined. Density matrices may be regarded as states on the von Neumann algebra$\mathfrak{B}(\mathcal{H})$ (in the sense of positive linear functionals which map $\mathbb{I}$ to 1). As we shall see, there are von Neumann algebras on infinite-dimensional Hilbert spaces which do admit states of minimal information that generalize $\mathbb{I}/n$, viz. the factors of type II_1 (see below).

Furthermore, von Neumann hoped that the divergences in quantum field theory might be removed by considering algebras of observables different from $\mathfrak{B}(\mathcal{H})$. This hope has not materialized, although in algebraic quantum field theory the basic algebras of local observables are, indeed, not of the form $\mathfrak{B}(\mathcal{H})$, but are all isomorphic to the unique hyperfinite factor of type III_1 (see below).

Motivation from a different direction came from the structure theory of algebras. In the present context, a theorem of Wedderburn says that a von Neumann algebra on a finite-dimensional Hilbert space is (isomorphic to) a direct sum of matrix algebras. Von Neumann wondered if this, or a similar result in which direct sums are replaced by direct integrals (see below), still holds when the dimension of $\mathcal{H}$ is infinite. (As we shall see, it does not.)

Finally, von Neumann's motivation came from group representations. Von Neumann's bicommutant theorem implies a useful alternative characterization of von Neumann algebras; from now on we add to the definition of a von Neumann algebra the condition that $\mathfrak{M}$ contains $\mathbb{I}$.

The commutant of a group $\mathfrak{U}$ of unitary operators on a Hilbert space is a von Neumann algebra, and, conversely, every von Neumann algebra arises in this way. In one direction, one trivially verifies that the commutant of any set of bounded operators is weakly closed, whereas the commutant of a set of bounded operators which is closed under the involution is a *-algebra. In the opposite direction, given $\mathfrak{M}$, one takes $\mathfrak{U}$ to be the set of all unitaries in $\mathfrak{M}'$.

This alternative characterization indicates why von Neumann algebras are important in physics: the set of bounded operators on $\mathcal{H}$ which are invariant under a given group representation $U(G)$ on $\mathcal{H}$ is automatically a von Neumann algebra. (Note that a given group $\mathfrak{U}$ of unitaries on $\mathcal{H}$ may be regarded as a representation U of $\mathfrak{U}$ itself, where U is the identity map.)

1.3 Reduction of unitary group representations

The (possible) reduction of $U(G)$ is determined by the von Neumann algebras $U(G)''$ and $U(G)'$. For example, U is irreducible iff $U(G)' = \mathbb{C}\mathbb{I}$ (Schur's lemma). The representation U is called **primary** when $U(G)''$ has a trivial center, that is, when $U(G)'' \cap U(G)' = \mathbb{C}\mathbb{I}$. When G is compact, so that U is discretely reducible, this implies that U is a multiple of a fixed irreducible representation U_γ on a Hilbert space $\mathcal{H}_\gamma$, so that $\mathcal{H} \simeq \mathcal{H}_\gamma \otimes \mathcal{K}$, and $U \simeq U_\gamma \otimes \mathbb{I}_\mathcal{K}$.

When G is not compact, but still assumed to be locally compact, unitary representations may be reducible without containing any irreducible subrepresentation. This occurs already in the simplest possible cases, such as the regular representation of $G = \mathbb{R}$ on $\mathcal{H} = L^2(\mathbb{R})$; that is, one puts $U(x)\Psi(y) = \Psi(y - x)$. The irreducible would-be subspaces of $\mathcal{H}$ would be spanned by the vectors $\Psi_p(y) := \exp(ipy)$, but these functions do not lie in $L^2(\mathbb{R})$. The solution to this problem was given by von Neumann in a paper published in 1949, but written in the thirties (the ideas in it must have guided von Neumann from at least 1936 on).

Instead of decomposing $\mathcal{H}$ as a direct sum, one should decompose it as a **direct integral**. (To do so, one needs to assume that $\mathcal{H}$ is separable.) This means that firstly one has a measure space (Λ, μ) and a family of Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda\in\Lambda}$. A **section** of this family is a function $\Psi : \Lambda \to \{\mathcal{H}_\lambda\}_{\lambda\in\Lambda}$ for which $\Psi(\lambda) \in \mathcal{H}_\lambda$. To define the direct integral of the $\mathcal{H}_\lambda$ with respect to the measure μ, one needs a sequence of sections $\{\Psi_n\}$ satisfying the two conditions that firstly the function $\lambda \to (\Psi_n(\lambda), \Psi_m(\lambda))_\lambda$ be measurable for all n, m, and secondly that for each fixed λ the Ψ_n span $\mathcal{H}_\lambda$. There then exists a unique maximal linear subspace Γ_0 of the space Γ of all sections which contains all Ψ_n, and for which all sections $\lambda \to (\Psi_\lambda, \Phi_\lambda)_\lambda$ are measurable.

For $\Psi, \Phi \in \Gamma_0$ it then makes sense to define

$$(\Psi, \Phi) := \int_\Lambda d\mu(\lambda)\, (\Psi(\lambda), \Phi(\lambda))_\lambda.$$

The **direct integral**

$$\int_\Lambda^\oplus d\mu(\lambda)\, \mathcal{H}_\lambda$$

is then by definition the subset of Γ_0 of functions Ψ for which $(\Psi, \Psi) < \infty$. When Λ is discrete, the direct integral reduces to a direct sum.

An operator A on this direct integral Hilbert space is said to be **diagonal** when

$$A\Psi(\lambda) = A_\lambda \Psi(\lambda)$$

for some (suitably measurable) family of operators A_λ on $\mathcal{H}_\lambda$. We then write

$$A = \int_\Lambda^\oplus d\mu(\lambda)\, A_\lambda.$$

Thus a unitary group representation $U(G)$ on $\mathcal{H}$ is diagonal when

$$U(x)\Psi(\lambda) = U_\lambda(x)\Psi_\lambda$$

for all $x \in G$, in which case we, of course, write

$$U = \int_\Lambda^\oplus d\mu(\lambda)\, U_\lambda.$$

Reducing a given representation U on some Hilbert space then amounts to finding a unitary map V between $\mathcal{H}$ and some direct integral Hilbert space, such that each $\mathcal{H}_\lambda$ carries a representation U_λ, and $VU(x)V^*$ is diagonal in the above sense, with $A_\lambda = U_\lambda(x)$. When $\mathcal{H}$ is separable, one may always reduce a unitary representation in such a way that the U_λ occurring in the decomposition are primary, and this **central decomposition** of U is essentially unique.

To completely reduce U, one needs the U_λ to be irreducible, so that Λ is the space $\hat{G}$ of all equivalence classes of irreducible unitary representations of G. Complete reduction therefore calls for a further direct integral decomposition of primary representations; this will be discussed below.

For example, one may take $\Lambda = \mathbb{R}$ with Lebesgue measure μ, and take the sequence $\{\Psi_n\}$ to consist of a single strictly positive measurable function. This leads to the direct integral decomposition

$$L^2(\mathbb{R}) = \int_\mathbb{R}^\oplus dp\, \mathcal{H}_p,$$

in which each $\mathcal{H}_p$ is $\mathbb{C}$. To reduce the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$, one simply performs a Fourier transform $V : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, i.e.,

$$V\Psi(p) = \int_{\mathbb{R}} dy\, e^{-ipy}\Psi(y).$$

This leads to $VU(x)V^*\Psi(p) = \exp(ipx)\Psi(p)$, so that U has been diagonalized: the $U_\lambda(x)$ above are now the one-dimensional operators $U_p(x) = \exp(ipx)$ on $\mathcal{H}_p = \mathbb{C}$. We have therefore completely reduced U.

As far as the reduction of unitary representations is concerned, there exist two radically different classes of locally compact groups (the class of all locally compact groups includes, for example, all finite-dimensional Lie groups and all discrete groups). A primary representation is said to be of **type I** when it may be decomposed as the direct sum of irreducible subrepresentations; these subrepresentations are necessarily equivalent. A locally compact group is said to be **type I** or **tame** when every primary representation is a multiple of a fixed irreducible representation; in other words, a group is type I when all its primary representations are of type I. If not, the group is called **non-type I** or **wild**. An example of a wild group, well known to von Neumann, is the free group on two generators. Another example, discovered at a later stage, is the group of matrices of the form

$$\begin{pmatrix} e^{it} & 0 & z \\ 0 & e^{i\alpha t} & w \\ 0 & 0 & 1 \end{pmatrix},$$

where α is an irrational real number, $t \in \mathbb{R}$, and $z, w \in \mathbb{C}$.

When G is wild, curious phenomena may occur. By definition, a wild group has primary unitary representations which contain no irreducible subrepresentations. More bizarrely, representations of the latter type may be decomposed in two alternative ways

$$U = \int_{\hat{G}}^{\oplus} d\mu_1(\gamma)\, U_\gamma = \int_{\hat{G}}^{\oplus} d\mu_2(\gamma)\, U_\gamma,$$

where the measures μ_1 and μ_2 are disjoint (that is, supported by disjoint subsets of $\hat{G}$).

A reducible primary representation U may always be decomposed as $U = U_h \oplus U_h$. In case that U is not equivalent to U_h, and U is not of type I, it is said to be a representation of **type II**. When U is neither of type I nor of type II, it is of **type III**. In that case U is equivalent to U_h; indeed, all (proper) subrepresentations of a primary type III representation are equivalent.

1.4 The classification of factors

Between 1936 and 1953 von Neumann wrote 5 lengthy, difficult, and profound papers (3 of which were in collaboration with Murray) in which the study of his 'rings of operators' was initiated. (According to I.E. Segal, these papers form 'perhaps the most original major work in mathematics in this century'.)

The analysis of Murray and von Neumann is based on the study of the projections in a von Neumann algebra $\mathfrak{M}$ (a projection is an operator p for which $p^2 = p^* = p$); indeed, $\mathfrak{M}$ is generated by its projections. They noticed that one may define an equivalence relation $\sim$ on the set of all projections in $\mathfrak{M}$, in which $p \sim q$ iff there exists a partial isometry V in $\mathfrak{M}$ such that $V^*V = p$ and $VV^* = q$. When $\mathfrak{M} \subseteq \mathfrak{B}(\mathcal{H})$, the operator V is unitary from $p\mathcal{H}$ to $q\mathcal{H}$, and annihilates $p\mathcal{H}^\perp$. Hence when $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ one has $p \sim q$ iff $p\mathcal{H}$ and $q\mathcal{H}$ have the same dimension, for in that case one may take any V with the above properties.

An equivalent characterization of $\sim$ arises when we write $\mathfrak{M} = U(G)'$ for some unitary representation U of a group G (as we have seen, this always applies); then $p \sim q$ iff the subrepresentations pU and qU (on $p\mathcal{H}$ and $q\mathcal{H}$, respectively), are unitarily equivalent.

Moreover, Murray and von Neumann define a partial orderering on the collection of all projections in $\mathfrak{M}$ by declaring that $p \leq q$ when $pq = p$, that is, when $p\mathcal{H} \subseteq q\mathcal{H}$. This induces a partial orderering on the set of equivalence classes of projections by putting $[p] \leq [q]$ when the equivalence classes $[p]$ and $[q]$ contain representatives $\tilde{p}$ and $\tilde{q}$ such that $\tilde{p} \leq \tilde{q}$. For $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ this actually defines a total ordering on the equivalence classes, in which $[p] \leq [q]$ when $p\mathcal{H}$ has the same dimension as $q\mathcal{H}$; as we just saw, this is independent of the choice of $p \in [p]$ and $q \in [q]$.

More generally, Murray and von Neumann showed that the set of equivalence classes of projections in $\mathfrak{M}$ is totally ordered by $\leq$ whenever $\mathfrak{M}$ is a **factor**. A von Neumann algebra $\mathfrak{M}$ is a factor when $\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}\mathbb{I}$; when $\mathfrak{M} = U(G)'$ this means that $\mathfrak{M}$ is a factor iff the representation U is primary. The study of von Neumann algebras acting on separable Hilbert spaces $\mathcal{H}$ reduces to the study of factors, for von Neumann proved that every von Neumann algebra $\mathfrak{M} \subseteq \mathfrak{B}(\mathcal{H})$ may be uniquely decomposed, as in

$$\begin{aligned} \mathcal{H} &= \int_{\Lambda}^{\oplus} d\mu(\lambda)\, \mathcal{H}_{\lambda}; \\ \mathfrak{M} &= \int_{\Lambda}^{\oplus} d\mu(\lambda)\, \mathfrak{M}_{\lambda}, \end{aligned}$$

where (almost) each $\mathfrak{M}_{\lambda}$ is a factor. For $\mathfrak{M} = U(G)'$ the decomposition of $\mathcal{H}$ amounts to the central decomposition of $U(G)$.

As we have seen, for the factor $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ the dimension d of a projection is a complete invariant, distinguishing the equivalence classes $[p]$. The dimension is a function from the set of all projections in $\mathfrak{B}(\mathcal{H})$ to $\mathbb{R}^{+} \cup \infty$, satisfying

1. $d(p) > 0$ when $p \neq 0$, and $d(0) = 0$;
2. $d(p) = d(q)$ iff $[p] \sim [q]$;
3. $d(p+q) = d(p) + d(q)$ when $pq = 0$ (i.e., when $p\mathcal{H}$ and $q\mathcal{H}$ are orthogonal;
4. $d(p) < \infty$ iff p is finite.

Here a projection in $\mathfrak{B}(\mathcal{H})$ is called finite when $p\mathcal{H}$ is finite-dimensional. Murray and von Neumann now proved that on any factor $\mathfrak{M}$ (acting on a separable Hilbert space) there exists a function d from the set of all projections in $\mathfrak{M}$ to $\mathbb{R}^{+} \cup \infty$, satisfying the above properties. Moreover, d is unique up to finite rescaling. For this to be the possible, Murray and von Neumann define a projection to be **finite** when it is not equivalent to any of its (proper) sub-projections; an **infinite** projection is then a projection which has proper sub-projections to which it is equivalent. For $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ this generalized notion of finiteness coincides with the usual one, but in other factors all projections may be infinite in the usual sense, yet some are finite in the sense of Murray and von Neumann. One may say that, in order to distinguish infinite-dimensional but inequivalent projections, the dimension function d is a 'renormalized' version of the usual one.

A first classification of factors (on a separable Hilbert space) is now performed by considering the possible finiteness of its projections and the range of d. A projection p is called **minimal** or **atomic** when there exists no $q < p$ (i.e., $q \leq p$ and $q \neq p$). One then has the following possibilities for a factor $\mathfrak{M}$.

- **type** $\mathbf{I}_n$, where $n < \infty$: $\mathfrak{M}$ has minimal projections, all projections are finite, and d takes the values $\{0, 1, \ldots, n\}$. A factor of type I_n is isomorphic to the algebra of $n \times n$ matrices.
- **type** $\mathbf{I}_{\infty}$: $\mathfrak{M}$ has minimal projections, and d takes the values $\{0, 1, \ldots, \infty\}$. Such a factor is isomorphic to $\mathfrak{B}(\mathcal{H})$ for separable infinite-dimensional $\mathcal{H}$.
- **type** $\mathbf{II}_1$: $\mathfrak{M}$ has no minimal projections, all projections are infinite-dimensional in the usual sense, and $\mathbb{I}$ is finite. Normalizing d such that $d(\mathbb{I}) = 1$, the range of d is the interval $[0, 1]$.
- **type** $\mathbf{II}_{\infty}$: $\mathfrak{M}$ has no minimal projections, all nonzero projections are infinite-dimensional in the usual sense, but $\mathfrak{M}$ has finite-dimensional projections in the sense of Murray and von Neumann, and $\mathbb{I}$ is infinite. The range of d is $[0, \infty]$.

- **type III**: $\mathfrak{M}$ has no minimal projections, all nonzero projections are infinite-dimensional and equivalent in the usual sense as well as in the sense of Murray and von Neumann, and d assumes the values $\{0, \infty\}$.

With $\mathfrak{M} = U(G)'$, where, as we have seen, the representation U is primary iff $\mathfrak{M}$ is a factor, U is of a given type iff $\mathfrak{M}$ is of the same type.

One sometimes says that a factor is **finite** when $\mathbb{I}$ is finite (so that $d(\mathbb{I}) < \infty$); hence type I_n and type II_1 factors are finite. Factors of type I_∞ and II_∞ are then called **semifinite**, and type III factors are **purely infinite**.

It is hard to construct an example of a II_1 factor, and even harder to write down a type III factor. Murray and von Neumann managed to do the former, and von Neumann did the latter by himself, but only 5 years after he and Murray had recognized that the existence of type III factors was a logical possibility. However, they were unable to provide a further classification of all factors, and they admitted having no tools to study type III factors.

Von Neumann was fascinated by II_1 factors. In view of the range of d, he believed these defined some form of continuous geometry. Moreover, the existence of a II_1 factor solved one of the problems that worried him in quantum mechanics. For he showed that on a II_1 factor $\mathfrak{M}$ the dimension function d, defined on the projections in $\mathfrak{M}$, may be extended to a positive linear functional tr on $\mathfrak{M}$, with the property that $tr(UAU^*) = tr(A)$ for all $A \in \mathfrak{M}$ and all unitaries U in $\mathfrak{M}$. This 'trace' satisfies $tr(\mathbb{I}) = d(\mathbb{I}) = 1$, and gave von Neumann the state of minimal information he had sought. Partly for this reason he believed that physics should be described by II_1 factors.

At the time not many people were familiar with the difficult papers of Murray and von Neumann, and until the sixties only a handful of mathematicians worked on operator algebras (e.g., Segal, Kaplansky, Kadison, Dixmier, Sakai, and others). The precise connection between von Neumann algebras and the decomposition of unitary group representations envisaged by von Neumann was worked out by Mackey, Mautner, Godement, and Adel'son-Vel'skii.

In the sixties, a group of physicists, led by Haag, realized that operator algebras could be a useful tool in quantum field theory and in the quantum statistical mechanics of infinite systems. This has led to an extremely fruitful intercation between physics and mathematics, which has helped both subjects. In particular, in 1957 Haag observed a formal similarity between the collection of all von Neumann algebras on a Hilbert space and the set of all causally closed subsets of Minkowksi space-time. Here a region $\mathcal{O}$ in space-time is said to be **causally closed** when $\mathcal{O}^{\perp\perp} = \mathcal{O}$, where $\mathcal{O}^{\perp}$ consists of all points that are spacelike separated from $\mathcal{O}$. The operation $\mathcal{O} \rightarrow \mathcal{O}^{\perp}$ on causally closed regions in space-time is somewhat analogous to the operation $\mathfrak{M} \rightarrow \mathfrak{M}'$ on von Neumann algebras. Thus Haag proposed that a quantum field theory should be defined by a **net of local observables**; this is a map $\mathcal{O} \rightarrow \mathfrak{M}(\mathcal{O})$ from the set of all causally closed regions in space-time to the set of all von Neumann algebras on some Hilbert space, such that $\mathfrak{M}(\mathcal{O}_1) \subseteq \mathfrak{M}(\mathcal{O}_2)$ when $\mathcal{O}_1 \subseteq \mathcal{O}_2$, and $\mathfrak{M}(\mathcal{O})' = \mathfrak{M}(\mathcal{O}^{\perp})$.

This idea initiated **algebraic quantum field theory**, a subject that really got off the ground with papers by Haag's pupil Araki in 1963 and by Haag and Kastler in 1964. From then till the present day, algebraic quantum field theory has attracted a small but dedicated group of mathematical physicists. One of the result has been that in realistic quantum field theories the local algebras $\mathfrak{M}(\mathcal{O})$ must all be isomorphic to the unique hyperfinite factor of type III_1 discussed below. (Hence von Neumann's belief that physics should use II_1 factors has not been vindicated.)

A few years later (1967), an extraordinary coincidence took place, which was to play an essential role in the classification of factors of type III. On the mathematics side, Tomita developed a technique in the study of von Neumann algebras, which nowadays is called **modular theory** or **Tomita-Takesaki theory** (apart from clarifying Tomita's work, Takesaki made essential contributions to this theory). Among other things, this theory leads to a natural time-evolution on certain factors. On the physics side, Haag, Hugenholtz, and Winnink characterized states of thermal equilibrium of infinite quantum systems by an algebraic condition that had previously been introduced in a heuristic setting by Kubo, Martin, and Schwinger, and is therefore called the **KMS condition**. This condition leads to type III factors equipped with a time-evolution which coincided with the one of the Tomita-Takesaki theory.

In the hands of Connes, the Tomita-Takesaki theory and the examples of type III factors provided by physicists (Araki, Woods, Powers, and others) eventually led to the classification of all **hyperfinite factors** of type II and III (the complete classification of all factors of type I is already given by the list presented earlier). These are factors containing a sequence of finite-dimensional subalgebras $\mathfrak{M}_1 \subset \mathfrak{M}_2 \ldots \subset \mathfrak{M}$, such that $\mathfrak{M}$ is the weak closure of $\cup_n \mathfrak{M}_n$. (Experience shows that all factors playing a role in physics are hyperfinite, and many natural examples of factors constructed by purely mathematical techniques are hyperfinite as well.) The work of Connes, for which he was awarded the Fields Medal in 1982, and others, led to the following classification of hyperfinite factors of type II and III (up to isomorphism):

- There is a unique hyperfinite factor of type II_1. (In physics this factor occurs when one considers KMS-states at infinite temperature.)
- There is a unique hyperfinite factor of type II_∞, namely the tensor product of the hyperfinite II_∞-factor with $\mathfrak{B}(\mathcal{K})$, for an infinite-dimensional separable Hilbert space $\mathcal{K}$.
- There is a family of type III factors, labeled by $\lambda \in [0,1]$. For $\lambda \neq 0$ the factor of type III_λ is unique. There is a family of type III_0 factors, which in turn is has been classified in terms of concepts from ergodic theory.

As we have mentioned already, the unique hyperfinite III_1 factor plays a central role in algebraic quantum field theory. The unique hyperfinite II_1 factor was crucial in a spectacular development, in which the theory of inclusions of II_1 factors was related to knot theory, and even led to a new knot invariant. In 1990 Jones was awarded a Fields medal for this work, the second one to be given to the once obscure field of operator algebras.

1.5 C^*-algebras

In the midst of the Murray-von Neumann series of papers, Gel'fand initiated a separate development, combining operator algebras with the theory of Banach spaces. In 1941 he defined the concept of a Banach algebra, in which multiplication is (separately) continuous in the norm-topology. He proceeded to define an intrinsic spectral theory, and proved most basic results in the theory of commutative Banach algebras.

In 1943 Gel'fand and Neumark defined what is now called a C^*-algebra (some of their axioms were later shown to be superfluous), and proved the basic theorem that each C^*-algebra is isomorphic to the norm-closed *-algebra of operators on a Hilbert space. Their paper also contained the rudiments of what is now called the GNS construction, connecting states to representations. In its present form, this construction is due to Segal (1947), a great admirer of von Neumann, who generalized von Neumann's idea of a state as a positive normalized linear functional from $\mathfrak{B}(\mathcal{H})$ to arbitrary C^*-algebras. Moreover, Segal returned to von Neumann's motivation of relating operator algebras to quantum mechanics.

As with von Neumann algebras, the sixties brought a fruitful interaction between C^*-algebras and quantum physics. Moreover, the theory of C^*-algebras turned out to be interesting both for intrinsic reasons (structure and representation theory of C^*-algebras), as well as because of its connections with a number of other fields of mathematics. Here the strategy is to take a given mathematical structure, try and find a C^*-algebra which encodes this structure in some way, and then obtain information about the structure through proving theorems about the C^*-algebra of the structure.

The first instance where this led to a deep result which has not been proved in any other way is the theorem of Gel'fand and Raikov (1943), stating that the unitary representations of a locally compact group separate the points of the group (that is, for each pair $x \neq y$ there exists a unitary representation U for which $U(x) \neq U(y)$. This was proved by constructing a C^*-algebra $C^*(G)$ of the group G, showing that representations of $C^*(G)$ bijectively correspond to unitary representations of G, and finally showing that the states of an arbitrary C^*-algebra $\mathfrak{A}$ separate the elements of $\mathfrak{A}$.

Other examples of mathematical structures that may be analyzed through an appropriate C^*-algebra are group actions, groupoids, foliations, and complex domains. The same idea lies at the basis of **non-commutative geometry** and **non-commutative topology**. Here the starting point is another theorem of Gel'fand, stating that any commutative C^*-algebra (with unit) is isomorphic to $C(X)$, where X is a compact Hausdorff space. The strategy is now that the basic tools in the topology of X, and, when appropriate, in its differential geometry, should be translated into tools pertinent to the C^*-algebra $C(X)$, and that subsequently these tools should be generalized to non-commutative C^*-algebras.

This strategy has been successful in K-theory, whose non-commutative version is even simpler than its usual incarnation, and in (de Rham) cohomology theory, whose non-commutative version is called **cyclic cohomology**. Finally, homology, cohomology, K-theory, and index theory haven been unified and made non-commutative in the **KK-theory** of Kasparov. The basic tool in KK-theory is the concept of a **Hilbert C^*-module**, which we will study in detail in these lectures.

2 Elementary theory of C^*-algebras

2.1 Basic definitions

All vector spaces will be defined over $\mathbb{C}$, and all functions will be $\mathbb{C}$-valued, unless we explicitly state otherwise. The abbreviation 'iff' means 'if and only if', which is the same as the symbol $\Leftrightarrow$. An equation of the type $a := b$ means that a is by definition equal to b.

Definition 2.1.1 *A* **norm** *on a vector space $\mathcal{V}$ is a map $\| \, \| : \mathcal{V} \to \mathbb{R}$ such that*

1. $\| v \| \geq 0$ *for all* $v \in \mathcal{V}$;
2. $\| v \| = 0$ *iff* $v = 0$;
3. $\| \lambda v \| = |\lambda| \; \| v \|$ *for all* $\lambda \in \mathbb{C}$ *and* $v \in \mathcal{V}$;
4. $\| v + w \| \leq \| v \| + \| w \|$ *(triangle inequality).*

A norm on $\mathcal{V}$ defines a metric d on $\mathcal{V}$ by $d(v, w) := \| v - w \|$. A vector space with a norm which is complete in the associated metric (in the sense that every Cauchy sequence converges) is called a **Banach space**. *We will denote a generic Banach space by the symbol $\mathcal{B}$.*

The two main examples of Banach spaces we will encounter are Hilbert spaces and certain collections of operators on Hilbert spaces.

Definition 2.1.2 *A* **pre-inner product** *on a vector space $\mathcal{V}$ is a map $(\, , \,) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that*

1. $(\lambda_1 v_1 + \lambda_2 v_2, \mu_1 w_1 + \mu_2 w_2) = \overline{\lambda_1}\mu_1 (v_1, w_1) + \overline{\lambda_1}\mu_2 (v_1, w_2) + \overline{\lambda_2}\mu_1 (v_2, w_1) + \overline{\lambda_2}\mu_2 (v_2, w_2)$ *for all* $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ *and* $v_1, v_2, w_1, w_2 \in \mathcal{V}$;
2. $(v, v) \geq 0$ *for all* $v \in \mathcal{V}$.

An equivalent set of conditions is

1. $\overline{(v, w)} = (w, v)$ *for all* $v, w \in \mathcal{V}$;
2. $(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 (v, w_1) + \lambda_2 (v, w_2)$ *for all* $\lambda_1, \lambda_2 \in \mathbb{C}$ *and* $v, w_1, w_2 \in \mathcal{V}$;
3. $(v, v) \geq 0$ *for all* $v \in \mathcal{V}$.

A pre-inner product for which $(v, v) = 0$ iff $v = 0$ is called an **inner product**.

The equivalence between the two definitions of a pre-inner product is elementary; in fact, to derive the first axiom of the second characterization from the first set of conditions, it is enough to assume that $(v, v) \in \mathbb{R}$ for all v (use this reality with $v \to v + iw$). Either way, one derives the **Cauchy-Schwarz inequality**

$$|(v, w)|^2 \leq (v, v)(w, w), \tag{2.1}$$

for all $v, w \in \mathcal{V}$. Note that this inequality is valid even when $(\, , \,)$ is not an inner product, but merely a pre-inner product.

It follows from these properties that an inner product on $\mathcal{V}$ defines a norm on $\mathcal{V}$ by $\parallel v \parallel := \sqrt{(v,v)}$; the triangle inequality is automatic.

Definition 2.1.3 *A* **Hilbert space** *is a vector space with inner product which is complete in the associated norm. We will usually denote Hilbert spaces by the symbol $\mathcal{H}$.*

A Hilbert space is completely characterized by its dimension (i.e., by the cardinality of an arbitrary orthogonal basis). To obtain an interesting theory, one therefore studies operators on a Hilbert space, rather than the Hilbert space itself. To obtain a satisfactory mathematical theory, it is wise to restrict oneself to bounded operators. We recall this concept in the more general context of arbitrary Banach spaces.

Definition 2.1.4 *A* **bounded operator** *on a Banach space $\mathcal{B}$ is a linear map $A : \mathcal{B} \to \mathcal{B}$ for which*

$$\parallel A \parallel := \sup \{ \parallel Av \parallel \; | v \in \mathcal{B}, \parallel v \parallel = 1 \} < \infty. \tag{2.2}$$

The number $\parallel A \parallel$ is the **operator norm**, or simply the norm, of A. This terminology is justified, as it follows almost immediately from its definition (and from the properties of the the norm on $\mathcal{B}$) that the operator norm is indeed a norm.

(It is easily shown that a linear map on a Banach space is continuous iff it is a bounded operator, but we will never use this result. Indeed, in arguments involving continuous operators on a Banach space one almost always uses boundedness rather than continuity.)

When $\mathcal{B}$ is a Hilbert space $\mathcal{H}$ the expression (2.2) becomes

$$\parallel A \parallel := \sup \{ (A\Psi, A\Psi)^{\frac{1}{2}} \, | \, \Psi \in \mathcal{H}, \, (\Psi, \Psi) = 1 \}. \tag{2.3}$$

When A is bounded, it follows that

$$\parallel Av \parallel \leq \parallel A \parallel \parallel v \parallel \tag{2.4}$$

for all $v \in \mathcal{B}$. Conversely, when for $A \neq 0$ there is a $C > 0$ such that $\parallel Av \parallel \leq C \parallel v \parallel$ for all v, then A is bounded, with operator norm $\parallel A \parallel$ equal to the smallest possible C for which the above inequality holds.

Proposition 2.1.5 *The space $\mathfrak{B}(\mathcal{B})$ of all bounded operators on a Banach space $\mathcal{B}$ is itself a Banach space in the operator norm.*

In view of the comments following (2.3), it only remains to be shown that $\mathfrak{B}(\mathcal{B})$ is complete in the operator norm. Let $\{A_n\}$ be a Cauchy sequence in $\mathfrak{B}(\mathcal{B})$. In other words, for any $\epsilon > 0$ there is a natural number $N(\epsilon)$ such that $\parallel A_n - A_m \parallel < \epsilon$ when $n, m > N(\epsilon)$. For arbitrary $v \in \mathcal{B}$, the sequence $\{A_n v\}$ is a Cauchy sequence in $\mathcal{B}$, because

$$\parallel A_n v - A_m v \parallel \leq \parallel A_n - A_m \parallel \parallel v \parallel \leq \epsilon \parallel v \parallel \tag{2.5}$$

for $n, m > N(\epsilon)$. Since $\mathcal{B}$ is complete by assumption, the sequence $\{A_n v\}$ converges to some $w \in \mathcal{B}$. Now define a map A on $\mathcal{B}$ by $Av := w = \lim_n A_n v$. This map is obviously linear. Taking $n \to \infty$ in (2.5), we obtain

$$\parallel Av - A_m v \parallel \leq \epsilon \parallel v \parallel \tag{2.6}$$

for all $m > N(\epsilon)$ and all $v \in \mathcal{B}$. It now follows from (2.2) that $A - A_m$ is bounded. Since $A = (A - A_m) + A_m$, and $\mathfrak{B}(\mathcal{B})$ is a linear space, we infer that A is bounded. Moreover, (2.6) and (2.2) imply that $\parallel A - A_m \parallel \leq \epsilon$ for all $m > N(\epsilon)$, so that $\{A_n\}$ converges to A. Since we have just seen that $A \in \mathfrak{B}(\mathcal{B})$, this proves that $\mathfrak{B}(\mathcal{B})$ is complete. ■

We define a **functional** on a Banach space $\mathcal{B}$ as a linear map $\rho : \mathcal{B} \to \mathbb{C}$ which is continuous in that $\rho(v)| \leq C \parallel v \parallel$ for some C, and all $v \in \mathcal{B}$. The smallest such C is the norm

$$\parallel \rho \parallel := \sup \{ |\rho(v)|, \, v \in \mathcal{B}, \, \parallel v \parallel = 1 \}. \tag{2.7}$$

The **dual** $\mathcal{B}^*$ of $\mathcal{B}$ is the space of all functionals on $\mathcal{B}$. Similarly to the proof of 2.1.5, one shows that $\mathfrak{B}^*$ is a Banach space. For later use, we quote, without proof, the fundamental **Hahn-Banach theorem**.

Theorem 2.1.6 *For a functional ρ_0 on a linear subspace $\mathcal{B}_0$ of a Banach space $\mathcal{B}$ there exists a functional ρ on $\mathcal{B}$ such that $\rho = \rho_0$ on $\mathcal{B}_0$ and $\| \rho \| = \| \rho_0 \|$. In other words, each functional defined on a linear subspace of $\mathcal{B}$ has an extension to $\mathcal{B}$ with the same norm.*

Corollary 2.1.7 *When $\rho(v) = 0$ for all $\rho \in \mathcal{B}^*$ then $v = 0$.*

For $v \neq 0$ we may define a functional ρ_0 on $\mathbb{C}v$ by $\rho_0(\lambda v) = \lambda$, and extend it to a functional ρ on $\mathcal{B}$ with norm 1. ■

Recall that an **algebra** is a vector space with an associative bilinear operation ('multiplication') $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$; we usually write AB for $A \cdot B$. It is clear that $\mathfrak{B}(\mathcal{B})$ is an algebra under operator multiplication. Moreover, using (2.4) twice, for each $v \in \mathcal{B}$ one has

$$\| ABv \| \leq \| A \| \| Bv \| \leq \| A \| \| B \| \| v \| .$$

Hence from (2.2) we obtain $\| AB \| \leq \| A \| \| B \|$.

Definition 2.1.8 *A* **Banach algebra** *is a Banach space $\mathfrak{A}$ which is at the same time an algebra, in which for all $A, B \in \mathfrak{A}$ one has*

$$\| AB \| \leq \| A \| \| B \| \quad . \tag{2.8}$$

It follows that multiplication in a Banach algebra is separately continuous in each variable.

As we have just seen, for any Banach space $\mathcal{B}$ the space $\mathfrak{B}(\mathcal{B})$ of all bounded operators on $\mathcal{B}$ is a Banach algebra. In what follows, we will restrict ourselves to the case that $\mathcal{B}$ is a Hilbert space $\mathcal{H}$; this leads to the Banach algebra $\mathfrak{B}(\mathcal{H})$. This algebra has additional structure.

Definition 2.1.9 *An* **involution** *on an algebra $\mathfrak{A}$ is a real-linear map $A \to A^*$ such that for all $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ one has*

$$A^{**} = A; \tag{2.9}$$

$$(AB)^* = B^* A^*; \tag{2.10}$$

$$(\lambda A)^* = \overline{\lambda} A^*. \tag{2.11}$$

A ***-algebra** *is an algebra with an involution.*

The operator adjoint $A \to A^*$ on a Hilbert space, defined by the property $(\Psi, A^*\Phi) := (A\Psi, \Phi)$, defines an involution on $\mathfrak{B}(\mathcal{H})$. Hence $\mathfrak{B}(\mathcal{H})$ is a *-algebra. As in this case, an element A of a C^*-algebra $\mathfrak{A}$ is called **self-adjoint** when $A^* = A$; we sometimes denote the collection of all self-adjoint elements by

$$\mathfrak{A}_{\mathbb{R}} := \{A \in \mathfrak{A} | \, A^* = A\}. \tag{2.12}$$

Since one may write

$$A = A' + iA'' := \frac{A + A^*}{2} + i\frac{A - A^*}{2i}, \tag{2.13}$$

every element of $\mathfrak{A}$ is a linear combination of two self-adjoint elements.

To see how the norm in $\mathfrak{B}(\mathcal{H})$ is related to the involution, we pick $\Psi \in \mathcal{H}$, and use the Cauchy-Schwarz inequality and (2.4) to estimate

$$\| A\Psi \|^2 = (A\Psi, A\Psi) = (\Psi, A^*A\Psi) \leq \| \Psi \| \| A^*A\Psi \| \leq \| A^*A \| \| \Psi \|^2 .$$

Using (2.3) and (2.8), we infer that

$$\| A \|^2 \leq \| A^*A \| \leq \| A^* \| \| A \| . \tag{2.14}$$

This leads to $\| A \| \leq \| A^* \|$. Replacing A by A^* and using (2.9) yields $\| A^* \| \leq \| A \|$, so that $\| A^* \| = \| A \|$. Substituting this in (2.14), we derive the crucial property $\| A^*A \| = \| A \|^2$.

This motivates the following definition.

Definition 2.1.10 *A* C^***-algebra** *is a complex Banach space* $\mathfrak{A}$ *which is at the same time a* **-algebra, such that for all* $A, B \in \mathfrak{A}$ *one has*

$$\| AB \| \leq \| A \| \, \| B \|; \tag{2.15}$$
$$\| A^*A \| = \| A \|^2 . \tag{2.16}$$

In other words, a C^**-algebra is a Banach* **-algebra in which (2.16) holds.*

Here a Banach *-algebra is, of course, a Banach algebra with involution. Combining (2.16) and (2.15), one derives $\| A \| \leq \| A^* \|$; as in the preceding paragraph, we infer that for all elements A of a C^*-algebra one has the equality

$$\| A^* \| = \| A \| . \tag{2.17}$$

The same argument proves the following.

Lemma 2.1.11 *A Banach* **-algebra in which* $\| A \|^2 \leq \| A^*A \|$ *is a* C^**-algebra.*

We have just shown that $\mathfrak{B}(\mathcal{H})$ is a C^*-algebra. Moreover, each (operator) norm-closed *-algebra in $\mathfrak{B}(\mathcal{H})$ is a C^*-algebra by the same argument. A much deeper result, which we will formulate precisely and prove in due course, states the converse of this: each C^*-algebra is isomorphic to a norm-closed *-algebra in $\mathfrak{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Hence the axioms in 2.1.10 characterize norm-closed *-algebras on Hilbert spaces, although the axioms make no reference to Hilbert spaces at all.

For later use we state some self-evident definitions.

Definition 2.1.12 *A* **morphism** *between* C^**-algebras* $\mathfrak{A}, \mathfrak{B}$ *is a (complex-) linear map* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *such that*

$$\varphi(AB) = \varphi(A)\varphi(B); \tag{2.18}$$
$$\varphi(A^*) = \varphi(A)^* \tag{2.19}$$

for all $A, B \in \mathfrak{A}$*. An* **isomorphism** *is a bijective morphism. Two* C^**-algebras are* **isomorphic** *when there exists an isomorphism between them.*

One immediately checks that the inverse of a bijective morphism is a morphism. It is remarkable, however, that an injective morphism (and hence an isomorphism) between C^*-algebras is automatically isometric. For this reason the condition that an isomorphism be isometric is not included in the definition.

2.2 Banach algebra basics

The material in this section is not included for its own interest, but because of its role in the theory of C^*-algebras. Even in that special context, it is enlightening to see concepts such as the spectrum in their general and appropriate setting.

Recall Definition 2.1.8. A **unit** in a Banach algebra $\mathfrak{A}$ is an element $\mathbb{I}$ satisfying $\mathbb{I}A = A\mathbb{I} = A$ for all $A \in \mathfrak{A}$, and

$$\| \mathbb{I} \| = 1. \tag{2.20}$$

A Banach algebra with unit is called **unital**. We often write z for $z\mathbb{I}$, where $z \in \mathbb{C}$. Note that in a C^*-algebra the property $\mathbb{I}A = A\mathbb{I} = A$ already implies, (2.20); take $A = \mathbb{I}^*$, so that $\mathbb{I}^*\mathbb{I} = \mathbb{I}^*$; taking the adjoint, this implies $\mathbb{I}^* = \mathbb{I}$, so that (2.20) follows from (2.16).

When a Banach algebra $\mathfrak{A}$ does not contain a unit, we can always add one, as follows. Form the vector space

$$\mathfrak{A}_{\mathbb{I}} := \mathfrak{A} \oplus \mathbb{C}, \tag{2.21}$$

and make this into an algebra by means of

$$(A + \lambda\mathbb{I})(B + \mu\mathbb{I}) := AB + \lambda B + \mu A + \lambda\mu\mathbb{I}, \tag{2.22}$$

where we have written $A+\lambda\mathbb{I}$ for (A,λ), etc. In other words, the number 1 in $\mathbb{C}$ is identified with $\mathbb{I}$. Furthermore, define a norm on $\mathfrak{A}_{\mathbb{I}}$ by

$$\| A+\lambda\mathbb{I} \| := \| A \| + |\lambda|. \tag{2.23}$$

In particular, $\| \mathbb{I} \| = 1$. Using (2.15) in $\mathfrak{A}$, as well as 2.1.1.3, one sees from (2.22) and (2.23) that

$$\| (A+\lambda\mathbb{I})(B+\mu\mathbb{I}) \| \leq \| A \| \| B \| + |\lambda| \, \| B \| + |\mu| \, \| A \| + |\lambda|\,|\mu| = \| A+\lambda\mathbb{I} \| \| B+\mu\mathbb{I} \|,$$

so that $\mathfrak{A}_{\mathbb{I}}$ is a Banach algebra with unit. Since by (2.23) the norm of $A \in \mathfrak{A}$ in $\mathfrak{A}$ coincides with the norm of $A+0\mathbb{I}$ in $\mathfrak{A}_{\mathbb{I}}$, we have shown the following.

Proposition 2.2.1 *For every Banach algebra without unit there exists a unital Banach algebra $\mathfrak{A}_{\mathbb{I}}$ and an isometric (hence injective) morphism $\mathfrak{A} \to \mathfrak{A}_{\mathbb{I}}$, such that $\mathfrak{A}_{\mathbb{I}}/\mathfrak{A} \simeq \mathbb{C}$.*

As we shall see at the end of section 2.4, the **unitization** $\mathfrak{A}_{\mathbb{I}}$ with the given properties is not unique.

Definition 2.2.2 *Let $\mathfrak{A}$ be a unital Banach algebra. The* **resolvent** *$\rho(A)$ of $A \in \mathfrak{A}$ is the set of all $z \in \mathbb{C}$ for which $A - z\mathbb{I}$ has a (two-sided) inverse in $\mathfrak{A}$.*

The **spectrum** *$\sigma(A)$ of $A \in \mathfrak{A}$ is the complement of $\rho(A)$ in $\mathbb{C}$; in other words, $\sigma(A)$ is the set of all $z \in \mathbb{C}$ for which $A - z\mathbb{I}$ has no (two-sided) inverse in $\mathfrak{A}$.*

When $\mathfrak{A}$ has no unit, the resolvent and the spectrum are defined through the embedding of $\mathfrak{A}$ in $\mathfrak{A}_{\mathbb{I}} = \mathfrak{A} \oplus \mathbb{C}$.

When $\mathfrak{A}$ is the algebra of $n \times n$ matrices, the spectrum of A is just the set of eigenvalues. For $A = \mathfrak{B}(\mathcal{H})$, Definition 2.2.2 reproduces the usual notion of the spectrum of an operator on a Hilbert space.

When $\mathfrak{A}$ has no unit, the spectrum $\sigma(A)$ of $A \in \mathfrak{A}$ always contains zero, since it follows from (2.22) that A never has an inverse in $\mathfrak{A}_{\mathbb{I}}$.

Theorem 2.2.3 *The spectrum $\sigma(A)$ of any element A of a Banach algebra is*

1. *contained in the set $\{z \in \mathbb{C} | \, |z| \leq \| A \|\}$;*
2. *compact;*
3. *not empty.*

The proof uses two lemmas. We assume that $\mathfrak{A}$ is unital.

Lemma 2.2.4 *When $\| A \| < 1$ the sum $\sum_{k=0}^{n} A^k$ converges to $(\mathbb{I} - A)^{-1}$.*
Hence $(A - z\mathbb{I})^{-1}$ always exists when $|z| > \| A \|$.

We first show that the sum is a Cauchy sequence. Indeed, for $n > m$ one has

$$\| \sum_{k=0}^{n} A^k - \sum_{k=0}^{m} A^k \| = \| \sum_{k=m+1}^{n} A^k \| \leq \sum_{k=m+1}^{n} \| A^k \| \leq \sum_{k=m+1}^{n} \| A \|^k .$$

For $n, m \to \infty$ this goes to 0 by the theory of the geometric series. Since $\mathfrak{A}$ is complete, the Cauchy sequence $\sum_{k=0}^{n} A^k$ converges for $n \to \infty$. Now compute

$$\sum_{k=0}^{n} A^k(\mathbb{I} - A) = \sum_{k=0}^{n} (A^k - A^{k+1}) = \mathbb{I} - A^{n+1}.$$

Hence

$$\| \mathbb{I} - \sum_{k=0}^{n} A^k(\mathbb{I} - A) \| = \| A^{n+1} \| \leq \| A \|^{n+1},$$

which $\to 0$ for $n \to \infty$, as $\| A \| < 1$ by assumption. Thus

$$\lim_{n\to\infty} \sum_{k=0}^{n} A^k(\mathbb{I} - A) = \mathbb{I}.$$

By a similar argument,

$$\lim_{n\to\infty} (\mathbb{I} - A) \sum_{k=0}^{n} A^k = \mathbb{I}.$$

so that, by continuity of multiplication in a Banach algebra, one finally has

$$\lim_{n\to\infty} \sum_{k=0}^{n} A^k = (\mathbb{I} - A)^{-1}. \tag{2.24}$$

The second claim of the lemma follows because $(A - z)^{-1} = -z^{-1}(\mathbb{I} - A/z)^{-1}$, which exists because $\| A/z \| < 1$ when $|z| > \| A \|$. ■

To prove that $\sigma(A)$ is compact, it remains to be shown that it is closed.

Lemma 2.2.5 *The set*

$$G(\mathfrak{A}) := \{A \in \mathfrak{A} |\, A^{-1} \text{ exists}\} \tag{2.25}$$

of invertible elements in $\mathfrak{A}$ *is open in* $\mathfrak{A}$.

Given $A \in G(\mathfrak{A})$, take a $B \in \mathfrak{A}$ for which $\| B \| < \| A^{-1} \|^{-1}$. By (2.8) this implies

$$\| A^{-1}B \| \leq \| A^{-1} \| \, \| B \| < 1. \tag{2.26}$$

Hence $A + B = A(\mathbb{I} + A^{-1}B)$ has an inverse, namely $(\mathbb{I} + A^{-1}B)^{-1}A^{-1}$, which exists by (2.26) and Lemma 2.2.4. It follows that all $C \in \mathfrak{A}$ for which $\| A - C \| < \epsilon$ lie in $G(\mathfrak{A})$, for $\epsilon \leq \| A^{-1} \|^{-1}$. ■

To resume the proof of Theorem 2.2.3, given $A \in \mathfrak{A}$ we now define a function $f : \mathbb{C} \to \mathfrak{A}$ by $f(z) := z - A$. Since $\| f(z + \delta) - f(z) \| = \delta$, we see that f is continuous (take $\delta = \epsilon$ in the definition of continuity). Because $G(\mathfrak{A})$ is open in $\mathfrak{A}$ by Lemma 2.2.5, it follows from the topological definition of a continuous function that $f^{-1}(G(\mathfrak{A}))$ is open in $\mathfrak{A}$. But $f^{-1}(G(\mathfrak{A}))$ is the set of all $z \in \mathbb{C}$ where $z - A$ has an inverse, so that $f^{-1}(G(\mathfrak{A})) = \rho(A)$. This set being open, its complement $\sigma(A)$ is closed.

Finally, define $g : \rho(A) \to \mathfrak{A}$ by $g(z) := (z - A)^{-1}$. For fixed $z_0 \in \rho(A)$, choose $z \in \mathbb{C}$ such that $|z - z_0| < \| (A - z_0)^{-1} \|^{-1}$. From the proof of Lemma 2.2.5, with $A \to A - z_0$ and $C \to A - z$, we see that $z \in \rho(A)$, as $\| A - z_0 - (A - z) \| = |z - z_0|$. Moreover, the power series

$$\frac{1}{z_0 - A} \sum_{k=0}^{n} \left(\frac{z_0 - z}{z_0 - A} \right)$$

converges for $n \to \infty$ by Lemma 2.2.4, because

$$\| (z_0 - z)(z_0 - A)^{-1} \| = |z_0 - z| \; \| (z_0 - A)^{-1} \| < 1.$$

By Lemma 2.2.4, the limit $n \to \infty$ of this power series is

$$\frac{1}{z_0 - A} \sum_{k=0}^{\infty} \left(\frac{z_0 - z}{z_0 - A} \right) = \frac{1}{z_0 - A} \left(1 - \left(\frac{z_0 - z}{z_0 - A} \right)^{-1} \right) = \frac{1}{z - A} = g(z).$$

Hence

$$g(z) = \sum_{k=0}^{\infty} (z_0 - z)^k (z_0 - A)^{k-1} \tag{2.27}$$

is a norm-convergent power series in z. For $z \neq 0$ we write $\| g(z) \| = |z|^{-1} \| (\mathbb{I} - A/z)^{-1} \|$ and observe that $\lim_{z\to\infty} \mathbb{I} - A/z = \mathbb{I}$, since $\lim_{z\to\infty} \| A/z \| = 0$ by 2.1.1.3. Hence $\lim_{z\to\infty}(\mathbb{I} - A/z)^{-1} = \mathbb{I}$, and

$$\lim_{z\to\infty} \| g(z) \| = 0. \tag{2.28}$$

Let $\rho \in \mathfrak{A}^*$ be a functional on $\mathfrak{A}$; since ρ is bounded, (2.27) implies that the function $g_\rho : z \to \rho(g(z))$ is given by a convergent power series, and (2.28) implies that

$$\lim_{z\to\infty} g_\rho(z) = 0. \tag{2.29}$$

Now suppose that $\sigma(A) = \emptyset$, so that $\rho(A) = \mathbb{C}$. The function g, and hence g_ρ, is then defined on $\mathbb{C}$, where it is analytic and vanishes at infinity. In particular, g_ρ is bounded, so that by Liouville's theorem it must be constant. By (2.29) this constant is zero, so that $g = 0$ by Corollary 2.1.7. This is absurd, so that $\rho(A) \neq \mathbb{C}$ hence $\sigma(A) \neq \emptyset$. ■

The fact that the spectrum is never empty leads to the following **Gel'fand-Mazur theorem**, which will be essential in the characterization of commutative C^*-algebras.

Corollary 2.2.6 *If every element (except 0) of a unital Banach algebra $\mathfrak{A}$ is invertible, then $\mathfrak{A} \simeq \mathbb{C}$ as Banach algebras.*

Since $\sigma(A) \neq \emptyset$, for each $A \neq 0$ there is a $z_A \in \mathbb{C}$ for which $A - z_A\mathbb{I}$ is not invertible. Hence $A - z_A\mathbb{I} = 0$ by assumption, and the map $A \to z_A$ is the desired algebra isomorphism. Since $\| A \| = \| z\mathbb{I} \| = |z|$, this isomorphism is isometric. ■

Define the **spectral radius** $r(A)$ of $A \in \mathfrak{A}$ by

$$r(A) := \sup\{|z|, z \in \sigma(A)\}. \tag{2.30}$$

From Theorem 2.2.3.1 one immediately infers

$$r(A) \leq \| A \| . \tag{2.31}$$

Proposition 2.2.7 *For each A in a unital Banach algebra one has*

$$r(A) = \lim_{n\to\infty} \| A^n \|^{1/n} . \tag{2.32}$$

By Lemma 2.2.4, for $|z| > \| A \|$ the function g in the proof of Lemma 2.2.5 has the norm-convergent power series expansion

$$g(z) = \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{A}{z}\right)^k . \tag{2.33}$$

On the other hand, we have seen that for any $z \in \rho(A)$ one may find a $z_0 \in \rho(A)$ such that the power series (2.27) converges. If $|z| > r(A)$ then $z \in \rho(A)$, so (2.27) converges for $|z| > r(A)$. At this point the proof relies on the theory of analytic functions with values in a Banach space, which says that, accordingly, (2.33) is norm-convergent for $|z| > r(A)$, uniformly in z. Comparing with (2.31), this sharpens what we know from Lemma 2.2.4. The same theory says that (2.33) cannot norm-converge uniformly in z unless $\| A^n \| / |z|^n < 1$ for large enough n. This is true for all z for which $|z| > r(A)$, so that

$$\limsup_{n\to\infty} \| A \|^{1/n} \leq r(A). \tag{2.34}$$

To derive a second inequality we use the following **polynomial spectral mapping property**.

Lemma 2.2.8 *For a polynomial p on $\mathbb{C}$, define $p(\sigma(A))$ as $\{p(z) | z \in \sigma(A)\}$. Then*

$$p(\sigma(A)) = \sigma(p(A)). \tag{2.35}$$

To prove this equality, choose $z, \alpha \in \mathbb{C}$ and compare the factorizations

$$\begin{aligned} p(z) - \alpha &= c\prod_{i=1}^{n}(z - \beta_i(\alpha)); \\ p(A) - \alpha\mathbb{I} &= c\prod_{i=1}^{n}(A - \beta_i(\alpha)\mathbb{I}). \end{aligned} \tag{2.36}$$

Here the coefficients c and $\beta_i(\alpha)$ are determined by p and α. When $\alpha \in \rho(p(A))$ then $p(A) - \alpha\mathbb{I}$ is invertible, which implies that all $A - \beta_i(\alpha)\mathbb{I}$ must be invertible. Hence $\alpha \in \sigma(p(A))$ implies that at least one of the $A - \beta_i(\alpha)\mathbb{I}$ is not invertible, so that $\beta_i(\alpha) \in \sigma(A)$ for at least one i. Hence $p(\beta_i(\alpha)) - \alpha = 0$, i.e., $\alpha \in p(\sigma(A))$. This proves the inclusion $\sigma(p(A)) \subseteq p(\sigma(A))$.

Conversely, when $\alpha \in p(\sigma(A))$ then $\alpha = p(z)$ for some $z \in \sigma(A)$, so that for some i one must have $\beta_i(\alpha) = z$ for this particular z. Hence $\beta_i(\alpha) \in \sigma(A)$, so that $A - \beta_i(\alpha)$ is not invertible, implying that $p(A) - \alpha\mathbb{I}$ is not invertible, so that $\in \sigma(p(A))$. This shows that $p(\sigma(A)) \subseteq \sigma(p(A))$, and (2.35) follows. ■

To conclude the proof of Proposition 2.2.7, we note that since $\sigma(A)$ is closed there is an $\alpha \in \sigma(A)$ for which $|\alpha| = r(A)$. Since $\alpha^n \in \sigma(A^n)$ by Lemma 2.2.8, one has $|\alpha^n| \leq \| A^n \|$ by (2.31). Hence $\| A^n \|^{1/n} \geq |\alpha| = r(A)$. Combining this with (2.34) yields

$$\limsup_{n\to\infty} \| A \|^{1/n} \leq r(A) \leq \| A^n \|^{1/n}.$$

Hence the limit must exist, and

$$\lim_{n\to\infty} \| A \|^{1/n} = \inf_n \| A^n \|^{1/n} = r(A). \tag{2.37}$$ ■

Definition 2.2.9 *An* **ideal** *in a Banach algebra $\mathfrak{A}$ is a closed linear subspace $\mathfrak{J} \subseteq \mathfrak{A}$ such that $A \in \mathfrak{J}$ implies $AB \in \mathfrak{J}$ and $BA \in \mathfrak{J}$ for all $B \in \mathfrak{A}$.*

A **left-ideal** *of $\mathfrak{A}$ is a closed linear subspace $\mathfrak{J}$ for which $A \in \mathfrak{J}$ implies $BA \in \mathfrak{J}$ for all $B \in \mathfrak{A}$.*

A **right-ideal** *of $\mathfrak{A}$ is a closed linear subspace $\mathfrak{J}$ for which $A \in \mathfrak{J}$ implies $AB \in \mathfrak{J}$ for all $B \in \mathfrak{A}$.*

A **maximal ideal** *is an ideal $\mathfrak{J} \neq \mathfrak{A}$ for which no ideal $\tilde{\mathfrak{J}} \neq \mathfrak{A}$, $\tilde{\mathfrak{J}} \neq \mathfrak{J}$, exists which contains $\mathfrak{J}$.*

In particular, an ideal is itself a Banach algebra. An ideal $\mathfrak{J}$ that contains an invertible element A must coincide with $\mathfrak{A}$, since $A^{-1}A = \mathbb{I}$ must lie in $\mathfrak{J}$, so that all $B = B\mathbb{I}$ must lie in $\mathfrak{J}$. This shows the need for considering Banach algebras with and without unit; it is usually harmless to add a unit to a Banach algebra $\mathfrak{A}$, but a given proper ideal $\mathfrak{J} \neq \mathfrak{A}$ does not contain $\mathbb{I}$, and one cannot add $\mathbb{I}$ to $\mathfrak{J}$ without ruining the property that it is a proper ideal.

Proposition 2.2.10 *If $\mathfrak{J}$ is an ideal in a Banach algebra $\mathfrak{A}$ then the quotient $\mathfrak{A}/\mathfrak{J}$ is a Banach algebra in the norm*

$$\| \tau(A) \| := \inf_{J\in\mathfrak{J}} \| A + J \| \tag{2.38}$$

and the multiplication

$$\tau(A)\tau(B) := \tau(AB). \tag{2.39}$$

Here $\tau : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ is the canonical projection. If $\mathfrak{A}$ is unital then $\mathfrak{A}/\mathfrak{J}$ is unital, with unit $\tau(\mathbb{I})$.

We omit the standard proof that $\mathfrak{A}/\mathfrak{J}$ is a Banach space in the norm (2.38). As far as the Banach algebra structure is concerned, first note that (2.39) is well defined: when $J_1, J_2 \in \mathfrak{J}$ one has

$$\tau(A + J_1)\tau(B + J_2) = \tau(AB + AJ_2 + J_1B + J_1J_2) = \tau(AB) = \tau(A)\tau(B),$$

since $AJ_2 + J_1B + J_1J_2 \in \mathfrak{J}$ by definition of an ideal, and $\tau(J) = 0$ for all $J \in \mathfrak{J}$. To prove (2.8), observe that, by definition of the infimum, for given $A \in \mathfrak{A}$, for each $\epsilon > 0$ there exists a $J \in \mathfrak{J}$ such that

$$\| \tau(A) \| + \epsilon \geq \| A + J \|. \tag{2.40}$$

For if such a J would not exist, the norm in $\mathfrak{A}/\mathfrak{I}$ could not be given by (2.38). On the other hand, for any $J \in \mathfrak{I}$ it is clear from (2.38) that

$$\| \tau(A) \| = \| \tau(A+J) \| \leq \| A+J \| . \tag{2.41}$$

For $A, B \in \mathfrak{A}$ choose $\epsilon > 0$ and $J_1, J_2 \in \mathfrak{I}$ such that (2.40) holds for A, B, and estimate

$$\begin{aligned} \| \tau(A)\tau(B) \| &= \| \tau(A+J_1)\tau(B+J_2) \| = \| \tau((A+J_1)(B+J_2)) \| \\ &\leq \| (A+J_1)(B+J_2) \| \leq \| A+J_1 \| \| B+J_2 \| \\ &\leq (\| \tau(A) \| +\epsilon)(\| \tau(B) \| +\epsilon). \end{aligned} \tag{2.42}$$

Letting $\epsilon \to 0$ yields $\| \tau(A)\tau(B) \| \leq \| \tau(A) \| \| \tau(B) \|$.

When $\mathfrak{A}$ has a unit, it is obvious from (2.39) that $\tau(\mathbb{I})$ is a unit in $\mathfrak{A}/\mathfrak{I}$. By (2.41) with $A = \mathbb{I}$ one has $\| \tau(\mathbb{I}) \| \leq \| \mathbb{I} \| = 1$. On the other hand, from (2.8) with $B = \mathbb{I}$ one derives $\| \tau(\mathbb{I}) \| \geq 1$. Hence $\| \tau(\mathbb{I}) \| = 1$. ∎

2.3 Commutative Banach algebras

We now assume that the Banach algebra $\mathfrak{A}$ is **commutative** (that is, $AB = BA$ for all $A, B \in \mathfrak{A}$).

Definition 2.3.1 *The* **structure space** $\Delta(\mathfrak{A})$ *of a commutative Banach algebra* $\mathfrak{A}$ *is the set of all nonzero linear maps* $\omega : \mathfrak{A} \to \mathbb{C}$ *for which*

$$\omega(AB) = \omega(A)\omega(B) \tag{2.43}$$

for all $A, B \in \mathfrak{A}$. *We say that such an* ω *is* **multiplicative**.

In other words, $\Delta(\mathfrak{A})$ *consists of all nonzero homomorphisms from* $\mathfrak{A}$ *to* $\mathbb{C}$.

Proposition 2.3.2 *Let* $\mathfrak{A}$ *have a unit* $\mathbb{I}$.

1. *Each* $\omega \in \Delta(\mathfrak{A})$ *satisfies*
$$\omega(\mathbb{I}) = 1; \tag{2.44}$$

2. *each* $\omega \in \Delta(\mathfrak{A})$ *is continuous, with norm*
$$\| \omega \| = 1; \tag{2.45}$$
hence
$$|\omega(A)| \leq \| A \| \tag{2.46}$$
for all $A \in \mathfrak{A}$.

The first claim is obvious, since $\omega(\mathbb{I}A) = \omega(\mathbb{I})\omega(A) = \omega(A)$, and there is an A for which $\omega(A) \neq 0$ because ω is not identically zero.

For the second, we know from Lemma 2.2.4 that $A - z$ is invertible when $|z| > \| A \|$, so that $\omega(A - z) = \omega(A) - z \neq 0$, since ω is a homomorphism. Hence $|\omega(A)| \neq |z|$ for $|z| > A \|$, and (2.46) follows. ∎

Theorem 2.3.3 *Let* $\mathfrak{A}$ *be a unital commutative Banach algebra. There is a bijective correspondence between* $\Delta(\mathfrak{A})$ *and the set of all maximal ideals in* $\mathfrak{A}$, *in that the kernel* $\ker(\omega)$ *of each* $\omega \in \Delta(\mathfrak{A})$ *is a maximal ideal* $\mathfrak{I}_\omega$, *each maximal ideal is the kernel of some* $\omega \in \Delta(\mathfrak{A})$, *and* $\omega_1 = \omega_2$ *iff* $\mathfrak{I}_{\omega_1} = \mathfrak{I}_{\omega_2}$.

The kernel of each $\omega \in \Delta(\mathfrak{A})$ is closed, since ω is continuous by 2.3.2.2. Furthermore, $\ker(\omega)$ is an ideal since ω satisfies (2.43). The kernel of every linear map $\omega : \mathcal{V} \to \mathbb{C}$ on a vector space $\mathcal{V}$ has codimension one (that is, $\dim(\mathcal{V}/\ker(\omega)) = 1$), so that $\ker(\omega)$ is a maximal ideal. Again on any vector space, when $\ker(\omega_1) = \ker(\omega_2)$ then ω_1 is a multiple of ω_2. For $\omega_i \in \Delta(\mathfrak{A})$ this implies $\omega_1 = \omega_2$ because of (2.44).

We now show that every maximal ideal $\mathfrak{J}$ of $\mathfrak{A}$ is the kernel of some $\omega \in \Delta(\mathfrak{A})$. Since $\mathfrak{J} \neq \mathfrak{A}$, there is a nonzero $B \in \mathfrak{A}$ which is not in $\mathfrak{J}$. Form

$$\mathfrak{J}_B := \{BA + J|, A \in \mathfrak{A}, J \in \mathfrak{J}\}.$$

This is clearly a left-ideal; since $\mathfrak{A}$ is commutative, $\mathfrak{J}_B$ is even an ideal. Taking $A = 0$ we see $\mathfrak{J} \subseteq \mathfrak{J}_B$. Taking $A = \mathbb{I}$ and $J = 0$ we see that $B \in \mathfrak{J}_B$, so that $\mathfrak{J}_B \neq \mathfrak{J}$. Hence $\mathfrak{J}_B = \mathfrak{A}$, as $\mathfrak{J}$ is maximal. In particular, $\mathbb{I} \in \mathfrak{J}_B$, hence $\mathbb{I} = BA + J$ for suitable $A \in \mathfrak{A}, J \in \mathfrak{J}$. Apply the canonical projection $\tau : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ to this equation, giving

$$\tau(\mathbb{I}) = \mathbb{I} = \tau(BA) = \tau(B)\tau(A),$$

because of (2.39) and $\tau(J) = 0$. hence $\tau(A) = \tau(B)^{-1}$ in $\mathfrak{A}/\mathfrak{J}$. Since B was arbitrary (though nonzero), this shows that every nonzero element of $\mathfrak{A}/\mathfrak{J}$ is invertible. By Corollary 2.2.6 this yields $\mathfrak{A}/\mathfrak{J} \simeq \mathbb{C}$, so that there is a homomorphism $\psi : \mathfrak{A}/\mathfrak{J} \to \mathbb{C}$. Now define a map $\omega : \mathfrak{A} \to \mathbb{C}$ by $\omega(A) := \psi(\tau(A))$. This map is clearly linear, since τ and ψ are. Also,

$$\omega(A)\omega(B) = \psi(\tau(A))\psi(\tau(B)) = \psi(\tau(A)\tau(B)) = \psi(\tau(AB)) = \omega(AB),$$

because of (2.39) and the fact that ψ is a homomorphism.

Therefore, ω is multiplicative; it is nonzero because $\omega(B) \neq 0$, or because $\omega(\mathbb{I}) = 1$. Hence $\omega \in \Delta(\mathfrak{A})$. Finally, $\mathfrak{J} \subseteq \ker(\omega)$ since $\mathfrak{J} = \ker(\tau)$; but if $B \notin \mathfrak{J}$ we saw that $\omega(B) \neq 0$, so that actually $\mathfrak{J} = \ker(\omega)$. ■

By 2.3.2.2 we have $\Delta(\mathfrak{A}) \subset \mathfrak{A}^*$. Recall that the **weak*-topology**, also called **w^*-topology**, on the dual $\mathcal{B}^*$ of a Banach space $\mathcal{B}$ is defined by the convergence $\omega_n \to \omega$ iff $\omega_n(v) \to \omega(v)$ for all $v \in \mathcal{B}$. The **Gel'fand topology** on $\Delta(\mathfrak{A})$ is the relative w^*-topology.

Proposition 2.3.4 *The structure space $\Delta(\mathfrak{A})$ of a unital commutative Banach algebra $\mathfrak{A}$ is compact and Hausdorff in the Gel'fand topology.*

The convergence $\omega_n \to \omega$ in the w^*-topology by definition means that $\omega_n(A) \to \omega(A)$ for all $A \in \mathfrak{A}$. When $\omega_n \in \Delta(\mathfrak{A})$ for all n, one has

$$|\omega(AB) - \omega(A)\omega(B)| = |\omega(AB) - \omega_n(AB) + \omega_n(A)\omega_n(B) - \omega(A)\omega(B)|$$
$$\leq |\omega(AB) - \omega_n(AB)| + |\omega_n(A)\omega_n(B) - \omega(A)\omega(B)|.$$

In the second term we write

$$\omega_n(A)\omega_n(B) - \omega(A)\omega(B) = (\omega_n(A) - \omega(A))\omega_n(B) + \omega(A)(\omega_n(B) - \omega(B)).$$

By (2.46) and the triangle inequality, the absolute value of the right-hand side is bounded by

$$\| B \| \ |\omega_n(A) - \omega(A)| + \| A \| \ |\omega_n(B) - \omega(B)|.$$

All in all, when $\omega_n \to \omega$ in the w^*-topology we obtain $|\omega(AB) - \omega(A)\omega(B)| = 0$, so that the limit $\omega \in \Delta(\mathfrak{A})$. Hence $\Delta(\mathfrak{A})$ is w^*-closed.

From (2.45) we have $\Delta(\mathfrak{A}) \in \mathfrak{A}^*_1$ (the unit ball in $\mathfrak{A}^*$, consisting of all functionals with norm ≤ 1). By the Banach-Alaoglu theorem, the unit ball in $\mathfrak{A}^*$ is w^*-compact. Being a closed subset of this unit ball, $\Delta(\mathfrak{A})$ is w^*-compact. Since the w^*-topology is Hausdorff (as is immediate from its definition), the claim follows. ■

We embed $\mathfrak{A}$ in $\mathfrak{A}^{**}$ by $A \to \hat{A}$, where

$$\hat{A}(\omega) := \omega(A). \tag{2.47}$$

When $\omega \in \Delta(\mathfrak{A})$, this defines $\hat{A}$ as a function on $\Delta(\mathfrak{A})$. By elementary functional analysis, the w^*-topology on $\mathfrak{A}^*$ is the weakest topology for which all $\hat{A}$, $A \in \mathfrak{A}$, are continuous. This implies that the Gel'fand topology on $\Delta(\mathfrak{A})$ is the weakest topology for which all functions $\hat{A}$ are continuous. In particular, a basis for this topology is formed by all open sets of the form

$$\hat{A}^{-1}(\mathcal{O}) = \{\omega \in \Delta(\mathfrak{A}) | \, \omega(A) \in \mathcal{O}\}, \tag{2.48}$$

where $A \in \mathfrak{A}$ and $\mathcal{O}$ is an open set in $\mathbb{C}$.

Seen as a map from $\mathfrak{A}$ to $C(\Delta(\mathfrak{A}))$, the map $A \to \hat{A}$ defined by (2.47) is called the **Gel'fand transform**.

For any compact Hausdorff space X, we regard the space $C(X)$ of all continuous functions on X as a Banach space in the **sup-norm** defined by

$$\| f \|_\infty := \sup_{x \in X} |f(x)|. \tag{2.49}$$

A basic fact of topology and analysis is that $C(X)$ is complete in this norm. Convergence in the sup-norm is the same as uniform convergence. What's more, it is easily verified that $C(X)$ is even a commutative Banach algebra under pointwise addition and multiplication, that is,

$$\begin{aligned} (\lambda f + \mu g)(x) &:= \lambda f(x) + \mu g(x); \\ (fg)(x) &:= f(x)g(x). \end{aligned} \tag{2.50}$$

Hence the function 1_X which is 1 for every x is the unit $\mathbb{I}$. One checks that the spectrum of $f \in C(X)$ is simply the set of values of f.

We regard $C(\Delta(\mathfrak{A}))$ as a commutative Banach algebra in the manner explained.

Theorem 2.3.5 *Let $\mathfrak{A}$ be a unital commutative Banach algebra.*

1. *The Gel'fand transform is a homomorphism from $\mathfrak{A}$ to $C(\Delta(\mathfrak{A}))$.*

2. *The image of $\mathfrak{A}$ under the Gel'fand transform separates points in $\Delta(\mathfrak{A})$.*

3. *The spectrum of $A \in \mathfrak{A}$ is the set of values of $\hat{A}$ on $\Delta(\mathfrak{A})$; in other words,*

$$\sigma(A) = \sigma(\hat{A}) = \{\hat{A}(\omega) | \, \omega \in \Delta(\mathfrak{A})\}. \tag{2.51}$$

4. *The Gel'fand transform is a contraction, that is,*

$$\| \hat{A} \|_\infty \leq \| A \| \, . \tag{2.52}$$

The first property immediately follows from (2.47) and (2.43). When $\omega_1 \neq \omega_2$ there is an $A \in \mathfrak{A}$ for which $\omega_1(A) \neq \omega_2(A)$, so that $\hat{A}(\omega_1) \neq \hat{A}(\omega_2)$. This proves 2.3.5.2.

If $A \in G(\mathfrak{A})$ (i.e., A is invertibe), then $\omega(A)\omega(A^{-1}) = 1$, so that $\omega(A) \neq 0$ for all $\omega \in \Delta(\mathfrak{A})$. When $A \notin G(\mathfrak{A})$ the ideal $\mathfrak{I}_A := \{AB|, B \in \mathfrak{A}\}$ does not contain $\mathbb{I}$, so that it is contained in a maximal ideal $\mathfrak{I}$ (this conclusion is actually nontrivial, relying on the axiom of choice in the guise of Hausdorff's maximality priciple). Hence by Theorem 2.3.3 there is a $\omega \in \Delta(\mathfrak{A})$ for which $\omega(A) = 0$. All in all, we have showed that $A \in G(\mathfrak{A})$ is equivalent to $\omega(A) \neq 0$ for all $\omega \in \Delta(\mathfrak{A})$. Hence $A - z \in G(\mathfrak{A})$ iff $\omega(A) \neq z$ for all $\omega \in \Delta(\mathfrak{A})$. Thus the resolvent is

$$\omega(A) = \{z \in \mathbb{C} | \, z \neq \omega(A) \, \forall \omega \in \Delta(\mathfrak{A})\}. \tag{2.53}$$

Taking the complement, and using (2.47), we obtain (2.51).

Eq. (2.52) then follows from (2.30), (2.31), (2.47), and (2.49). ■

We now look at an example, which is included for three reasons: firstly it provides a concrete illustration of the Gel'fand transform, secondly it concerns a commutative Banach algebra which is not a C^*-algebra, and thirdly the Banach algebra in question has no unit, so the example illustrates what happens to the structure theory in the absence of a unit. In this connection, let us note in general that each $\omega \in \Delta(\mathfrak{A})$ has a suitable extension $\tilde{\omega}$ to $\mathfrak{A}_{\mathbb{I}}$, namely

$$\omega(A + \lambda\mathbb{I}) := \omega(A) + \lambda. \tag{2.54}$$

The point is that $\tilde{\omega}$ remains multiplicative on $\mathfrak{A}_{\mathbb{I}}$, as can be seen from (2.22) and the definition (2.43). This extension is clearly unique. Even if one does not actually extend $\mathfrak{A}$ to $\mathfrak{A}_{\mathbb{I}}$, the existence of $\tilde{\omega}$ shows that ω satisfies (2.46), since this property (which was proved for the unital case) holds for $\tilde{\omega}$, and therefore certainly for the restriction ω of $\tilde{\omega}$ to $\mathfrak{A}$.

Consider $\mathfrak{A} = L^1(\mathbb{R})$, with the usual linear structure, and norm

$$\| f \|_1 := \int_{\mathbb{R}} dx\, |f(x)|. \tag{2.55}$$

The associative product $*$ defining the Banach algebra structure is convolution, that is,

$$f * g(x) := \int_{\mathbb{R}} dy\, f(x-y)g(y). \tag{2.56}$$

Strictly speaking, this should first be defined on the dense subspace $C_c(\mathbb{R})$, and subsequently be extended by continuity to $L^1(\mathbb{R})$, using the inequality below. Indeed, using Fubini's theorem on product integrals, we estimate

$$\begin{aligned} \| f * g \|_1 = \int_{\mathbb{R}} dx\, |\int_{\mathbb{R}} dy\, f(x-y)g(y)| &\leq \int_{\mathbb{R}} dy\, |g(y)| \int_{\mathbb{R}} dx\, |f(x-y)| \\ &= \int_{\mathbb{R}} dy\, |g(y)| \int_{\mathbb{R}} dx\, |f(x)| = \| f \|_1 \| g \|_1, \end{aligned}$$

which is (2.8).

There is no unit in $L^1(\mathbb{R})$, since from (2.56) one sees that the unit should be Dirac's delta-function (i.e., the measure on $\mathbb{R}$ which assigns 1 to $x=0$ and 0 to all other x), which does not lie in $L^1(\mathbb{R})$.

We know from the discussion following (2.54) that every multiplicative functional $\omega \in \Delta(L^1(\mathbb{R}))$ is continuous. Standard Banach space theory says that the dual of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$. Hence for each $\omega \in \Delta(L^1(\mathbb{R}))$ there is a function $\hat{\omega} \in L^\infty(\mathbb{R})$ such that

$$\omega(f) = \int_{\mathbb{R}} dx\, f(x)\hat{\omega}(x). \tag{2.57}$$

The multiplicativity condition (2.43) then implies that $\hat{\omega}(x+y) = \hat{\omega}(x)\hat{\omega}(y)$ for almost all $x, y \in \mathbb{R}$. This implies

$$\hat{\omega}(x) = \exp(ipx) \tag{2.58}$$

for some $p \in \mathbb{C}$, and since $\hat{\omega}$ is bounded (being in $L^\infty(\mathbb{R})$) it must be that $p \in \mathbb{R}$. The functional ω corresponding to (2.58) is simply called p. It is clear that different p's yield different functionals, so that $\Delta(L^1(\mathbb{R}))$ may be identified with $\mathbb{R}$. With this notation, we see from (2.57) and (2.58) that the Gel'fand transform (2.47) reads

$$\hat{f}(p) = \int_{\mathbb{R}} dx\, f(x)e^{ipx}. \tag{2.59}$$

Hence the Gel'fand transform is nothing but the Fourier transform (more generally, many of the integral transforms of classical analysis may be seen as special cases of the Gel'fand transform). The well-known fact that the Fourier transform maps the convolution product (2.56) into the pointwise product is then a restatement of Theorem 2.3.5.1. Moreover, we see from 2.3.5.3 that the spectrum $\sigma(f)$ of f in $L^1(\mathbb{R})$ is just the set of values of its Fourier transform.

Note that the Gel'fand transform is strictly a contraction, i.e., there is no equality in the bound (2.52). Finally, the Riemann-Lebesgue lemma states that $f \in L^1(\mathbb{R})$ implies $\hat{f} \in C_0(\mathbb{R})$, which is the space of continuous functions on $\mathbb{R}$ that go to zero when $|x| \rightarrow \infty$. This is an important function space, whose definition may be generalized as follows.

Definition 2.3.6 *Let X be a Hausdorff space X which is* **locally compact** *(in that each point has a compact neighbourhood). The space $C_0(X)$ consists of all continuous functions on X which* **vanish at infinity** *in the sense that for each $\epsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for all x outside K.*

So when X is compact one trivially has $C_0(X) = C(X)$. When X is not compact, the sup-norm (2.49) can still be defined, and just as for $C(X)$ one easily checks that $C_0(X)$ is a Banach algebra in this norm.

We see that in the example $\mathfrak{A} = L^1(\mathbb{R})$ the Gel'fand transform takes values in $C_0(\Delta(\mathfrak{A}))$. This may be generalized to arbitrary commutative non-unital Banach algebras. The non-unital version of Theorem 2.3.5 is

Theorem 2.3.7 *Let $\mathfrak{A}$ be a non-unital commutative Banach algebra.*

1. *The structure space $\Delta(\mathfrak{A})$ is locally compact and Hausdorff in the Gel'fand topology.*
2. *The space $\Delta(\mathfrak{A}_{\mathbb{I}})$ is the one-point compactification of $\Delta(\mathfrak{A})$.*
3. *The Gel'fand transform is a homomorphism from $\mathfrak{A}$ to $C_0(\Delta(\mathfrak{A}))$.*
4. *The spectrum of $A \in \mathfrak{A}$ is the set of values of $\hat{A}$ on $\Delta(\mathfrak{A})$, with zero added (if 0 is not already contained in this set).*
5. *The claims 2 and 4 in Theorem 2.3.5 hold.*

Recall that the **one-point compactification** $\tilde{X}$ of a non-compact topological space X is the set $X \cup \infty$, whose open sets are the open sets in X plus those subsets of $X \cup \infty$ whose complement is compact in X. If, on the other hand, $\tilde{X}$ is a compact Hausdorff space, the removal of some point '∞' yields a locally compact Hausdorff space $X = \tilde{X}\backslash\{\infty\}$ in the relative topology (i.e., the open sets in X are the open sets in $\tilde{X}$ minus the point ∞), whose one-point compactification is, in turn, $\tilde{X}$.

To prove 2.3.7 we add a unit to $\mathfrak{A}$, and note that

$$\Delta(\mathfrak{A}_{\mathbb{I}}) = \Delta(\mathfrak{A}) \cup \infty, \tag{2.60}$$

where each $\omega \in \Delta(\mathfrak{A})$ is seen as a functional $\tilde{\omega}$ on $\mathfrak{A}_{\mathbb{I}}$ by (2.54), and the functional ∞ is defined by

$$\infty(A + \lambda\mathbb{I}) := \lambda. \tag{2.61}$$

There can be no other elements φ of $\Delta(\mathfrak{A}_{\mathbb{I}})$, because the restriction of φ has a unique multiplicative extension (2.54) to $\mathfrak{A}_{\mathbb{I}}$, unless it identically vanishes on $\Delta(\mathfrak{A})$. In the latter case (2.61) is clearly the only multiplicative possibility.

By Proposition 2.3.4 the space $\Delta(\mathfrak{A}_{\mathbb{I}})$ is compact and Hausdorff; by (2.61) one has

$$\Delta(\mathfrak{A}) = \Delta(\mathfrak{A}_{\mathbb{I}})\backslash\{\infty\} \tag{2.62}$$

as a set. In view of the paragraph following 2.3.7, in order to prove 2.3.7.1 and 2, we need to show that the Gel'fand topology of $\Delta(\mathfrak{A}_{\mathbb{I}})$ restricted to $\Delta(\mathfrak{A})$ coincides with the Gel'fand topology of $\Delta(\mathfrak{A})$ itself. Firstly, it is clear from (2.48) that any open set in $\Delta(\mathfrak{A})$ (in its own Gel'fand topology) is the restriction of some open set in $\Delta(\mathfrak{A}_{\mathbb{I}})$, because $\mathfrak{A} \subset \mathfrak{A}_{\mathbb{I}}$. Secondly, for any $A \in \mathfrak{A}$, $\lambda \in \mathbb{C}$, and open set $\mathcal{O} \subset \mathbb{C}$, from (2.54) we evidently have

$$\{\varphi \in \Delta(\mathfrak{A}_{\mathbb{I}}) | \, \varphi(A + \lambda\mathbb{I}) \in \mathcal{O}\}\backslash\{\infty\} = \{\omega \in \Delta(\mathfrak{A}) | \, \omega(A) \in \mathcal{O} - \lambda\}.$$

(When ∞ does not lie in the set $\{\ldots\}$ on the left-hand side, one should here omit the "$\backslash\{\infty\}$".) With (2.48), this shows that the restriction of any open set in $\Delta(\mathfrak{A}_{\mathbb{I}})$ to $\Delta(\mathfrak{A})$ is always open in the Gel'fand topology of $\Delta(\mathfrak{A})$. This establishes 2.3.7.1 and 2.

It follows from (2.3.5) and (2.61) that

$$\hat{A}(\infty) = 0 \tag{2.63}$$

for all $A \in \mathfrak{A}$, which by continuity of $\hat{A}$ leads to 2.3.7.3.

The comment preceding Theorem 2.2.3 implies 2.3.7.4. The final claim follows from the fact that it holds for $\mathfrak{A}_{\mathbb{I}}$. ■

2.4 Commutative C^*-algebras

The Banach algebra $C(X)$ considered in the previous section is more than a Banach algebra. Recall Definition 2.1.9. The map $f \to f^*$, where

$$f^*(x) := \overline{f(x)}, \tag{2.64}$$

evidently defines an involution on $C(X)$, in which $C(X)$ is a commutative C^*-algebra with unit. The main goal of this section is to prove the converse statement; cf. Definition 2.1.12

Theorem 2.4.1 *Let $\mathfrak{A}$ be a commutative C^*-algebra with unit. Then there is a compact Hausdorff space X such that $\mathfrak{A}$ is (isometrically) isomorphic to $C(X)$. This space is unique up to homeomorphism.*

The isomorphism in question is the Gel'fand transform, so that $X = \Delta(\mathfrak{A})$, equipped with the Gel'fand topology, and the isomorphism $\varphi : \mathfrak{A} \to C(X)$ is given by

$$\varphi(A) := \hat{A}. \tag{2.65}$$

We have already seen in 2.3.5.1 that this transform is a homomorphism, so that (2.18) is satisfied. To show that (2.19) holds as well, it suffices to show that a self-adjoint element of $\mathfrak{A}$ is mapped into a real-valued function, because of (2.13), (2.64), and the fact that the Gel'fand transform is complex-linear.

We pick $A \in \mathfrak{A}_{\mathbb{R}}$ and $\omega \in \Delta(\mathfrak{A})$, and suppose that $\omega(A) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. By (2.44) one has $\omega(B) = i\beta$, where $B := A - \alpha\mathbb{I}$ is self-adjoint. Hence for $t \in \mathbb{R}$ one computes

$$|\omega(B + it\mathbb{I})|^2 = \beta^2 + 2t\beta + t^2. \tag{2.66}$$

On the other hand, using (2.46) and (2.16) we estimate

$$|\omega(B + it\mathbb{I})|^2 \leq \parallel B + it\mathbb{I} \parallel^2 = \parallel (B + it\mathbb{I})^*(B + it\mathbb{I}) \parallel = \parallel B^2 + t^2 \parallel \leq \parallel B \parallel^2 + t^2.$$

Using (2.66) then yields $\beta^2 + t\beta \leq \parallel B \parallel^2$ for all $t \in \mathbb{R}$. For $\beta > 0$ this is impossible. For $\beta < 0$ we repeat the argument with $B \to -B$, finding the same absurdity. Hence $\beta = 0$, so that $\omega(A)$ is real when $A = A^*$. Consequently, by (2.47) the function $\hat{A}$ is real-valued, and (2.19) follows as announced.

We now prove that the Gel'fand transform, and therefore the morphism φ in (2.65), is isometric. When $A = A^*$, the axiom (2.16) reads $\parallel A^2 \parallel = \parallel A \parallel^2$. This implies that $\parallel A^{2^m} \parallel = \parallel A \parallel^{2^m}$ for all $m \in \mathbb{N}$. Taking the limit in (2.32) along the subsequence $n = 2^m$ then yields

$$r(A) = \parallel A \parallel . \tag{2.67}$$

In view of (2.30) and (2.51), this implies

$$\parallel \hat{A} \parallel_{\infty} = \parallel A \parallel . \tag{2.68}$$

For general $A \in \mathfrak{A}$ we note that A^*A is self-adjoint, so that we may use the previous result and (2.16) to compute

$$\parallel A \parallel^2 = \parallel A^*A \parallel = \parallel \widehat{A^*A} \parallel_{\infty} = \parallel \hat{A}^*\hat{A} \parallel_{\infty} = \parallel \hat{A} \parallel_{\infty}^2 .$$

In the third equality we used $\widehat{A^*} = \hat{A}^*$, which we just proved, and in the fourth we exploited the fact that $C(X)$ is a C^*-algebra, so that (2.16) is satisfied in it. Hence (2.68) holds for all $A \in \mathfrak{A}$.

It follows that φ in (2.65) is injective, because if $\varphi(A) = 0$ for some $A \neq 0$, then φ would fail to be an isometry. (A commutative Banach algebra for which the Gel'fand transform is injective is called semi-simple. Thus commutative C^*-algebraa are semi-simple.)

We finally prove that the morphism φ is surjective. We know from (2.68) that the image $\varphi(\mathfrak{A}) = \hat{\mathfrak{A}}$ is closed in $C(\Delta(\mathfrak{A}))$, because $\mathcal{A}$ is closed (being a C^*-algebra, hence a Banach space). In addition, we know from 2.3.5.2 that $\varphi(\mathfrak{A})$ separates points on $\Delta(\mathfrak{A})$. Thirdly, since the Gel'fand transform was just shown to preserve the adjoint, $\varphi(\mathfrak{A})$ is closed under complex conjugation by (2.64). Finally, since $\hat{I} = 1_X$ by (2.44) and (2.47), the image $\varphi(\mathfrak{A})$ contains 1_X. The surjectivity of φ now follows from the following **Stone-Weierstrass theorem**, which we state without proof.

Lemma 2.4.2 *Let X be a compact Hausdorff space, and regard $C(X)$ as a commutative C^*-algebra as explained above. A C^*-subalgebra of $C(X)$ which separates points on X and contains 1_X coincides with $C(X)$.*

Being injective and surjective, the morphism φ is bijective, and is therefore an isomorphism. The uniqueness of X is the a consequence of the following result.

Proposition 2.4.3 *Let X be a compact Hausdorff space, and regard $C(X)$ as a commutative C^*-algebra as explained above. Then $\Delta(C(X))$ (equipped with the Gel'fand topology) is homeomorphic to X.*

Each $x \in X$ defines a linear map $\omega_x : C(X) \to \mathbb{C}$ by $\omega_x(f) := f(x)$, which is clearly multiplicative and nonzero. Hence $x \to \omega_x$ defines a map E (for Evaluation) from X to $\Delta(C(X))$, given by

$$E(x) : f \to f(x). \tag{2.69}$$

Since a compact Hausdorff space is normal, Urysohn's lemma says that $C(X)$ separates points on X (i.e., for all $x \neq y$ there is an $f \in C(X)$ for which $f(x) \neq f(y)$). This shows that E is injective.

We now use the compactness of X and Theorem 2.3.3 to prove that E is surjective. The maximal ideal $\mathfrak{I}_x := \mathfrak{I}_{\omega_x}$ in $C(X)$ which corresponds to $\omega_x \in \Delta(C(X))$ is obviously

$$\mathfrak{I}_x = \{f \in C(X) | \, f(x) = 0\}. \tag{2.70}$$

Therefore, when E is not surjective there exists a maximal ideal $\mathfrak{I} \subset C(X)$ which for each $x \in X$ contains at a function f_x for which $f_x(x) \neq 0$ (if not, $\mathfrak{I}$ would contain an ideal $\mathfrak{I}_x$ which thereby would not be maximal). For each x, the set $\mathcal{O}_x$ where f_x is nonzero is open, because f is continuous. This gives a covering $\{\mathcal{O}_x\}_{x\in X}$ of X. By compactness, there exists a finite subcovering $\{\mathcal{O}_{x_i}\}_{i=1,\ldots,N}$. Then form the function $g := \sum_{i=1}^N |f_{x_i}|^2$. This function is strictly positive by construction, so that it is invertible (note that $f \in C(X)$ is invertible iff $f(x) \neq 0$ for all $x \in X$, in which case $f^{-1}(x) = 1/f(x)$). But $\mathfrak{I}$ is an ideal, so that, with all $f_{x_i} \in \mathfrak{I}$ (since all $f_x \in \mathfrak{I}$) also $g \in \mathfrak{I}$. But an ideal containing an invertible element must coincide with $\mathfrak{A}$ (see the comment after 2.2.9), contradicting the assumption that $\mathfrak{I}$ is a maximal ideal.

Hence E is surjective; since we already found it is injective, E must be a bijection. It remains to be shown that E is a homeomorphism. Let X_o denote X with its originally given topology, and write X_G for X with the topology induced by E^{-1}. Since $\hat{f} \circ E = f$ by (2.69) and (2.47), and the Gel'fand topology on $\Delta(C(X))$ is the weakest topology for which all functions $\hat{f}$ are continuous, we infer that X_G is weaker than X_o (since f, lying in $C(X_o)$, is continuous). Here a topology $\mathcal{T}_1$ is called weaker than a topology $\mathcal{T}_2$ on the same set if any open set of $\mathcal{T}_1$ contains an open set of $\mathcal{T}_2$. This includes the possibility $\mathcal{T}_1 = \mathcal{T}_2$.

Without proof we now state a result from topology.

Lemma 2.4.4 *Let a set X be Hausdorff in some topology $\mathcal{T}_1$ and compact in a topology $\mathcal{T}_2$. If $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ then $\mathcal{T}_1 = \mathcal{T}_2$.*

Since X_o and X_G are both compact and Hausdorff (the former by assumption, and the latter by Proposition 2.3.4), we conclude from this lemma that $X_0 = X_G$; in other words, E is a homeomorphism. This concludes the proof of 2.4.3. ■

Proposition 2.4.3 shows that X as a topological space may be extracted from the Banach-algebraic structure of $C(X)$, up to homeomorphism. Hence if $C(X) \simeq C(Y)$ as a C^*-algebra, where Y is a second compact Hausdorff space, then $X \simeq Y$ as topological spaces. Given the isomorphism $\mathfrak{A} \simeq C(X)$ constructed above, a second isomorphism $\mathfrak{A} \simeq C(Y)$ is therefore only possible if $X \simeq Y$. This proves the final claim of Theorem 2.4.1. ■

The condition that a compact topological space be Hausdorff is sufficient, but not necessary for the completeness of $C(X)$ in the sup-norm. However, when X is not Hausdorff yet $C(X)$ is complete, the map E may fail to be injective since in that case $C(X)$ may fail to separate points on X.

On the other hand, suppose X is locally compact but not compact, and consider $\mathfrak{A} = C_b(X)$; this is the space of all continuous bounded functions on X. Equipped with the operations (2.49), (2.50), and (2.64) this is a commutative C^*-algebra. The map $E : X \to \Delta(C_b(X))$ is now injective, but fails to be surjective (this is suggested by the invalidity of the proof we gave for $C(X)$). Indeed, it can be shown that $\Delta(C_b(X))$ is homeomorphic to the Ceh-Stone compactification of X.

Let us now consider what happens to Theorem 2.4.1 when $\mathfrak{A}$ has no unit. Following the strategy we used in proving Theorem 2.3.7, we would like to add a unit to $\mathfrak{A}$. As in the case of a general Banach algebra (cf. section 2.2), we form $\mathfrak{A}_{\mathbb{I}}$ by (2.21), define multiplication by (2.22), and use the natural involution

$$(A + \lambda\mathbb{I})^* := A^* + \overline{\lambda}\mathbb{I}. \tag{2.71}$$

However, the straightforward norm (2.23) cannot be used, since it is not a C^*-norm in that axiom (2.16) is not satisfied. Recall Definition 2.1.4.

Lemma 2.4.5 *Let $\mathfrak{A}$ be a C^*-algebra.*

1. *The map $\rho : \mathfrak{A} \to \mathfrak{B}(\mathfrak{A})$ given by*
$$\rho(A)B := AB \tag{2.72}$$
establishes an isomorphism between $\mathfrak{A}$ and $\rho(\mathfrak{A}) \subset \mathfrak{B}(\mathfrak{A})$.

2. *When $\mathfrak{A}$ has no unit, define a norm on $\mathfrak{A}_{\mathbb{I}}$ by*
$$\| A + \lambda\mathbb{I} \| := \| \rho(A) + \lambda\mathbb{I} \|, \tag{2.73}$$
where the norm on the right-hand side is the operator norm (2.2) in $\mathfrak{B}(\mathfrak{A})$, and $\mathbb{I}$ on the right-hand side is the unit operator in $\mathfrak{B}(\mathfrak{A})$. With the operations (2.22) and (2.71), the norm (2.73) turns $\mathfrak{A}_{\mathbb{I}}$ into a C^*-algebra with unit.

By (2.15) we have $\| \rho(A)B \| = \| AB \| \leq \| A \| \| B \|$ for all B, so that $\| \rho(A) \| \leq \| A \|$ by (2.2). On the other hand, using (2.16) and (2.17) we can write

$$\| A \| = \| AA^* \| / \| A \| = \| \rho(A) \frac{A^*}{\| A \|} \| \leq \rho(A) \|;$$

in the last step we used (2.4) and $\| (A^* / \| A \|) \| = 1$. Hence

$$\| \rho(A) \| = \| A \| . \tag{2.74}$$

Being isometric, the map ρ must be injective; it is clearly a homomorphism, so that we have proved 2.4.5.1.

It is clear from (2.22) and (2.71) that the map $A + \lambda\mathbb{I} \to \rho(A) + \lambda\mathbb{I}$ (where the symbol $\mathbb{I}$ on the left-hand side is defined below (2.22), and the $\mathbb{I}$ on the right-hand side is the unit in $\mathfrak{B}(\mathfrak{A})$) is a morphism. Hence the norm (2.73) satisfies (2.15), because (2.8) is satisfied in $\mathfrak{B}(\mathfrak{A})$. Moreover, in order to prove that the norm (2.73) satisfies (2.16), by Lemma 2.1.11 it suffices to prove that

$$\| \rho(A) + \lambda\mathbb{I} \|^2 \leq \| (\rho(A) + \lambda\mathbb{I})^*(\rho(A) + \lambda\mathbb{I}) \| \tag{2.75}$$

for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. To do so, we use a trick similar to the one involving (2.40), but with inf replaced by sup. Namely, in view of (2.2), for given $A \in \mathfrak{B}(\mathcal{B})$ and $\epsilon > 0$ there exists a $v \in \mathcal{V}$, with $\| v \| = 1$, such that $\| A \|^2 - \epsilon \leq \| Av \|^2$. Applying this with $\mathcal{B} \to \mathfrak{A}$ and $A \to \rho(A) + \lambda\mathbb{I}$, we infer that for every $\epsilon > 0$ there exists a $B \in \mathfrak{A}$ with norm 1 such that

$$\| \rho(A) + \lambda\mathbb{I} \|^2 - \epsilon \leq \| (\rho(A) + \lambda\mathbb{I})B \|^2 = \| AB + \lambda B \|^2 = \| (AB + \lambda B)^*(AB + \lambda B) \| .$$

Here we used (2.16) in $\mathfrak{A}$. Using (2.72), the right-hand side may be rearranged as

$$\| \rho(B^*)\rho(A^* + \overline{\lambda}\mathbb{I})\rho(A + \lambda\mathbb{I})B \| \leq \| \rho(B^*) \| \| (\rho(A) + \lambda\mathbb{I})^*(\rho(A) + \lambda\mathbb{I}) \| \| B \| .$$

Since $\| \rho(B^*) \| = \| B^* \| = \| B \| = 1$ by (2.74) and (2.17), and $\| B \| = 1$ also in the last term, the inequality (2.75) follows by letting $\epsilon \to 0$. ■

Hence the C^*-algebraic version of Theorem 2.2.1 is

Proposition 2.4.6 *For every C^*-algebra without unit there exists a unique unital C^*-algebra $\mathfrak{A}_{\mathbb{I}}$ and an isometric (hence injective) morphism $\mathfrak{A} \to \mathfrak{A}_{\mathbb{I}}$, such that $\mathfrak{A}_{\mathbb{I}}/\mathfrak{A} \simeq \mathbb{C}$.*

The uniqueness of $\mathfrak{A}_{\mathbb{I}}$ follows from Corollary 2.5.3 below. On the other hand, in view of the fact that both (2.23) and (2.73) define a norm on $\mathfrak{A}_{\mathbb{I}}$ satisfying the claims of Proposition 2.2.1, we conclude that the unital Banach algebra $\mathfrak{A}_{\mathbb{I}}$ called for in that proposition is not, in general, unique.

In any case, having established the existence of the unitization of an arbitrary non-unital C^*-algebra, we see that, in particular, a commutative non-unital C^*-algebra has a unitization. The passage from Theorem 2.3.5 to Theorem 2.3.7 may then be repeated in the C^*-algebraic setting; the only nontrivial point compared to the situation for Banach algebras is the generalization of Lemma 2.4.2. This now reads

Lemma 2.4.7 *Let X be a locally compact Hausdorff space, and regard $C_0(X)$ as a commutative C^*-algebra as explained below Definition 2.3.6.*

A C^*-subalgebra $\mathfrak{A}$ of $C_0(X)$ which separates points on X, and is such that for each $x \in X$ there is an $f \in \mathfrak{A}$ such that $f(x) \neq 0$, coincides with $C_0(X)$.

At the end of the day we then find

Theorem 2.4.8 *Let $\mathfrak{A}$ be a commutative C^*-algebra without unit. There is a locally compact Hausdorff space X such that $\mathfrak{A}$ is (isometrically) isomorphic to $C_0(X)$. This space is unique up to homeomorphism.*

2.5 Spectrum and functional calculus

We return to the general case in which a C^*-algebra $\mathfrak{A}$ is not necessarily commutative (but assumed unital), but analyze properties of $\mathfrak{A}$ by studying certain commutative subalgebras. This will lead to important results.

For each element $A \in \mathfrak{A}$ there is a smallest C^*-subalgebra $C^*(A, \mathbb{I})$ of $\mathfrak{A}$ which contains A and $\mathbb{I}$, namely the closure of the linear span of $\mathbb{I}$ and all operators of the type $A_1 \ldots A_n$, where A_i is A or A^*. Following the terminology for operators on a Hilbert space, an element $A \in \mathfrak{A}$ is called **normal** when $[A, A^*] = 0$. The crucial property of a normal operator is that $C^*(A, \mathbb{I})$ is commutative. In particular, when A is self-adjoint, $C^*(A, \mathbb{I})$ is simply the closure of the space of all polynomials in A. It is sufficient for our purposes to restrict ourselves to this case.

Theorem 2.5.1 *Let $A = A^*$ be a self-adjoint element of a unital C^*-algebra.*

1. *The spectrum $\sigma_{\mathfrak{A}}(A)$ of A in $\mathfrak{A}$ coincides with the spectrum $\sigma_{C^*(A,\mathbb{I})}(A)$ of A in $C^*(A, \mathbb{I})$ (so that we may unambiguously speak of the spectrum $\sigma(A)$).*

2. *The spectrum $\sigma(A)$ is a subset of $\mathbb{R}$.*

3. *The structure space $\Delta(C^*(A, \mathbb{I}))$ is homeomorphic with $\sigma(A)$, so that $C^*(A, \mathbb{I})$ is isomorphic to $C(\sigma(A))$. Under this isomorphism the Gel'fand transform $\hat{A} : \sigma(A) \to \mathbb{R}$ is the identity function $\mathrm{id}_{\sigma(A)} : t \to t$.*

Recall (2.25). Let $A \in G(\mathfrak{A})$ be normal in $\mathfrak{A}$, and consider the C^*-algebra $C^*(A, A^{-1}, \mathbb{I})$ generated by A, A^{-1}, and $\mathbb{I}$. One has $(A^{-1})^* = (A^*)^{-1}$, and A, A^*, A^{-1}, $(A^*)^{-1}$ and $\mathbb{I}$ all commute with each other. Hence $C^*(A, A^{-1}, \mathbb{I})$ is commutative; it is the closure of the space of all polynomials in A, A^*, A^{-1}, $(A^*)^{-1}$, and $\mathbb{I}$. By Theorem 2.4.1 we have $C^*(A, A^{-1}, \mathbb{I}) \simeq C(X)$ for some compact Hausdorff space X. Since A is invertible and the Gel'fand transform (2.47) is an isomorphism, $\hat{A}$ is invertible in $C(X)$ (i.e., $\hat{A}(x) \neq 0x$ for all $x \in X$). However, for any $f \in C(X)$ that is nonzero throughout X we have $0 < \parallel f \parallel_\infty^{-2} ff^* \leq 1$ pointwise, so that $0 \leq 1_X - \parallel f \parallel_\infty^{-2} ff* < 1$ pointwise, hence

$$\parallel 1_X - ff * / \parallel f \parallel_\infty^2 \parallel_\infty < 1.$$

Here f^* is given by (2.64). Using Lemma 2.2.4, in terms of $\mathbb{I} = 1_X$ we may therefore write

$$\frac{1}{f} = \frac{f^*}{\parallel f \parallel_\infty^2} \sum_{k=0}^{\infty} \left(\mathbb{I} - \frac{ff^*}{\parallel f \parallel_\infty^2} \right)^k. \tag{2.76}$$

Hence $\hat{A}^{-1}$ is a norm-convergent limit of a sequence of polynomials in $\hat{A}$ and $\hat{A}^*$. Gel'fand transforming this result back to $C^*(A, A^{-1}, \mathbb{I})$, we infer that A^{-1} is a norm-convergent limit of a sequence of polynomials in A and A^*. Hence A^{-1} lies in $C^*(A, \mathbb{I})$, and $C^*(A, A^{-1}, \mathbb{I}) = C^*(A, \mathbb{I})$.

Now replace A by $A - z$, where $z \in \mathbb{C}$. When A is normal $A - z$ is normal. So if we assume that $A - z \in G(\mathfrak{A})$ the argument above applies, leading to the conclusion that the resolvent $\rho_{\mathfrak{A}}(A)$ in $\mathfrak{A}$ coincides with the resolvent $\rho_{C^*(A,\mathbb{I})}(A)$ in $C^*(A, \mathbb{I})$. By Definition 2.2.2 we then conclude that $\sigma_{\mathfrak{A}}(A) = \sigma_{C^*(A,\mathbb{I})}(A)$.

According to Theorem 2.4.1, the function $\hat{A}$ is real-valued when $A = A^*$. Hence by 2.3.5.3 the spectrum $\sigma_{C^*(A,\mathbb{I})}(A)$ is real, so that by the previous result $\sigma(A)$ is real.

Finally, given the isomorphism $C^*(A, \mathbb{I}) \simeq C(X)$ of Theorem 2.4.1 (where $X = \Delta(C^*(A, \mathbb{I}))$), according to 2.3.5.3 the function $\hat{A}$ is a surjective map from X to $\sigma(A)$. We now prove injectivity. When $\omega_1, \omega_2 \in X$ and $\omega_1(A) = \omega_2(A)$, then, for all $n \in \mathbb{N}$, we have

$$\omega_1(A^n) = \omega_1(A)^n = \omega_2(A)^n = \omega_2(A^n)$$

by iterating (2.43) with $B = A$. Since also $\omega_1(\mathbb{I}) = \omega_2(\mathbb{I}) = 1$ by (2.44), we conclude by linearity that $\omega_1 = \omega_2$ on all polynomials in A. By continuity (cf. 2.3.2.2) this implies that $\omega_1 = \omega_2$ on $C^*(A, \mathbb{I})$, since the linear span of all polynomials is dense in $C^*(A, \mathbb{I})$. Using (2.47), we have proved that $\hat{A}(\omega_1) = \hat{A}(\omega_2)$ implies $\omega_1 = \omega_2$.

Since $\hat{A} \in C(X)$ by 2.3.5.1, $\hat{A}$ is continuous. To prove continuity of the inverse, one checks that for $z \in \sigma(A)$ the functional $\hat{A}^{-1}(z) \in \Delta(C^*(A, \mathbb{I}))$ maps A to z (and hence A^n to z^n, etc.). Looking at (2.48), one then sees that $\hat{A}^{-1}$ is continuous. In conclusion, $\hat{A}$ is a homeomorphism. The final claim in 2.5.1.3 is then obvious. ■

An immediate consequence of this theorem is the **continuous functional calculus**.

Corollary 2.5.2 *For each self-adjoint element $A \in \mathfrak{A}$ and each $f \in C(\sigma(A))$ there is an operator $f(A) \in \mathfrak{A}$, which is the obvious expression when f is a polynomial (and in general is given via the uniform approximation of f by polynomials), such that*

$$\sigma(f(A)) = f(\sigma(A)); \tag{2.77}$$

$$\| f(A) \| = \| f \|_\infty . \tag{2.78}$$

In particular, the norm of $f(A)$ in $C^*(A, \mathbb{I})$ coincides with its norm in $\mathfrak{A}$.

Theorem 2.5.1.3 yields an isomorphism $C(\sigma(A)) \to C^*(A, \mathbb{I})$, which is precisely the map $f \to f(A)$ of the continuous functional calculus. The fact that this isomorphism is isometric (see 2.4.1) yields (2.78). Since $f(\sigma(A))$ is the set of values of f on $\sigma(A)$, (2.77) follows from (2.51), with $A \to f(A)$.

The last claim follows by combining 2.5.1.1 with (2.77) and (2.78). ■

Corollary 2.5.3 *The norm in a C^*-algebra is unique (that is, given a C^*-algebra $\mathfrak{A}$ there is no other norm in which $\mathfrak{A}$ is a C^*-algebra).*

First assume $A = A^*$, and apply (2.78) with $f = \mathrm{id}_{\sigma(A)}$. By definition (cf. (2.31)), the sup-norm of $\mathrm{id}_{\sigma(A)}$ is $r(A)$, so that

$$\| A \| = r(A) \quad (A = A^*). \tag{2.79}$$

Since A^*A is self-adjoint for any A, for general $A \in \mathfrak{A}$ we have, using (2.16),

$$\| A \| = \sqrt{r(A^*A)}. \tag{2.80}$$

Since the spectrum is determined by the algebraic structure alone, (2.80) shows that the norm is determined by the algebraic structure as well. ■

Note that Corollary 2.5.3 does not imply that a given *-algebra can be normed only in one way so as to be completed into a C^*-algebra (we will, in fact, encounter an example of the opposite situation). In 2.5.3 the completeness of $\mathfrak{A}$ is assumed from the outset.

Corollary 2.5.4 *A morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras satisfies*

$$\| \varphi(A) \| \leq \| A \|, \tag{2.81}$$

and is therefore automatically continuous.

When $z \in \rho(A)$, so that $(A - z)^{-1}$ exists in $\mathfrak{A}$, then $\varphi(A - z)$ is certainly invertible in $\mathfrak{B}$, for (2.18) implies that $(\varphi(A - z))^{-1} = \varphi((A - z)^{-1})$. Hence $\rho(A) \subseteq \rho(\varphi(A))$, so that

$$\sigma(\varphi(A)) \subseteq \sigma(A). \tag{2.82}$$

Hence $r(\varphi(A)) \leq r(A)$, so that (2.81) follows from (2.80). ∎

For later use we note

Lemma 2.5.5 *When $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a morphism and $A = A^*$ then*

$$f(\varphi(A)) = \varphi(f(A)) \tag{2.83}$$

for all $f \in C(\sigma(A))$ (here $f(A)$ is defined by the continuous functional calculus, and so is $f(\varphi(A))$ in view of (2.82)).

The property is true for polynomials by (2.18), since for those f has its naive meaning. For general f the result then follows by continuity. ∎

2.6 Positivity in C^*-algebras

A bounded operator $A \in \mathfrak{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called positive when $(\Psi, A\Psi) \geq 0$ for all $\Psi \in \mathcal{H}$; this property is equivalent to $A^* = A$ and $\sigma(A) \subseteq \mathbb{R}^+$, and clearly also applies to closed subalgebras of $\mathfrak{B}(\mathcal{H})$. In quantum mechanics this means that the expectation value of the observable A is always positive.

Classically, a function f on some space X is positive simply when $f(x) \geq 0$ for all $x \in X$. This applies, in particular, to elements of the commutative C^*-algebra $C_0(X)$ (where X is a locally compact Hausdorff space). Hence we have a notion of positivity for certain concrete C^*-algebras, which we would like to generalize to arbitrary abstract C^*-algebras. Positivity is one of the most important features in a C^*-algebra; it will, for example, play a central role in the proof of the Gel'fand Neumark theorem. In particular, one is interested in finding a number of equivalent characterizations of positivity.

Definition 2.6.1 *An element A of a C^*-algebra $\mathfrak{A}$ is called* **positive** *when $A = A^*$ and its spectrum is positive; i.e., $\sigma(A) \subset \mathbb{R}^+$. We write $A \geq 0$ or $A \in \mathfrak{A}^+$, where*

$$\mathfrak{A}^+ := \{A \in \mathfrak{A}_{\mathbb{R}} | \, \sigma(A) \subset \mathbb{R}^+\}. \tag{2.84}$$

It is immediate from Theorems 2.3.5.3 and 2.5.1.3 that $A \in \mathfrak{A}_{\mathbb{R}}$ is positive iff its Gel'fand transform $\hat{A}$ is pointwise positive in $C(\sigma(A))$.

Proposition 2.6.2 *The set $\mathfrak{A}^+$ of all positive elements of a C^*-algebra $\mathfrak{A}$ is a* **convex cone**; *that is,*

1. *when $A \in \mathfrak{A}^+$ and $t \in \mathbb{R}^+$ then $tA \in \mathfrak{A}^+$;*

2. *when $A, B \in \mathfrak{A}^+$ then $A + B \in \mathfrak{A}^+$;*

3. *$\mathfrak{A}^+ \cap -\mathfrak{A}^+ = 0$.*

The first property follows from $\sigma(tA) = t\sigma(A)$, which is a special case of (2.77).

Since $\sigma(A) \subseteq [0, r(A)]$, we have $|c - t| \leq c$ for all $t \in \sigma(A)$ and all $c \geq r(A)$. Hence $\sup_{t\in\sigma(A)} |c1_{\sigma(A)} - \hat{A}| \leq c$ by 2.3.5.3 and 2.5.1.3, so that $\| c1_{\sigma(A)} - \hat{A} \|_\infty \leq c$. Gel'fand transforming back to $C^*(A, \mathbb{I})$, this implies $\| c\mathbb{I} - A \| \leq c$ for all $c \geq \| A \|$ by 2.5.2. Inverting this argument, one sees that if $\| c\mathbb{I} - A \| \leq c$ for some $c \geq \| A \|$, then $\sigma(A) \subset \mathbb{R}^+$.

Use this with $A \to A + B$ and $c = \| A \| + \| B \|$; clearly $c \geq \| A + B \|$ by 2.1.1.4. Then

$$\| c\mathbb{I} - (A + B) \| \leq \| (\| A \| - A) \| + \| (\| B \| - B) \| \leq c,$$

where in the last step we used the previous paragraph for A and for B separately. As we have seen, this inequality implies $A + B \in \mathfrak{A}^+$.

Finally, when $A \in \mathfrak{A}^+$ and $A \in -\mathfrak{A}^+$ it must be that $\sigma(A) = 0$, hence $A = 0$ by (2.79) and (2.30). ■

This is important, because a convex cone in a real vector space is equivalent to a **linear partial ordering**, i.e., a partial ordering $\leq$ in which $A \leq B$ implies $A + C \leq B + C$ for all C and $\lambda A \leq \lambda B$ for all $\lambda \in \mathbb{R}^+$. The real vector space in question is the space $\mathfrak{A}_\mathbb{R}$ of all self-adjoint elements of $\mathfrak{A}$. The equivalence between these two structures is as follows: given $\mathfrak{A}^+_\mathbb{R} := \mathfrak{A}^+$ one defines $A \leq B$ if $B - A \in \mathfrak{A}^+_\mathbb{R}$, and given $\leq$ one puts $\mathfrak{A}^+_\mathbb{R} = \{A \in \mathfrak{A}_\mathbb{R} \,|\, 0 \leq A\}$.

For example, when $A = A^*$ one checks the validity of

$$- \| A \| \; \mathbb{I} \leq A \leq \| A \| \; \mathbb{I} \tag{2.85}$$

by taking the Gel'fand transform of $C^*(A, \mathbb{I})$. The implication

$$-B \leq A \leq B \implies \| A \| \leq \| B \| \tag{2.86}$$

then follows, because $-B \leq A \leq B$ and (2.85) for $A \to B$ yield $- \| B \| \; \mathbb{I} \leq A \leq \| B \| \; \mathbb{I}$, so that $\sigma(A) \subseteq [- \| B \|, \| B \|]$, hence $\| A \| \leq \| B \|$ by (2.79) and (2.30). For later use we also record

Lemma 2.6.3 *When $A, B \in \mathfrak{A}^+$ and $\| A + B \| \leq k$ then $\| A \| \leq k$.*

By (2.85) we have $A + B \leq k\mathbb{I}$, hence $0 \leq A \leq k\mathbb{I} - B$ by the linearity of the partial ordering, which also implies that $k\mathbb{I} - B \leq k\mathbb{I}$, as $0 \leq B$. Hence, using $-k\mathbb{I} \leq 0$ (since $k \geq 0$) we obtain $-k\mathbb{I} \leq A \leq k\mathbb{I}$, from which the lemma follows by (2.86). ■

We now come to the central result in the theory of positivity in C^*-algebras, which generalizes the cases $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$ and $\mathfrak{A} = C_0(X)$.

Theorem 2.6.4 *One has*

$$\begin{aligned} \mathfrak{A}^+ &= \{A^2 | \, A \in \mathfrak{A}_\mathbb{R}\} & (2.87) \\ &= \{B^*B | \, B \in \mathfrak{A}\}. & (2.88) \end{aligned}$$

When $\sigma(A) \subset \mathbb{R}^+$ and $A = A^*$ then $\sqrt{A} \in \mathfrak{A}_\mathbb{R}$ is defined by the continuous functional calculus for $f = \sqrt{\cdot}$, and satisfies $\sqrt{A}^2 = A$. Hence $\mathfrak{A}^+ \subseteq \{A^2 | \, A \in \mathfrak{A}_\mathbb{R}\}$. The opposite inclusion follows from (2.77) and 2.5.1.2. This proves (2.87).

The inclusion $\mathfrak{A}^+ \subseteq \{B^*B | \, B \in \mathfrak{A}\}$ is is trivial from (2.87).

Lemma 2.6.5 *Every self-adjoint element A has a decomposition $A = A_+ - A_-$, where $A_+, A_- \in \mathfrak{A}^+$ and $A_+A_- = 0$. Moreover, $\| A_\pm \| \leq \| A \|$.*

Apply the continuous functional calculus with $f = \mathrm{id}_{\sigma(A)} = f_+ - f_-$, where $\mathrm{id}_{\sigma(A)}(t)$, $f_+(t) = \max\{t, 0\}$, and $f_-(t) = \max\{-t, 0\}$. Since $\| f_\pm \|_\infty \leq r(A) = \| A \|$ (where we used (2.79)), the bound follows from (2.78) with $A \to A_\pm$. ■

We use this lemma to prove that $\{B^*B | \, B \in \mathfrak{A}\} \subseteq \mathfrak{A}^+$. Apply the lemma to $A = B^*B$ (noting that $A = A^*$). Then

$$(A_-)^3 = -A_-(A_+ - A_-)A_- = -A_-AA_- = -A_-B^*BA_- = -(BA_-)^*BA_-.$$

Since $\sigma(A_-) \subset \mathbb{R}^+$ because A_- is positive, we see from (2.77) with $f(t) = t^3$ that $(A_-)^3 \geq 0$. Hence $-(BA_-)^*BA_- \geq 0$.

Lemma 2.6.6 *If $-C^*C \in \mathfrak{A}^+$ for some $C \in \mathfrak{A}$ then $C = 0$.*

By (2.13) we can write $C = D + iE$, $D, E \in \mathfrak{A}_{\mathbb{R}}$, so that

$$C^*C = 2D^2 + 2E^2 - CC^*. \tag{2.89}$$

Now for any $A, B \in \mathfrak{A}$ one has

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}. \tag{2.90}$$

This is because for $z \neq 0$ the invertibility of $AB - z$ implies the invertibility of $BA - z$. Namely, one computes that $(BA - z)^{-1} = B(AB - z)^{-1}A - z^{-1}\mathbb{I}$. Applying this with $A \to C$ and $B \to C^*$ we see that the assumption $\sigma(C^*C) \subset \mathbb{R}^-$ implies $\sigma(CC^*) \subset \mathbb{R}^-$, hence $\sigma(-CC^*) \subset \mathbb{R}^+$. By (2.89), (2.87), and 2.6.2.2 we see that $C^*C \geq 0$, i.e., $\sigma(C^*C) \subset \mathbb{R}^+$, so that the assumption $-C^*C \in \mathfrak{A}^+$ now yields $\sigma(C^*C) = 0$. Hence $C = 0$ by 2.6.2.3. ■

The last claim before the lemma therefore implies $BA_- = 0$. As $(A_-)^3 = -(BA_-)^*BA_- = 0$ we see that $(A_-)^3 = 0$, and finally $A_- = 0$ by the continuous functional calculus with $f(t) = t^{1/3}$. Hence $B^*B = A_+$, which lies in $\mathfrak{A}^+$. ■

An important consequence of (2.88) is the fact that inequalities of the type $A_1 \leq A_2$ for $A_1, A_2 \in \mathfrak{A}_{\mathbb{R}}$ are stable under conjugation by arbitrary elements $B \in \mathfrak{A}$, so that $A_1 \leq A_2$ implies $B^*A_1B \leq B^*A_2B$. This is because $A_1 \leq A_2$ is the same as $A_2 - A_1 \geq 0$; by (2.88) there is an $A_3 \in \mathfrak{A}$ such that $A_2 - A_1 = A_3^*A_3$. But clearly $(A_3B)^*A_3B \geq 0$, and this is nothing but $B^*AB \leq B^*A_2B$. For example, replace A in (2.85) by A^*A, and use (2.16), yielding $A^*A \leq\parallel A \parallel^2 \mathbb{I}$. Applying the above principle gives

$$B^*A^*AB \leq\parallel A \parallel^2 B^*B \tag{2.91}$$

for all $A, B \in \mathfrak{A}$.

2.7 Ideals in C^*-algebras

An ideal $\mathfrak{I}$ in a C^*-algebra $\mathfrak{A}$ is defined by 2.2.9. As we have seen, a proper ideal cannot contain $\mathbb{I}$; in order to prove properties of ideals we need a suitable replacement of a unit.

Definition 2.7.1 *An* **approximate unit** *in a non-unital C^*-algebra $\mathfrak{A}$ is a family $\{\mathbb{I}_\lambda\}_{\lambda\in\Lambda}$, where Λ is some directed set (i.e., a set with a partial order and a sense in which $\lambda \to \infty$), with the following properties:*

1.
$$\mathbb{I}_\lambda^* = \mathbb{I}_\lambda \tag{2.92}$$
and $\sigma(\mathbb{I}_\lambda) \subset [0, 1]$, so that
$$\parallel \mathbb{I}_\lambda \parallel \leq 1; \tag{2.93}$$

2.
$$\lim_{\lambda\to\infty} \parallel \mathbb{I}_\lambda A - A \parallel = \lim_{\lambda\to\infty} \parallel A\mathbb{I}_\lambda - A \parallel = 0 \tag{2.94}$$
for all $A \in \mathfrak{A}$.

For example, the C^*-algebra $C_0(\mathbb{R})$ has no unit (the unit would be $1_{\mathbb{R}}$, which does not vanish at infinity because it is constant), but an approximate unit may be constructed as follows: take $\Lambda = \mathbb{N}$, and take $\mathbb{I}_n$ to be a continuous function which is 1 on $[-n, n]$ and vanishes for $|x| > n + 1$. One checks the axioms, and notes that one certainly does not have $\mathbb{I}_n \to 1_{\mathbb{R}}$ in the sup-norm.

Proposition 2.7.2 *Every non-unital C^*-algebra $\mathfrak{A}$ has an approximate unit. When $\mathfrak{A}$ is separable (in containing a countable dense subset) then Λ may be taken to be countable.*

One takes Λ to be the set of all finite subsets of $\mathfrak{A}$, partially ordered by inclusion. Hence $\lambda \in \Lambda$ is of the form $\lambda = \{A_1, \ldots, A_n\}$, from which we build the element $B_\lambda := \sum_i A_i^* A_i$. Clearly B_λ is self-adjoint, and according to 2.6.4 and 2.6.2.2 one has $\sigma(B) \subset \mathbb{R}^+$, so that $n^{-1}\mathbb{I} + B_\lambda$ is invertible in $\mathfrak{A}_\mathbb{I}$. Hence we may form

$$\mathbb{I}_\lambda := B_\lambda(n^{-1}\mathbb{I} + B_\lambda)^{-1}. \tag{2.95}$$

Since B_λ is self-adjoint and B_λ commutes with functions of itself (such as $(n^{-1}\mathbb{I} + B_\lambda)^{-1}$), one has $\mathbb{I}_\lambda^* = \mathbb{I}_\lambda$. Although $(n^{-1}\mathbb{I} + B_\lambda)^{-1}$ is computed in $\mathfrak{A}_\mathbb{I}$, so that it is of the form $C + \mu\mathbb{I}$ for some $C \in \mathfrak{A}$ and $\mu \in \mathbb{C}$, one has $I_\lambda = B_\lambda C + \mu B_\lambda$, which lies in $\mathfrak{A}$. Using the continuous functional calculus on B, with $f(t) = t/(n+t)$, one sees from (2.77) and the positivity of B_λ that $\sigma(\mathbb{I}_\lambda) \subset [0,1]$.

Putting $C_i := \mathbb{I}_\lambda A_i - A_i$, a simple computation shows that

$$\sum_i C_i C_i^* = n^{-2} B_\lambda (n^{-1}\mathbb{I} + B_\lambda)^{-2}. \tag{2.96}$$

We now apply (2.78) with $A \to B_\lambda$ and $f(t) = n^{-2}t(n^{-1} + t)^{-2}$. Since $f \geq 0$ and f assumes its maximum at $t = 1/n$, one has $\sup_{t\in\mathbb{R}^+} |f(t)| = 1/4n$. As $\sigma(B) \subset \mathbb{R}^+$, it follows that $\parallel f \parallel_\infty \leq 1/4n$, hence $\parallel n^{-2}B_\lambda(n^{-1}\mathbb{I} + B_\lambda)^{-2} \parallel \leq 1/4n$ by (2.78), so that $\parallel \sum_i C_i C_i^* \parallel \leq 1/4n$ by (2.96). Lemma 2.6.3 then shows that $\parallel C_i C_i^* \parallel \leq 1/4n$ for each $i = 1, \ldots, n$. Since any $A \in \mathfrak{A}$ sits in some directed subset of Λ with $n \to \infty$, it follows from (2.16) that

$$\lim_{\lambda\to\infty} \parallel \mathbb{I}_\lambda A - A \parallel^2 = \lim_{\lambda\to\infty} \parallel (\mathbb{I}_\lambda A - A)^* \mathbb{I}_\lambda A - A \parallel = \lim_{\lambda\to\infty} \parallel C_i^* C_i \parallel = 0.$$

The other equality in (2.94) follows analogously.

Finally, when $\mathfrak{A}$ is separable one may draw all A_i occurring as elements of $\lambda \in \Lambda$ from a countable dense subset, so that Λ is countable. ■

The main properties of ideals in C^*-algebras are as follows.

Theorem 2.7.3 *Let $\mathfrak{I}$ be an ideal in a C^*-algebra $\mathfrak{A}$.*

1. *If $A \in \mathfrak{I}$ then $A^* \in \mathfrak{I}$; in other words, every ideal in a C^*-algebra is self-adjoint.*
2. *The quotient $\mathfrak{A}/\mathfrak{I}$ is a C^*-algebra in the norm (2.38), the multiplication (2.39), and the involution*

$$\tau(A)^* := \tau(A^*). \tag{2.97}$$

Note that (2.97) is well defined because of 2.7.3.1.

Put $\mathfrak{I}^* := \{A^* | A \in \mathfrak{I}\}$. Note that $J \in \mathfrak{I}$ implies $J^*J \in \mathfrak{I} \cap \mathfrak{I}^*$: it lies in $\mathfrak{I}$ because $\mathfrak{I}$ is an ideal, hence a left-ideal, and it lies in $\mathfrak{I}^*$ because $\mathfrak{I}^*$ is an ideal, hence a right-ideal. Since $\mathfrak{I}$ is an ideal, $\mathfrak{I} \cap \mathfrak{I}^*$ is a C^*-subalgebra of $\mathfrak{A}$. Hence by 2.7.2 it has an approximate unit $\{\mathbb{I}_\lambda\}$. Take $J \in \mathfrak{I}$. Using (2.16) and (2.92), we estimate

$$\parallel J^* - J^*\mathbb{I}_\lambda \parallel^2 = \parallel (J - \mathbb{I}_\lambda J)(J^* - J^*\mathbb{I}_\lambda) \parallel = \parallel (JJ^* - JJ^*\mathbb{I}_\lambda) - \mathbb{I}_\lambda(JJ^* - JJ^*\mathbb{I}_\lambda) \parallel$$

$$\leq \parallel (JJ^* - JJ^*\mathbb{I}_\lambda) \parallel + \parallel \mathbb{I}_\lambda(JJ^* - JJ^*\mathbb{I}_\lambda) \parallel \leq \parallel (J^*J - J^*J\mathbb{I}_\lambda) \parallel + \parallel \mathbb{I}_\lambda \parallel \parallel (JJ^* - JJ^*\mathbb{I}_\lambda) \parallel .$$

As we have seen, $J^*J \in \mathfrak{I} \cap \mathfrak{I}^*$, so that, also using (2.93), both terms vanish for $\lambda \to \infty$. Hence $\lim_{\lambda\to\infty} \parallel J^* - J^*\mathbb{I}_\lambda \parallel = 0$. But $\mathbb{I}_\lambda$ lies in $\mathfrak{I} \cap \mathfrak{I}^*$, so certainly $\mathbb{I}_\lambda \in \mathfrak{I}$, and since $\mathfrak{I}$ is an ideal it must be that $J^*\mathbb{I}_\lambda \in \mathfrak{I}$ for all λ. Hence J^* is a norm-limit of elements in $\mathfrak{I}$; since $\mathfrak{I}$ is closed, it follows that $J^* \in \mathfrak{I}$. This proves 2.7.3.1.

In view of 2.2.10, all we need to prove to establish 2.7.3.2 is the property (2.16). This uses

Lemma 2.7.4 *Let $\{\mathbb{I}_\lambda\}$ be an approximate unit in $\mathfrak{I}$, and let $A \in \mathfrak{A}$. Then*

$$\parallel \tau(A) \parallel = \lim_{\lambda\to\infty} \parallel A - A\mathbb{I}_\lambda \parallel . \tag{2.98}$$

It is obvious from (2.38) that

$$\| A - A\mathbb{I}_\lambda \| \geq \| \tau(A) \| . \tag{2.99}$$

To derive the opposite inequality, add a unit $\mathbb{I}$ to $\mathfrak{A}$ if necessary, pick any $J \in \mathfrak{J}$, and write

$$\| A - A\mathbb{I}_\lambda \| = \| (A+J)(\mathbb{I} - \mathbb{I}_\lambda) + J(\mathbb{I}_\lambda - \mathbb{I} \| \leq \| A + J \| \| \mathbb{I} - \mathbb{I}_\lambda \| + \| J\mathbb{I}_\lambda - J \| .$$

Note that

$$\| \mathbb{I} - \mathbb{I}_\lambda \| \leq 1 \tag{2.100}$$

by 2.7.1.1 and the proof of 2.6.2. The second term on the right-hand side goes to zero for $\lambda \to \infty$, since $J \in \mathfrak{J}$. Hence

$$\lim_{\lambda\to\infty} \| A - A\mathbb{I}_\lambda \| \leq \| A + J \| . \tag{2.101}$$

For each $\epsilon > 0$ we can choose $J \in \mathfrak{J}$ so that (2.40) holds. For this specific J we combine (2.99), (2.101), and (2.40) to find

$$\lim_{\lambda\to\infty} \| A - A\mathbb{I}_\lambda \| - \epsilon \leq \| \tau(A) \| \leq \| A - A\mathbb{I}_\lambda \| .$$

Letting $\epsilon \to 0$ proves (2.98). ■

We now prove (2.16) in $\mathfrak{A}/\mathfrak{J}$. Successively using (2.98), (2.16) in $\mathfrak{A}_{\mathbb{I}}$, (2.100), (2.98), (2.39), and (2.97), we find

$$\| \tau(A) \|^2 = \lim_{\lambda\to\infty} \| A - A\mathbb{I}_\lambda \|^2 = \lim_{\lambda\to\infty} \| (A - A\mathbb{I}_\lambda)^*(A - A\mathbb{I}_\lambda) \|$$
$$= \lim_{\lambda\to\infty} \| (\mathbb{I} - \mathbb{I}_\lambda)A^*A(\mathbb{I} - \mathbb{I}_\lambda) \| \leq \lim_{\lambda\to\infty} \| \mathbb{I} - \mathbb{I}_\lambda \| \| A^*A(\mathbb{I} - \mathbb{I}_\lambda) \| \leq \lim_{\lambda\to\infty} \| A^*A(\mathbb{I} - \mathbb{I}_\lambda) \|$$
$$= \| \tau(A^*A) \| = \| \tau(A)\tau(A^*) \| = \| \tau(A)\tau(A)^* \| .$$

Lemma 2.1.11 then implies (2.16). ■

This seemingly technical result is very important.

Corollary 2.7.5 *The kernel of a morphism between two C^*-algebras is an ideal. Conversely, every ideal in a C^*-algebra is the kernel of some morphism. Hence every morphism has norm 1.*

The first claim is almost trivial, since $\varphi(A) = 0$ implies $\varphi(AB) = \varphi(BA) = 0$ for all B by (2.18). Also, since φ is continuous (see 2.5.4) its kernel is closed.

The converse follows from Theorem 2.7.3, since $\mathfrak{J}$ is the kernel of the canonical projection $\tau : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$, where $\mathfrak{A}/\mathfrak{J}$ is a C^*-algebra, and τ is a morphism by (2.39), and (2.97).

The final claim follows from the preceding one, since $\| \tau \| = 1$. ■

For the next consequence of 2.7.3 we need a

Lemma 2.7.6 *An injective morphism between C^*-algebras is isometric. In particular, its range is closed.*

Assume there is an $B \in \mathfrak{A}$ for which $\| \varphi(B) \| \neq \| B \|$. By (2.16), (2.18), and (2.19) this implies $\| \varphi(B^*B) \| \neq \| B^*B \|$. Put $A := B^*B$, noting that $A^* = A$. By (2.80) and (2.30) we must have $\sigma(A) \neq \sigma(\varphi(A))$. Then (2.82) implies $\sigma(\varphi(A)) \subset \sigma(A)$. By Urysohn's lemma there is a nonzero $f \in C(\sigma(A))$ which vanishes on $\sigma(\varphi(A))$, so that $f(\varphi(A)) = 0$. By Lemma 2.5.5 we have $\varphi(f(A)) = 0$, contradicting the injectivity of φ. ■

Corollary 2.7.7 *The image of a morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras is closed. In particular, $\varphi(\mathfrak{A})$ is a C^*-subalgebra of $\mathfrak{B}$.*

Define $\psi : \mathfrak{A}/\ker(\varphi) \to \mathfrak{B}$ by $\psi([A]) = \varphi(A)$, where $[A]$ is the equivalence class in $\mathfrak{A}/\ker(\varphi)$ of $A \in \mathfrak{A}$. By the theory of vector spaces, ψ is a vector space isomorphism between $\mathfrak{A}/\ker(\varphi)$ and $\varphi(\mathfrak{A})$, and $\varphi = \psi \circ \tau$. In particular, ψ is injective. According to 2.7.5 and 2.7.3.2, the space $\mathfrak{A}/\ker(\varphi)$ is a C^*-algebra. Since φ and τ are morphisms, ψ is a C^*-algebra morphism. Hence $\psi(\mathfrak{A}/\ker(\varphi))$ has closed range in $\mathfrak{B}$ by 2.7.6. But $\psi(\mathfrak{A}/\ker(\varphi)) = \varphi(\mathfrak{A})$, so that φ has closed range in $\mathfrak{B}$. Since φ is a morphism, its image is a *-algebra in $\mathfrak{B}$, which by the preceding sentence is closed in the norm of $\mathfrak{B}$. Hence $\varphi(\mathfrak{A})$, inheriting all operations in $\mathfrak{B}$, is a C^*-algebra. ■

2.8 States

The notion of a state in the sense defined below comes from quantum mechanics, but states play a central role in the general theory of abstract C^*-algebras also.

Definition 2.8.1 *A* **state** *on a unital C^*-algebra $\mathfrak{A}$ is a linear map $\omega : \mathfrak{A} \to \mathbb{C}$ which is* **positive** *in that $\omega(A) \geq 0$ for all $A \in \mathfrak{A}^+$, and* **normalized** *in that*

$$\omega(\mathbb{I}) = 1. \tag{2.102}$$

The **state space** *$\mathcal{S}(\mathfrak{A})$ of $\mathfrak{A}$ consists of all states on $\mathfrak{A}$.*

For example, when $\mathfrak{A} \subseteq \mathfrak{B}(\mathcal{H})$ then every $\Psi \in \mathcal{H}$ with norm 1 defines a state ψ by

$$\psi(A) := (\Psi, A\Psi). \tag{2.103}$$

Positivity follows from Theorem 2.6.4, since $\psi(B^*B) = \| B\Psi \|^2 \geq 0$, and normalization is obvious from $\psi(\mathbb{I}) = (\Psi, \Psi) = 1$.

Theorem 2.8.2 *The state space of $\mathfrak{A} = C(X)$ consists of all probability measures on X.*

By the Riesz theorem of measure theory, each positive linear map $\omega : C(X) \to \mathbb{C}$ is given by a regular positive measure μ_ω on X. The normalization of ω implies that $\omega(1_X) = \mu_\omega(X) = 1$, so that μ_ω is a probability measure. ■

The positivity of ω with 2.6.4 implies that $(A, B)_\omega := \omega(A^*B)$ defines a pre-inner product on $\mathfrak{A}$. Hence from (2.1) we obtain

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B), \tag{2.104}$$

which will often be used. Moreover, for all $A \in \mathfrak{A}$ one has

$$\omega(A^*) = \overline{\omega(A)}, \tag{2.105}$$

as $\omega(A^*) = \omega(A^*\mathbb{I}) = (A, \mathbb{I})_\omega = \overline{(\mathbb{I}, A)_\omega} = \omega(A)$.

Partly in order to extend the definition of a state to non-unital C^*-algebras, we have

Proposition 2.8.3 *A linear map $\omega : \mathfrak{A} \to \mathbb{C}$ on a unital C^*-algebra is positive iff ω is bounded and*

$$\| \omega \| = \omega(\mathbb{I}). \tag{2.106}$$

In particular:

1. *A state on a unital C^*-algebra is bounded, with norm 1.*
2. *An element $\omega \in \mathfrak{A}^*$ for which $\| \omega \| = \omega(\mathbb{I}) = 1$ is a state on $\mathfrak{A}$.*

When ω is positive and $A = A^*$ we have, using (2.85), the bound $|\omega(A)| \leq \omega(\mathbb{I}) \| A \|$. For general A we use (2.104) with $A = \mathbb{I}$, (2.16), and the bound just derived to find

$$|\omega(B)|^2 \leq \omega(B^*B)\omega(\mathbb{I}) \leq \omega(\mathbb{I})^2 \| B^*B \| = \omega(\mathbb{I})^2 \| B \|^2 .$$

Hence $\| \omega \| \leq \omega(\mathbb{I})$. Since the upper bound is reached by $B = \mathbb{I}$, we have (2.106).

To prove the converse claim, we first note that the argument around (2.66) may be copied, showing that ω is real on $\mathfrak{A}_\mathbb{R}$. Next, we show that $A \geq 0$ implies $\omega(A) \geq 0$. Choose $s > 0$ small enough, so that $\| \mathbb{I} - sA \| \leq 1$. Then (assuming $\omega \neq 0$)

$$1 \geq \| \mathbb{I} - sA \| = \frac{\| \omega \|}{\omega(\mathbb{I})} \| \mathbb{I} - sA \| \geq \frac{|\omega(\mathbb{I} - sA)|}{\omega(\mathbb{I})}.$$

Hence $|\omega(\mathbb{I}) - s\omega(A)| \leq \omega(\mathbb{I})$, which is only possible when $\omega(A) \geq 0$. ■

We now pass to states on C^*-algebras without unit. Firstly, we look at a state in a more general context.

Definition 2.8.4 *A* **positive map** $\mathcal{Q} : \mathfrak{A} \rightarrow \mathfrak{B}$ *between two* C^**-algebras is a linear map with the property that* $A \geq 0$ *implies* $\mathcal{Q}(A) \geq 0$ *in* $\mathfrak{B}$.

Proposition 2.8.5 *A positive map between two* C^**-algebras is bounded (continuous).*

Let us first show that boundedness on $\mathfrak{A}^+$ implies boundedness on $\mathfrak{A}$. Using (2.13) and 2.6.5, we can write

$$A = A'_+ - A'_- + iA''_+ - iA''_-, \tag{2.107}$$

where A'_+ etc. are positive. Since $\| A' \| \leq \| A \|$ and $\| A'' \| \leq \| A \|$ by (2.13), we have $\| B \| \leq \| A \|$ for $B = A'_+, A'_-, A''_+$, or A''_- by 2.6.5. Hence if $\| \mathcal{Q}(B) \| \leq C \| B \|$ for all $B \in \mathfrak{A}^+$ and some $C > 0$, then

$$\| \mathcal{Q}(A) \| \leq \| \mathcal{Q}(A'_+) \| + \| \mathcal{Q}(A'_-) \| + \| \mathcal{Q}(A''_+) \| + \| \mathcal{Q}(A''_-) \| \leq 4C \| A \| .$$

Now assume that $\mathcal{Q}$ is not bounded; by the previous argument it is not bounded on $\mathfrak{A}^+$, so that for each $n \in \mathbb{N}$ there is an $A_n \in \mathfrak{A}^+_1$ so that $\| \mathcal{Q}(A_n) \| \geq n^3$ (here $\mathfrak{A}^+_1$ consists of all $A \in \mathfrak{A}^+$ with $\| A \| \leq 1$). The series $\sum_{n=0}^{\infty} n^{-2} A_n$ obviously converges to some $A \in \mathfrak{A}^+$. Since $\mathcal{Q}$ is positive, we have $\mathcal{Q}(A) \geq n^{-2}\mathcal{Q}(A_n) \geq 0$ for each n. Hence by (2.86)

$$\| \mathcal{Q}(A) \| \geq n^{-2} \| \mathcal{Q}(A_n) \| \geq n$$

for all $n \in \mathbb{N}$, which is impossible since $\| \mathcal{Q}(A) \|$ is some finite number. Thus $\mathcal{Q}$ is bounded on $\mathfrak{A}^+$, and therefore on $\mathfrak{A}$ by the previous paragraph. ■

Choosing $\mathfrak{B} = \mathbb{C}$, we see that a state on a unital C^*-algebra is a special case of a positive map between C^*-algebras; Proposition 2.8.5 then provides an alternative proof of 2.8.3.1. Hence in the non-unital case we may replace the normalization condition in 2.8.1 as follows.

Definition 2.8.6 *A* **state** *on a* C^**-algebra* $\mathfrak{A}$ *is a linear map* $\omega : \mathfrak{A} \rightarrow \mathbb{C}$ *which is positive and has norm 1.*

This definition is possible by 2.8.5, and is consistent with 2.8.1 because of (2.106). The following result is very useful; cf. 2.4.6.

Proposition 2.8.7 *A state* ω *on a* C^**-algebra without unit has a unique extension to a state* $\omega_{\mathbb{I}}$ *on the unitization* $\mathfrak{A}_{\mathbb{I}}$.

The extension in question is defined by

$$\omega_{\mathbb{I}}(A + \lambda\mathbb{I}) := \omega(A) + \lambda. \tag{2.108}$$

This obviously satisfies (2.102); it remains to prove positivity.

Since a state ω on $\mathfrak{A}$ is bounded by 2.8.5, we have $|\omega(A - A\mathbb{I}_\lambda)| \rightarrow 0$ for any approximate unit in $\mathfrak{A}$. The derivation of (2.104) and (2.105) may then be copied from the unital case; in particular, one still has $|\omega(A)|^2 \leq \omega(A^*A)$. Combining this with (2.105), we obtain from (2.108) that

$$\omega_{\mathbb{I}}((A + \lambda\mathbb{I})^*(A + \lambda\mathbb{I})) \geq |\omega(A) + \overline{\lambda}|^2 \geq 0.$$

Hence ω is positive by (2.88). ■

There are lots of states:

Lemma 2.8.8 *For every* $A \in \mathfrak{A}$ *and* $a \in \sigma(A)$ *there is a state* ω_a *on* $\mathfrak{A}$ *for which* $\omega(A) = a$. *When* $A = A^*$ *there exists a state* ω *such that* $|\omega(A)| = \| A \|$.

If necessary we add a unit to $\mathfrak{A}$ (this is justified by 2.8.7). Define a linear map $\tilde{\omega}_a : \mathbb{C}A+\mathbb{C}\mathbb{I} \to \mathbb{C}$ by $\tilde{\omega}_a(\lambda A + \mu\mathbb{I}) := \lambda a + \mu$. Since $a \in \sigma(A)$ one has $\lambda a + \mu \in \sigma(\lambda A + \mu\mathbb{I})$; this easily follows from the definition of σ. Hence (2.41) with $A \to \lambda A + \mu\mathbb{I}$ implies $|\tilde{\omega}_a(\lambda A + \mu\mathbb{I})| \leq\| (\lambda A + \mu\mathbb{I}) \|$. Since $\tilde{\omega}_a(\mathbb{I}) = 1$, it follows that $\| \tilde{\omega} \|= 1$. By the Hahn-Banach Theorem 2.1.6, there exists an extension ω_a of $\tilde{\omega}$ to $\mathfrak{A}$ of norm 1. By 2.8.3.2 ω_a is a state, which clearly satisfies $\omega_a(A) = \tilde{\omega}_a(A) = a$.

Since $\sigma(A)$ is closed by 2.2.3.2, there is an $a \in \sigma(A)$ for which $r(A) = |a|$. For this a one has $|\omega(A)| = |a| = r(A) =\| A \|$ by (2.79). ■

An important feature of a state space $\mathcal{S}(\mathfrak{A})$ is that it is a **convex set**. A convex set C in a vector space $\mathcal{C}$ is a subset of $\mathcal{V}$ such that the convex sum $\lambda v + (1-\lambda)w$ belongs to $\mathcal{C}$ whenever $v, w \in \mathcal{C}$ and $\lambda \in [0,1]$. Repeating this process, it follows that $\sum_i p_i v_i$ belongs to $\mathcal{C}$ when all $p_i \geq 0$ and $\sum_i p_i = 1$, and all $v_i \in \mathcal{C}$. In the unital case it is clear that $\mathcal{S}(\mathfrak{A})$ is convex, since both positivity and normalization are clearly preserved under convex sums. In the non-unital case one arrives at this conclusion most simply via 2.8.7.

We return to the unital case. Let $\mathcal{S}(\mathfrak{A})$ be the state space of a unital C^*-algebra $\mathfrak{A}$. We saw in 2.8.3 that each element ω of $\mathcal{S}(\mathfrak{A})$ is continuous, so that $\mathcal{S}(\mathfrak{A}) \subset \mathfrak{A}^*$. Since w^*-limits obviously preserve positivity and normalization, we see that $\mathcal{S}(\mathfrak{A})$ is closed in $\mathfrak{A}^*$ if the latter is equipped with the w^*-topology. Moreover, $\mathcal{S}(\mathfrak{A})$ is a closed subset of the unit ball of $\mathfrak{A}^*$ by 2.8.3.1, so that $\mathcal{S}(\mathfrak{A})$ is compact in the (relative) w^*-topology by the Banach-Alaoglu theorem.

It follows that the state space of a unital C^*-algebra is a **compact convex set**. The very simplest example is $\mathfrak{A} = \mathbb{C}$, in which case $\mathcal{S}(\mathfrak{A})$ is a point.

The next case is $\mathfrak{A} = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2$. The dual is $\mathbb{C}^2$ as well, so that each element of $(\mathbb{C}^2)^*$ is of the form $\omega(\lambda \dot{+} \mu) = c_1\lambda_1 + c_2\lambda_2$. Positive elements of $\mathbb{C} \oplus \mathbb{C}$ are fo the form $\lambda \dot{+} \mu$ with $\lambda \geq 0$ and $\mu \geq 0$, so that a positive functional must have $c_1 \geq 0$ and $c_2 \geq 0$. Finally, since $\mathbb{I} = 1 \dot{+} 1$, normalization yields $c_1 + c_2 = 1$. We conclude that $\mathcal{S}(\mathbb{C} \oplus \mathbb{C})$ may be identified with the interval $[0, 1]$.

Now consider $\mathfrak{A} = \mathfrak{M}^2(\mathbb{C})$. We identify $\mathfrak{M}^2(\mathbb{C})$ with its dual through the pairing $\omega(A) = \mathrm{Tr}\,\omega A$. It follows that $\mathcal{S}(\mathfrak{A})$ consists of all positive 2×2 matrices ρ with $\mathrm{Tr}\,\rho = 1$; these are the density matrices of quantum mechanics. To identify $\mathcal{S}(\mathfrak{A})$ with a familiar compact convex set, we parametrize

$$\rho = \tfrac{1}{2} \begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix}, \tag{2.109}$$

where $x, y, z \in \mathbb{R}$. The positivity of this matrix then corresponds to the constraint $x^2+y^2+z^2 \leq 1$. Hence $\mathcal{S}(\mathfrak{M}^2(\mathbb{C}))$ is the unit ball in $\mathbb{R}^3$.

2.9 Representations and the GNS-construction

The material of this section explains how the usual Hilbert space framework of quantum mechanics emerges from the C^*-algebraic setting.

Definition 2.9.1 *A* **representation** *of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ is a (complex) linear map $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ satisfying*

$$\begin{aligned} \pi(A \cdot B) &= \pi(A)\pi(B); \\ \pi(A^*) &= \pi(A)^* \end{aligned} \tag{2.110}$$

for all $A, B \in \mathfrak{A}$.

A representation π is automatically continuous, satisfying the bound

$$\| \pi(A) \| \leq \| A \| . \tag{2.111}$$

This is because π is a morphism; cf. (2.81). In particular, $\| \pi(A) \|=\| A \|$ when π is faithful by Lemma 2.7.6.

There is a natural equivalence relation in the set of all representations of $\mathfrak{A}$: two representations π_1, π_2 on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively, are called **equivalent** if there exists a unitary isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(A)U^* = \pi_2(A)$ for all $A \in \mathfrak{A}$.

The map $\pi(A) = 0$ for all $A \in \mathfrak{A}$ is a representation; more generally, such trivial π may occur as a summand. To exclude this possibility, one says that a representation is **non-degenerate** if 0 is the only vector annihilated by all representatives of $\mathfrak{A}$.

A representation π is called **cyclic** if its carrier space $\mathcal{H}$ contains a **cyclic vector** Ω for π; this means that the closure of $\pi(\mathfrak{A})\Omega$ (which in any case is a closed subspace of $\mathcal{H}$) coincides with $\mathcal{H}$.

Proposition 2.9.2 *Any non-degenerate representation π is a direct sum of cyclic representations.*

The proof uses a lemma which appears in many other proofs as well.

Lemma 2.9.3 *Let $\mathfrak{M}$ be a *-algebra in $\mathfrak{B}(\mathcal{H})$, take a nonzero vector $\Psi \in \mathcal{H}$, and let p be the projection onto the closure of $\mathfrak{M}\Psi$. Then $p \in \mathfrak{M}'$ (that is, $[p, A] = 0$ for all $A \in \mathfrak{M}$).*

If $A \in \mathfrak{M}$ then $Ap\mathcal{H} \subseteq p\mathcal{H}$ by definition of p. Hence $p^{\perp}Ap = 0$ with $p^{\perp} = \mathbb{I} - p$; this reads $Ap = pAp$. When $A = A^*$ then

$$(Ap)^* = pA = (pAp)^* = pAp = Ap,$$

so that $[A, p] = 0$. By (2.13) this is true for all $A \in \mathfrak{M}$. ■

Apply this lemma with $\mathfrak{M} = \pi(\mathfrak{A})$; the assumption of non-degeneracy guarantees that p is nonzero, and the conclusion implies that $A \to p\pi(A)$ defines a subrepresentation of $\mathfrak{A}$ on $p\mathcal{H}$. This subrepresentation is clearly cyclic, with cyclic vector Ψ. This process may be repeated on $p^{\perp}\mathcal{H}$, etc. ■

If π is a non-degenerate representation of $\mathfrak{A}$ on $\mathcal{H}$, then any unit vector $\Psi \in \mathcal{H}$ defines a state $\psi \in \mathcal{S}(\mathfrak{A})$, referred to as a **vector state** relative to π, by means of (2.103). Conversely, from any state $\omega \in \mathcal{S}(\mathfrak{A})$ one can construct a cyclic representation π_ω on a Hilbert space $\mathcal{H}_\omega$ with cyclic vector Ω_ω in the following way. We restrict ourselves to the unital case; the general case follows by adding a unit to $\mathfrak{A}$ and using 2.8.7.

Construction 2.9.4 1. *Given $\omega \in \mathcal{S}(\mathfrak{A})$, define the sesquilinear form $(\, ,\,)_0$ on $\mathfrak{A}$ by*

$$(A, B)_0 := \omega(A^*B). \tag{2.112}$$

Since ω is a state, hence a positive functional, this form is positive semi-definite (this means that $(A, A)_0 \geq 0$ for all A). Its null space

$$\mathcal{N}_\omega = \{A \in \mathfrak{A} \mid \omega(A^*A) = 0\} \tag{2.113}$$

is a closed left-ideal in $\mathfrak{A}$.

2. *The form $(\, ,\,)_0$ projects to an inner product $(\, ,\,)_\omega$ on the quotient $\mathfrak{A}/\mathcal{N}_\omega$. If $V : \mathfrak{A} \to \mathfrak{A}/\mathcal{N}_\omega$ is the canonical projection, then by definition*

$$(VA, VB)_\omega := (A, B)_0. \tag{2.114}$$

The Hilbert space $\mathcal{H}_\omega$ is the closure of $\mathfrak{A}/\mathcal{N}_\omega$ in this inner product.

3. *The representation $\pi_\omega(\mathfrak{A})$ is firstly defined on $\mathfrak{A}/\mathcal{N}_\omega \subset \mathcal{H}_\omega$ by*

$$\pi_\omega(A)VB := VAB; \tag{2.115}$$

it follows that π_ω is continuous. Hence $\pi_\omega(A)$ may be defined on all of $\mathcal{H}_\omega$ by continuous extension of (2.115), where it satisfies (2.110).

4. *The cyclic vector is defined by $\Omega_\omega = V\mathbb{I}$, so that*

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = \omega(A) \quad \forall A \in \mathfrak{A}. \tag{2.116}$$

We now prove the various claims made here. First note that the null space $\mathcal{N}_\omega$ of $(\,,\,)_0$ can be defined in two equivalent ways;

$$\mathcal{N}_\omega := \{A \in \mathfrak{A} \mid (A, A)_0 = 0\} = \{A \in \mathfrak{A} \mid (A, B)_0 = 0 \,\forall\, B \in \mathfrak{A}\}. \tag{2.117}$$

The equivalence follows from the Cauchy-Schwarz inequality (2.104). The equality (2.117) implies that $\mathcal{N}_\omega$ is a left-ideal, which is closed because of the continuity of ω. This is important, because it implies that the map $\rho(A) : \mathfrak{A} \to \mathfrak{A}$ defined in (2.72) quotients well to a map from $\mathfrak{A}/\mathcal{N}_\omega$ to $\mathfrak{A}/\mathcal{N}_\omega$; the latter map is π_ω defined in (2.115). Since ρ is a morphism, it is easily checked that π_ω is a morphism as well, satisfying (2.110) on the dense subspace $\mathfrak{A}/\mathcal{N}_\omega$ of $\mathcal{H}_\omega$.

To prove that π_ω is continuous on $\mathfrak{A}/\mathcal{N}_\omega$, we compute $\| \pi_\omega(A)\Psi \|^2$ for $\Psi = VB$, where $A, B \in \mathfrak{A}$. By (2.114) and step 2 above, one has $\| \pi_\omega(A)\Psi \|^2 = \omega(B^*AA * B)$. By (2.91) and the positivity of ω one has $\omega(B^*AA * B) \leq \| A \|^2 \omega(B^*B)$. But $\omega(B^*B) = \| \Psi \|^2$, so that $\| \pi_\omega(A)\Psi \| \leq \| A \| \| \Psi \|$, upon which

$$\| \pi_\omega(A) \| \leq \| A \| \tag{2.118}$$

follows from (2.3).

For later use we mention that the GNS-construction yields

$$(\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega) = \omega(A^*B). \tag{2.119}$$

Putting $B = A$ yields

$$\| \pi_\omega(A)\Omega_\omega \|^2 = \omega(A^*A), \tag{2.120}$$

which may alternatively be derived from (2.116) and the fact that π_ω is a representation.

Proposition 2.9.5 *If $(\pi(\mathfrak{A}), \mathcal{H})$ is cyclic then the GNS-representation $(\pi_\omega(\mathfrak{A}), \mathcal{H}_\omega)$ defined by any vector state Ω (corresponding to a cyclic unit vector $\Omega \in \mathcal{H}$) is unitarily equivalent to $(\pi(\mathfrak{A}), \mathcal{H})$.*

This is very simple to prove: the operator $U : \mathcal{H}_\omega \to \mathcal{H}$ implementing the equivalence is initially defined on the dense subspace $\pi_\omega(\mathfrak{A})\Omega_\omega$ by $U\pi_\omega(A)\Omega_\omega = \pi(A)\Omega$; this operator is well-defined, for $\pi_\omega(A)\Omega_\omega = 0$ implies $\pi(A)\Omega = 0$ by the GNS-construction. It follows from (2.116) that U is unitary as a map from $\mathcal{H}_\omega$ to $U\mathcal{H}_\omega$, but since Ω is cyclic for π the image of U is $\mathcal{H}$. Hence U is unitary. It is trivial to verify that U intertwines π_ω and π. ∎

Corollary 2.9.6 *If the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ of two cyclic representations π_1, π_2 each contain a cyclic vector $\Omega_1 \in \mathcal{H}_1$, $\Omega_2 \in \mathcal{H}_2$, and*

$$\omega_1(A) := (\Omega_1, \pi_1(A)\Omega_1) = (\Omega_2, \pi_2(A)\Omega_2) =: \omega_2(A)$$

for all $A \in \mathfrak{A}$, then $\pi_1(\mathfrak{A})$ and $\pi_2(\mathfrak{A})$ are equivalent.

By 2.9.5 the representation π_1 is equivalent to the GNS-representation π_{ω_1}, and π_2 is equivalent to π_{ω_2}. On the other hand, π_{ω_1} and π_{ω_2} are induced by the same state, so they must coincide. ∎

2.10 The Gel'fand-Neumark theorem

One of the main results in the theory of C^*-algebras is

Theorem 2.10.1 *A C^*-algebra is isomorphic to a subalgebra of $\mathfrak{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.*

The GNS-construction leads to a simple proof this theorem, which uses the following notion.

Definition 2.10.2 *The* **universal representation** π_{u} *of a C^*-algebra $\mathfrak{A}$ is the direct sum of all its GNS-representations π_ω, $\omega \in \mathcal{S}(\mathfrak{A})$; hence it is defined on the Hilbert space $\mathcal{H}_{\text{u}} = \oplus_{\omega \in \mathcal{S}(\mathfrak{A})} \mathcal{H}_\omega$.*

Theorem 2.10.1 then follows by taking $\mathcal{H} = \mathcal{H}_{\mathrm{u}}$; the desired isomorphism is π_{u}. To prove that π_{u} is injective, suppose that $\pi_{\mathrm{u}}(A) = 0$ for some $A \in \mathfrak{A}$. By definition of a direct sum, this implies $\pi_\omega(A) = 0$ for all states ω. Hence $\pi_\omega(A)\Omega_\omega = 0$, hence $\| \pi_\omega(A)\Omega_\omega \|^2 = 0$; by (2.120) this means $\omega(A^*A) = 0$ for all states ω, which implies $\| A^*A \| = 0$ by Lemma 2.8.8, so that $\| A \| = 0$ by (2.16), and finally $A = 0$ by the definition of a norm.

Being injective, the morphism π_{u} is isometric by Lemma 2.7.6. ∎

While the universal representation leads to a nice proof of 2.10.1, the Hilbert space $\mathcal{H}_{\mathrm{u}}$ is absurdly large; in practical examples a better way of obtaining a faithful representation always exists. For example, the best faithful representation of $\mathfrak{B}(\mathcal{H})$ is simply its defining one.

Another consequence of the GNS-construction, or rather of 2.10.2, is

Corollary 2.10.3 *An operator $A \in \mathfrak{A}$ is positive (that is, $A \in \mathfrak{A}^+_{\mathbb{R}}$) iff $\pi(A) \geq 0$ for all cyclic representations π.*

2.11 Complete positivity

We have seen that a positive map $\mathcal{Q}$ (cf. Definition 2.8.4 generalizes the notion of a state, in that the $\mathbb{C}$ in $\omega : \mathfrak{A} \to \mathbb{C}$ is replaced by a general C^*-algebra $\mathfrak{B}$ in $\mathcal{Q} : A \to \mathfrak{B}$. We would like to see if one can generalize the GNS-construction. It turns out that for this purpose one needs to impose a further condition on $\mathcal{Q}$.

We first introduce the C^*-algebra $\mathfrak{M}^n(\mathfrak{A})$ for a given C^*-algebra $\mathfrak{A}$ and $n \in \mathbb{N}$. The elements of $\mathfrak{M}^n(\mathfrak{A})$ are $n \times n$ matrices with entries in $\mathfrak{A}$; multiplication is done in the usual way, i.e, $(MN)_{ij} := \sum_k M_{ik}N_{kj}$, with the difference that one now multiplies elements of $\mathfrak{A}$ rather than complex numbers. In particular, the order has to be taken into account. The involution in $\mathfrak{M}^n(\mathfrak{A})$ is, of course, given by $(M^*)_{ij} = M^*_{ji}$, in which the involution in $\mathfrak{A}$ replaces the usual complex conjugation in $\mathbb{C}$. One may identify $\mathfrak{M}^n(\mathfrak{A})$ with $\mathfrak{A} \otimes \mathfrak{M}^n(\mathbb{C})$ in the obvious way.

When π is a faithful representation of $\mathfrak{A}$ (which exists by Theorem 2.10.1), one obtains a faithful realization π_n of $\mathfrak{M}^n(\mathfrak{A})$ on $\mathcal{H} \otimes \mathbb{C}^n$, defined by linear extension of $\pi_n(M)v_i := \pi(M_{ij})v_j$; we here look at elements of $\mathcal{H} \otimes \mathbb{C}^n$ as n-tuples $(v_1, \ldots, v_n)$, where each $v_i \in \mathcal{H}$. The norm $\| M \|$ of $M \in \mathfrak{M}^n(\mathfrak{A})$ is then simply defined to be the norm of $\pi_n(M)$. Since $\pi_n(\mathfrak{M}^n(\mathfrak{A}))$ is a closed *-algebra in $\mathfrak{B}(\mathcal{H} \otimes \mathbb{C}^n)$ (because $n < \infty$), it is obvious that $\mathfrak{M}^n(\mathfrak{A})$ is a C^*-algebra in this norm. The norm is unique by Corollary 2.5.3, so that this procedure does not depend on the choice of π.

Definition 2.11.1 *A linear map $\mathcal{Q} : \mathfrak{A} \to \mathfrak{B}$ between C^*-algebras is called* **completely positive** *if for all $n \in \mathbb{N}$ the map $\mathcal{Q}_n : \mathfrak{M}^n(\mathfrak{A}) \to \mathfrak{M}^n(\mathfrak{B})$, defined by $(\mathcal{Q}_n(M))_{ij} := \mathcal{Q}(M_{ij})$, is positive.*

For example, a morphism φ is a completely positive map, since when $\mathbb{A} = \mathbb{B}^*\mathbb{B}$ in $\mathfrak{M}^n(\mathfrak{A})$, then $\varphi(\mathbb{A}) = \varphi(\mathbb{B})^*\varphi(\mathbb{B})$, which is positive in $\mathfrak{M}^n(\mathfrak{B})$. In particular, any representation of $\mathfrak{A}$ on $\mathcal{H}$ is a completely positive map from $\mathfrak{A}$ to $\mathfrak{B}(\mathcal{H})$.

If we also assume that $\mathfrak{A}$ and $\mathfrak{B}$ are unital, and that $\mathcal{Q}$ is normalized, we get an interesting generalization of the GNS-construction, which is of central importance for quantization theory. This generalization will appear as the proof of the following **Stinespring theorem**.

Theorem 2.11.2 *Let $\mathcal{Q} : \mathfrak{A} \to \mathfrak{B}$ be a completely positive map between C^*-algebras with unit, such that $\mathcal{Q}(\mathbb{I}) = \mathbb{I}$. By Theorem 2.10.1, we may assume that $\mathfrak{B}$ is faithfully represented as a subalgebra $\mathfrak{B} \simeq \pi_\chi(\mathfrak{B}) \subseteq \mathfrak{B}(\mathcal{H}_\chi)$, for some Hilbert space $\mathcal{H}_\chi$.*

There exists a Hilbert space $\mathcal{H}^\chi$, a representation π^χ of $\mathfrak{A}$ on $\mathcal{H}^\chi$, and a partial isometry $W : \mathcal{H}_\chi \to \mathcal{H}^\chi$ (with $W^*W = \mathbb{I}$), such that

$$\pi_\chi(\mathcal{Q}(A)) = W^*\pi^\chi(A)W \quad \forall A \in \mathfrak{A}. \tag{2.121}$$

Equivalently, with $p := WW^*$ (the target projection of W on $\mathcal{H}^\chi$), $\tilde{\mathcal{H}}_\chi := p\mathcal{H}^\chi \subset \mathcal{H}^\chi$, and $U : \mathcal{H}_\chi \to \tilde{\mathcal{H}}_\chi$ defined as W, seen as map not from $\mathcal{H}_\chi$ to $\mathcal{H}^\chi$ but as a map from $\mathcal{H}_\chi$ to $\tilde{\mathcal{H}}_\chi$, so that U is unitary, one has

$$U\pi_\chi(\mathcal{Q}(A))U^{-1} = p\pi^\chi(A)p. \tag{2.122}$$

The proof consists of a modification of the GNS-construction. It uses the notion of a **partial isometry**. This is a linear map $W : \mathcal{H}_1 \to \mathcal{H}_2$ between two Hilbert spaces, with the property that $\mathcal{H}_1$ contains a closed subspace $\mathcal{K}_1$ such that $(W\Psi, W\Phi)_2 = (\Psi, \Phi)_1$ for all $\Psi, \Phi \in \mathcal{K}_1$, and $W = 0$ on $\mathcal{K}_1^{\perp}$. Hence W is unitary from $\mathcal{K}_1$ to $W\mathcal{K}_1$. It follows that $WW^* = [\mathcal{K}_2]$ and $W^*W = [\mathcal{K}_1]$ are projections onto the image and the kernel of W, respectively.

We denote elements of $\mathcal{H}_\chi$ by v, w, with inner product $(v, w)_\chi$.

Construction 2.11.3 *1. Define the sesquilinear form* $(\, , \,)_0^\chi$ *on* $\mathfrak{A} \otimes \mathcal{H}_\chi$ *(algebraic tensor product) by (sesqui-)linear extension of*

$$(A \otimes v, B \otimes w)_0^\chi := (v, \pi_\chi(\mathcal{Q}(A^*B))w)_\chi. \tag{2.123}$$

Since $\mathcal{Q}$ *is completely positive, this form is positive semi-definite; denote its null space by* $\mathcal{N}_\chi$.

2. The form $(\, , \,)_0^\chi$ *projects to an inner product* $(\, , \,)^\chi$ *on* $\mathfrak{A} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$. *If* $V_\chi : \mathfrak{A} \otimes \mathcal{H}_\chi \to \mathfrak{A} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$ *is the canonical projection, then by definition*

$$(V_\chi(A \otimes v), V_\chi(B \otimes w))^\chi := (A \otimes v, B \otimes w)_0^\chi. \tag{2.124}$$

The Hilbert space $\mathcal{H}^\chi$ *is the closure of* $\mathfrak{A} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$ *in this inner product.*

3. The representation $\pi^\chi(\mathfrak{A})$ *is initially defined on* $\mathfrak{A} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$ *by linear extension of*

$$\pi^\chi(A)V_\chi(B \otimes w) := V_\chi(AB \otimes w); \tag{2.125}$$

this is well-defined, because $\pi^\chi(A)\mathcal{N}_\chi \subseteq \mathcal{N}_\chi$. *One has the bound*

$$\| \pi^\chi(A) \| \leq \| A \|, \tag{2.126}$$

so that $\pi^\chi(A)$ *may be defined on all of* $\mathcal{H}^\chi$ *by continuous extension of (2.125). This extension satisfies* $\pi^\chi(A^*) = \pi^\chi(A)^*$.

4. The map $W : \mathcal{H}_\chi \to \mathcal{H}^\chi$, *defined by*

$$Wv := V_\chi \mathbb{I} \otimes v \tag{2.127}$$

is a partial isometry. Its adjoint $W^* : \mathcal{H}^\chi \to \mathcal{H}_\chi$ *is given by (continuous extension of)*

$$W^*V_\chi A \otimes v = \pi_\chi(\mathcal{Q}(A))v, \tag{2.128}$$

from which the properties $W^*W = \mathbb{I}$ *and (2.121) follow.*

To show that the form defined by (2.123) is positive, we write

$$\sum_{i,j} (A_i \otimes v_i, A_j \otimes v_j)_0^\chi = \sum_{i,j} (v_i, \pi_\chi(\mathcal{Q}(A_i^* A_j))v_j)_\chi. \tag{2.129}$$

Now consider the element $\mathbb{A}$ of $\mathfrak{M}^n(\mathfrak{A})$ with matrix elements $\mathbb{A}_{ij} = A_i^* A_j$. Looking in a faithful representation π_n as explained above, one sees that

$$(z, \mathbb{A}z) = \sum_{i,j} (z_i, \pi(A_i^* A_j)z_j) = \sum_{i,j} (\pi(A_i)z_i, \pi(A_j)z_j) = \| Az \|^2 \geq 0$$

where $Az = \sum_i A_i z_i$. Hence $\mathbb{A} \geq 0$. Since $\mathcal{Q}$ is completely positive, it must be that $\mathbb{B}$, defined by its matrix elements $\mathbb{B}_{ij} := \mathcal{Q}(A_i^* A_j)$, is positive in $\mathfrak{M}^n(\mathfrak{B})$. Repeating the above argument with $\mathbb{A}$ and π replaced by $\mathbb{B}$ and π_χ, respectively, one concludes that the right-hand side of (2.129) is positive.

To prove (2.126) one uses (2.91) in $\mathfrak{M}_n(\mathfrak{A})$. Namely, for arbitrary $A, B_1, \ldots, B_n \in \mathfrak{A}$ we conjugate the inequality $0 \leq A^*A\mathbb{I}_n \leq \| A \|^2 \, \mathbb{I}_n$ with the matrix $\mathbb{B}$, whose first row is $(B_1, \ldots, B_n)$, and which has zeros everywhere else; the adjoint $\mathbb{B}^*$ is then the matrix whose first column is $(B_1^*, \ldots, B_n^*)^T$, and all other entries zero. This leads to $0 \leq \mathbb{B}^* A^* A \mathbb{B} \leq \parallel A \parallel^2 \mathbb{B}^* \mathbb{B}$. Since $\mathcal{Q}$ is completely positive, one has $\mathcal{Q}_n(\mathbb{B}^* A^* A \mathbb{B}) \leq \parallel A \parallel^2 \mathcal{Q}_n(\mathbb{B}^* \mathbb{B})$. Hence in any representation $\pi_\chi(\mathfrak{B})$ and any vector $(v_1, \ldots v_n) \in \mathcal{H}_\chi \otimes \mathbb{C}^n$ one has

$$\sum_{i,j} (v_i, \pi_\chi(\mathcal{Q}(B_i^* A^* A B_j))v_j) \leq \parallel A \parallel^2 \sum_{i,j} (v_i, \pi_\chi(\mathcal{Q}(B_i^* B_j))v_j). \tag{2.130}$$

With $\Psi = \sum_i V_\chi B_i \otimes v_i$, from (2.123), (2.125), and (2.130) one then has

$$\parallel \pi^\chi(A)\Psi \parallel^2 = \sum_{i,j} (AB_i \otimes v_i, AB_j \otimes v_j)_0^\chi = \sum_{i,j} (v_i, \pi_\chi(\mathcal{Q}(B_i^* A^* A B_j))v_j)_\chi$$

$$\leq \parallel A \parallel^2 \sum_{i,j} (v_i, \pi_\chi(\mathcal{Q}(B_i^* B_j))v_j)_\chi = \parallel A \parallel^2 \sum_{i,j} (B_i \otimes v_i, B_j \otimes v_j)_0^\chi$$

$$= \parallel A \parallel^2 (V_\chi \sum_i B_i \otimes v_i, V_\chi \sum_j B_j \otimes v_j)^\chi = \parallel A \parallel^2 \parallel \Psi \parallel^2 .$$

To show that W is a partial isometry, use the definition to compute

$$(Wv, Ww)^\chi = (V_\chi \mathbb{I} \otimes v, V_\chi \mathbb{I} \otimes w)^\chi = (\mathbb{I} \otimes v, \mathbb{I} \otimes w)_0^\chi = (v, w)_\chi,$$

where we used (2.123) and $\mathcal{Q}(\mathbb{I}) = \mathbb{I}$.

To check (2.128), one merely uses the definition of the adjoint, viz. $(w, W^*\Psi)_\chi = (Ww, \Psi)^\chi$ for all $w \in \mathcal{H}_\chi$ and $\Psi \in \mathcal{H}^\chi$. This trivially verified.

To verify (2.121), we use (2.127) and (2.128) to compute

$$W^* \pi^\chi(A) W v = W^* \pi^\chi(A) V_\chi(\mathbb{I} \otimes v) = W^* V_\chi(A \otimes v) = \pi_\chi(\mathcal{Q}(A))v.$$

Being a partial isometry, one has $p = WW^*$ for the projection p onto the image of W, and, in this case, $W^*W = \mathbb{I}$ for the projection onto the subspace of $\mathcal{H}_\chi$ on which W is isometric; this subspace is $\mathcal{H}_\chi$ itself Hence (2.122) follows from (2.121), since

$$U\pi_\chi(\mathcal{Q}(A))U^{-1} = W\pi_\chi(\mathcal{Q}(A))W^* = WW^*\pi^\chi(A)WW^* = p\pi^\chi(A)p. \qquad \blacksquare$$

When $\mathcal{Q}$ fails to preserve the unit, the above construction still applies, but W is no longer a partial isometry; one rather has $\parallel W \parallel^2 = \parallel \mathcal{Q}(\mathbb{I}) \parallel$. Thus it is no longer possible to regard $\mathcal{H}_\chi$ as a subspace of $\mathcal{H}^\chi$.

If $\mathfrak{A}$ and perhaps $\mathfrak{B}$ are non-unital the theorem holds if $\mathcal{Q}$ can be extended (as a positive map) to the unitization of $\mathfrak{A}$, such that the extension preserves the unit $\mathbb{I}$ (perhaps relative to the unitization of $\mathfrak{B}$). When the extension exists but does not preserve the unit, one is in the situation of the previous paragraph.

The relevance of Stinespring's theorem for quantum mechanics stems from the following result.

Proposition 2.11.4 *Let $\mathfrak{A}$ be a commutative unital C^*-algebra. Then any positive map $\mathcal{Q} : \mathfrak{A} \rightarrow$ is completely positive.*

By Theorem 2.4.1 we may assume that $\mathfrak{A} = C(X)$ for some locally compact Hausdorff space X. We may then identify $\mathfrak{M}^n(C(X))$ with $C(X, \mathfrak{M}^n(\mathbb{C}))$. The proof then proceeds in the following steps:

1. Elements of the form F, where $F(x) = \sum_i f_i(x)M_i$ for $f_i \in C(X)$ and $M_i \in \mathfrak{M}^n(\mathbb{C})$, and the sum is finite, are dense in $C(X, \mathfrak{M}^n(\mathbb{C}))$.

2. Such F is positive iff all f_i and M_i are positive.

3. Positive elements G of $C(X, \mathfrak{M}^n(\mathbb{C}))$ can be norm-approximated by positive F's, i.e., when $G \geq 0$ there is a sequence $F_k \geq 0$ such that $\lim_k F_k = G$.

4. $\mathcal{Q}_n(F)$ is positive when F is positive.

5. $\mathcal{Q}_n$ is continuous.

6. If $F_k \to G \geq 0$ in $C(X, \mathfrak{M}^n(\mathbb{C}))$ then $\mathcal{Q}(G) = \lim_k \mathcal{Q}(F_k)$ is a norm-limit of positive elements, hence is positive.

We now prove each of these claims.

1. Take $G \in C(X, \mathfrak{M}^n(\mathbb{C}))$ and pick $\epsilon > 0$. Since G is continuous, the set

$$\mathcal{O}_x^\epsilon := \{y \in X, \parallel G(x) - G(y) \parallel < \epsilon\}$$

is open for each $x \in X$. This gives an open cover of X, which by the compactness of X has a finite subcover $\{\mathcal{O}_{x_1}^\epsilon, \ldots \mathcal{O}_{x_l}^\epsilon\}$. A **partition of unity** subordinate to the given cover is a collection of continuous positive functions $\phi_i \in C(X)$, where $i = 1, \ldots, l$, such that the support of φ_i lies in $\mathcal{O}_{x_i}^\epsilon$ and $\sum_{i=1}^l \varphi_i(x) = 1$ for all $x \in X$. Such a partition of unity exists. Now define $F_l \in C(X, \mathfrak{M}^n(\mathbb{C}))$ by

$$F_l(x) := \sum_{i=1}^{l} \varphi_i(x) G(x_i). \tag{2.131}$$

Since $\parallel G(x_i) - G(x) \parallel < \epsilon$ for all $x \in \mathcal{O}_{x_i}^\epsilon$, one has

$$\parallel F_l(x) - G(x) \parallel = \parallel \sum_{i=1}^{l} \varphi_i(x)(G(x_i) - G(x)) \parallel \leq \sum_{i=1}^{l} \varphi_i(x) \parallel G(x_i) - G(x) \parallel < \sum_{i=1}^{l} \varphi_i(x)\epsilon = \epsilon.$$

Here the norm is the matrix norm in $\mathfrak{M}^n(\mathbb{C})$. Hence

$$\parallel F_l - G \parallel = \sup_{x \in X} \parallel F_l(x) - G(x) \parallel < \epsilon.$$

2. An element $F \in C(X, \mathfrak{M}^n(\mathbb{C}))$ is positive iff $F(x)$ is positive in $\mathfrak{M}^n(\mathbb{C})$ for each $x \in X$. In particular, when $F(x) = f(x)M$ for some $f \in C(X)$ and $M \in \mathfrak{M}^n(\mathbb{C})$ then F is positive iff f is positive in $C(X)$ and M is positive in $\mathfrak{M}^n(\mathbb{C})$. By 2.6.2.2 we infer that F defined by $F(x) = \sum_i f_i(x) M_i$ is positive when all f_i and M_i are positive.

3. When G in item 1 is positive then each $G(x_i)$ is positive, as we have just seen.

4. On F as specified in 2.11.4.1 one has $\mathcal{Q}_n(F) = \sum_i \mathcal{Q}(f_i) \otimes M_i$. Now each operator $B_i \otimes M$ is positive in $\mathfrak{M}^n(\mathfrak{B})$ when B_i and M are positive (as can be checked in a faithful representation). Since $\mathcal{Q}$ is positive, it follows that $\mathcal{Q}_n$ maps each positive element of the form $F = \sum_i f_i M_i$ into a positive member of $\mathfrak{M}^n(\mathfrak{B})$.

5. We know from 2.8.5 that $\mathcal{Q}$ is continuous; the continuity of $\mathcal{Q}_n$ follows because $n < \infty$.

6. A norm-limit $A = \lim_n A_n$ of positive elements in a C^*-algebra is positive, because by (2.88) we have $A_n = B_n^* B_n$, and $\lim B_n = B$ exist because of (2.16). Finally, $A = B^* B$ by continuity of multiplication, i.e., by (2.15). ∎

2.12 Pure states and irreducible representations

We return to the discussion at the end of 2.8. One sees that the compact convex sets in the examples have a natural boundary. The intrinsic definition of this boundary is as follows.

Definition 2.12.1 *An* **extreme point** *in a convex set K (in some vector space) is a member ω of K which can only be decomposed as*

$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2, \tag{2.132}$$

$\lambda \in (0,1)$, if $\omega_1 = \omega_2 = \omega$. The collection $\partial_e K$ of extreme points in K is called the **extreme boundary** *of K. An extreme point in the state space $K = \mathcal{S}(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is called a* **pure state**. *A state that is not pure is called a* **mixed state**.

When $K = \mathcal{S}(\mathfrak{A})$ is a state space of a C^*-algebra we write $\mathcal{P}(\mathfrak{A})$, or simply $\mathcal{P}$, for $\partial_e K$, referred to as the **pure state space** *of $\mathfrak{A}$.*

Hence the pure states on $\mathfrak{A} = \mathbb{C} \oplus \mathbb{C}$ are the points 0 and 1 in $[0,1]$, where 0 is identified with the functional mapping $\lambda \dot{+} \mu$ to λ, whereas 1 maps it to μ. The pure states on $\mathfrak{A} = \mathfrak{M}^2(\mathbb{C})$ are the matrices ρ in (2.109) for which $x^2 + y^2 + z^2 = 1$; these are the projections onto one-dimensional subspaces of $\mathbb{C}^2$.

More generally, we will prove in 2.13.10 that the state space of $\mathfrak{M}^n(\mathbb{C})$ consists of all positive matrices ρ with unit trace; the pure state space of $\mathfrak{M}^n(\mathbb{C})$ then consists of all one-dimensional projections. This precisely reproduces the notion of a pure state in quantum mechanics. The first part of Definition 2.12.1 is due to Minkowski; it was von Neumann who recognized that this definition is applicable to quantum mechanics.

We may now ask what happens to the GNS-construction when the state ω one constructs the representation π_ω from is pure. In preparation:

Definition 2.12.2 *A representation π of a C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ is called* **irreducible** *if a closed subspace of $\mathcal{H}$ which is stable under $\pi(\mathfrak{A})$ is either $\mathcal{H}$ or 0.*

This definition should be familiar from the theory of group representations. It is a deep fact of C^*-algebras that the qualifier 'closed' may be omitted from this definition, but we will not prove this. Clearly, the defining representation π_d of the matrix algebra $\mathfrak{M}^N$ on $\mathbb{C}^N$ is irreducible. In the infinite-dimensional case, the defining representations π_d of $\mathfrak{B}(\mathcal{H})$ on $\mathcal{H}$ is irreducible as well.

Proposition 2.12.3 *Each of the following conditions is equivalent to the irreducibility of $\pi(\mathfrak{A})$ on $\mathcal{H}$:*

1. *The commutant of $\pi(\mathfrak{A})$ in $\mathfrak{B}(\mathcal{H})$ is $\{\lambda\mathbb{I} \mid \lambda \in \mathbb{C}\}$; in other words, $\pi(\mathfrak{A})'' = \mathfrak{B}(\mathcal{H})$ (***Schur's lemma***);*

2. *Every vector Ω in $\mathcal{H}$ is cyclic for $\pi(\mathfrak{A})$ (recall that this means that $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$).*

The commutant $\pi(\mathfrak{A})'$ is a *-algebra in $\mathfrak{B}(\mathcal{H})$, so when it is nontrivial it must contain a self-adjoint element A which is not a multiple of $\mathbb{I}$. Using Theorem 2.14.3 below and the spectral theorem, it can be shown that the projections in the spectral resolution of A lie in $\pi(\mathfrak{A})'$ if A does. Hence when $\pi(\mathfrak{A})'$ is nontrivial it contains a nontrivial projection p. But then $p\mathcal{H}$ is stable under $\pi(\mathfrak{A})$, contradicting irreducibility. Hence "π irreducible $\Rightarrow \pi(\mathfrak{A})' = \mathbb{C}\mathbb{I}$".

Conversely, when $\pi(\mathfrak{A})' = \mathbb{C}\mathbb{I}$ and π is reducible one finds a contradiction because the projection onto the alleged nontrivial stable subspace of $\mathcal{H}$ commutes with $\pi(\mathfrak{A})$. Hence "$\pi(\mathfrak{A})' = \mathbb{C}\mathbb{I} \Rightarrow \pi$ irreducible".

When there exists a vector $\Psi \in \mathcal{H}$ for which $\pi(\mathfrak{A})\Psi$ is not dense in $\mathcal{H}$, we can form the projection onto the closure of $\pi(\mathfrak{A})\Psi$. By Lemma 2.9.3, with $\mathfrak{M} = \pi(\mathfrak{A})$, this projection lies in $\pi(\mathfrak{A})'$, so that π cannot be irreducible by Schur's lemma. Hence "π irreducible $\Rightarrow$ every vector cyclic". The converse is trivial. ■

We are now in a position to answer the question posed before 2.12.2.

Theorem 2.12.4 *The GNS-representation $\pi_\omega(\mathfrak{A})$ of a state $\omega \in \mathcal{S}(\mathfrak{A})$ is irreducible iff ω is pure.*

When ω is pure yet $\pi_\omega(\mathfrak{A})$ reducible, there is a nontrivial projection $p \in \pi_\omega(\mathfrak{A})'$ by Schur's lemma. Let Ω_ω be the cyclic vector for π_ω. If $p\Omega_\omega = 0$ then $Ap\Omega_\omega = pA\Omega_\omega = 0$ for all $A \in \mathfrak{A}$, so that $p = 0$ as π_ω is cyclic. Similarly, $p^\perp\Omega_\omega = 0$ is impossible. We may then decompose $\omega = \lambda\psi + (1-\lambda)\psi^\perp$, where ψ and $\psi^\perp$ are states defined as in (2.103), with $\Psi := p\Omega_\omega / \parallel p\Omega_\omega \parallel$, $\Psi^\perp := p^\perp\Omega_\omega / \parallel p^\perp\Omega_\omega \parallel$, and $\lambda = \parallel p^\perp\Omega_\omega \parallel^2$. Hence ω cannot be pure. This proves "pure $\Rightarrow$ irreducible".

In the opposite direction, suppose π_ω is irreducible, with (2.132) for $\omega_1, \omega_2 \in \mathcal{S}(\mathfrak{A})$ and $\lambda \in [0,1]$. Then $\lambda\omega_1 - \omega = (1-\lambda)\omega_2$, which is positive; hence $\lambda\omega_1(A^*A) \leq \omega(A^*A)$ for all $A \in \mathfrak{A}$. By (2.104) this yields

$$|\lambda\omega_1(A^*B)|^2 \leq \lambda^2\omega_1(A^*A)\omega_1(B^*B) \leq \omega(A^*A)\omega(B^*B) \tag{2.133}$$

for all A, B. This allows us to define a quadratic form (i.e., a sesquilinear map) $\hat{Q}$ on $\pi_\omega(\mathfrak{A})\Omega_\omega$ by

$$\hat{Q}(\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega) := \lambda\omega_1(A^*B). \tag{2.134}$$

This is well defined: when $\pi_\omega(A_1)\Omega_\omega = \pi_\omega(A_2)\Omega_\omega$ then $\omega((A_1-A_2)^*(A_1-A_2)) = 0$ by (2.120), so that

$$|\hat{Q}(\pi_\omega(A_1)\Omega_\omega, \pi_\omega(B)\Omega_\omega) - \hat{Q}(\pi_\omega(A_2)\Omega_\omega, \pi_\omega(B)\Omega_\omega)|^2 = |\lambda\omega_1((A_1-A_2)^*B)|^2 \leq 0$$

by (2.133); in other words, $\hat{Q}(\pi_\omega(A_1)\Omega_\omega, \pi_\omega(B)\Omega_\omega) = \hat{Q}(\pi_\omega(A_2)\Omega_\omega, \pi_\omega(B)\Omega_\omega)$. Similarly for B. Furthermore, (2.133) and (2.120) imply that $\hat{Q}$ is bounded in that

$$|\hat{Q}(\Psi, \Phi)| \leq C \parallel \Psi \parallel \parallel \Phi \parallel, \tag{2.135}$$

for all $\Psi, \Phi \in \pi_\omega(\mathfrak{A})\Omega_\omega$, with $C = 1$. It follows that $\hat{Q}$ can be extended to all of $\mathcal{H}_\omega$ by continuity. Moreover, one has

$$\hat{Q}(\Phi, \Psi) = \overline{\hat{Q}(\Psi, \Phi)} \tag{2.136}$$

by (2.105) with $A \to A^*B$ and $\omega \to \omega_1$.

Lemma 2.12.5 *Let a quadratic form $\hat{Q}$ on a Hilbert space $\mathcal{H}$ be bounded, in that (2.135) holds for all $\Psi, \Phi \in \mathcal{H}$, and some constant $C \geq 0$. There is a bounded operator Q on $\mathcal{H}$ such that $\hat{Q}(\Psi, \Phi) = (\Psi, Q\Phi)$ for all $\Psi, \Phi \in \mathcal{H}$, and $\parallel Q \parallel \leq C$. When (2.136) is satisfied Q is self-adjoint.*

Hold Ψ fixed. The map $\Phi \to \hat{Q}(\Psi, \Phi)$ is then bounded by (2.135), so that by the Riesz-Fischer theorem there exists a unique vector Ω such that $\hat{Q}(\Psi, \Phi) = (\Omega, \Phi)$. Define Q by $Q\Psi = \Omega$. The self-adjointness of Q in case that (2.136) holds is obvious.

Now use (2.135) to estimate

$$\parallel Q\Psi \parallel^2 = (Q\Psi, Q\Psi) = \hat{Q}(Q\Psi, \Psi) \leq C \parallel Q \parallel \parallel \Psi \parallel^2;$$

taking the supremum over all Ψ in the unit ball yields $\parallel Q \parallel^2 \leq C \parallel Q \parallel^2$, whence $\parallel Q \parallel \leq C$. ■

Continuing with the proof of 2.12.4, we see that there is a self-adjoint operator Q on $\mathcal{H}_\omega$ such that

$$(\pi_\omega(A)\Omega_\omega, Q\pi_\omega(B)\Omega_\omega) = \lambda\omega_1(A^*B). \tag{2.137}$$

It is the immediate from (2.110) that $[Q, \pi_\omega(C)] = 0$ for all $C \in \mathfrak{A}$. Hence $Q \in \pi_\omega(\mathfrak{A})'$; since π_ω is irreducible one must have $Q = t\mathbb{I}$ for some $t \in \mathbb{R}$; hence (2.137), (2.134), and (2.119) show that ω_1 is proportional to ω, and therefore equal to ω by normalization, so that ω is pure. ■

From 2.9.5 we have the

Corollary 2.12.6 *If $(\pi(\mathfrak{A}), \mathcal{H})$ is irreducible then the GNS-representation $(\pi_\omega(\mathfrak{A}), \mathcal{H}_\omega)$ defined by any vector state ψ (corresponding to a unit vector $\Psi \in \mathcal{H}$) is unitarily equivalent to $(\pi(\mathfrak{A}), \mathcal{H})$.*

Combining this with 2.12.4 yields

Corollary 2.12.7 *Every irreducible representation of a C^*-algebra comes from a pure state via the GNS-construction.*

A useful reformulation of the notion of a pure state is as follows.

Proposition 2.12.8 *A state is pure iff $0 \leq \rho \leq \omega$ for a positive functional ρ implies $\rho = t\omega$ for some $t \in \mathbb{R}^+$.*

We assume that $\mathfrak{A}$ is unital; if not, use 2.4.6 and 2.8.7. For $\rho = 0$ or $\rho = \omega$ the claim is obvious. When ω is pure and $0 \leq \rho \leq \omega$, with $0 \neq \rho \neq \omega$, then $0 < \rho(\mathbb{I}) < 1$, since $\omega - \rho$ is positive, hence $\| \omega - \rho \| = \omega(\mathbb{I}) - \rho(I) = 1 - \rho(\mathbb{I})$. Hence $\rho(\mathbb{I})$ would imply $\omega = \rho$, whereas $\rho(\mathbb{I}) = 0$ implies $\rho = 0$, contrary to assumption. Hence $(\omega - \rho)/(1 - \rho(\mathbb{I}))$ and $\rho/\rho(\mathbb{I})$ are states, and

$$\omega = \lambda \frac{\omega - \rho}{1 - \rho(\mathbb{I})} + (1 - \lambda) \frac{\rho}{\rho(\mathbb{I})}$$

with $\lambda = 1 - \rho(\mathbb{I})$. Since ω is pure, by 2.12.1 we have $\rho = \rho(\mathbb{I})\omega$.

Conversely, if (2.132) holds then $0 \leq \lambda\omega_1 \leq \omega$ (cf. the proof of 2.12.4), so that $\lambda\omega_1 = t\omega$ by assumption; normalization gives $t = \lambda$, hence $\omega_1 = \omega = \omega_2$, and ω is pure. ■

The simplest application of this proposition is

Theorem 2.12.9 *The pure state space of the commutative C^*-algebra $C_0(X)$ (equipped with the relative w^*-topology) is homeomorphic to X.*

In view of Proposition 2.4.3 and Theorems 2.4.1 and 2.4.8, we merely need to establish a bijective correspondence between the pure states and the multiplicative functionals on $C_0(X)$. The case that X is not compact may be reduced to the compact case by passing from $\mathfrak{A} = C_0(X)$ to $\mathfrak{A}_\mathbb{I} = C(\tilde{X})$; cf. 2.4.6 and 2.3.7 etc. This is possible because the unique extension of a pure state on $C_0(X)$ to a state on $C(\tilde{X})$ guaranteed by 2.8.7 remains pure. Moreover, the extension of a multiplicative functional defined in (2.54) coincides with the extension $\omega_\mathbb{I}$ of a state defined in (2.108), and the functional ∞ in (2.61) clearly defines a pure state.

Thus we put $\mathfrak{A} = C(X)$. Let $\omega_x \in \Delta(C(X))$ (cf. the proof of 2.4.3), and suppose a functional ρ satisfies $0 \leq \rho \leq \omega_x$. Then $\ker(\omega_x) \subseteq \ker(\rho)$, and $\ker(\rho)$ is an ideal. But $\ker(\omega_x)$ is a maximal ideal, so when $\rho \neq 0$ it must be that $\ker(\omega_x) = \ker(\rho)$. Since two functionals on any vector space are proportional when they have the same kernel, it follows from 2.12.8 that ω_x is pure.

Conversely, let ω be a pure state, and pick a $g \in C(X)$ with $0 \leq g \leq 1_X$. Define a functional ω_g on $C(X)$ by $\omega_g(f) := \omega(fg)$. Since $\omega(f) - \omega_g(f) = \omega(f(1-g))$, and $0 \leq 1 - g \leq 1_X$, one has $0 \leq \omega_g \leq \omega$. Hence $\omega_g = t\omega$ for some $t \in \mathbb{R}^+$ by 2.12.8. In particular, $\ker(\omega_g) = \ker(\omega)$. It follows that when $f \in \ker(\omega)$, then $fg \in \ker(\omega)$ for all $g \in C(X)$, since any function is a linear combination of functions for which $0 \leq g \leq 1_X$. Hence $\ker(\omega)$ is an ideal, which is maximal because the kernel of a functional on any vector space has codimension 1. Hence ω is multiplicative by Theorem 2.3.3. ■

It could be that no pure states exist in $\mathcal{S}(\mathfrak{A})$; think of an open convex cone. It would follow that such a C^*-algebra has no irreducible representations. Fortunately, this possibility is excluded by the **Krein-Milman theorem** in functional analysis, which we state without proof. The **convex hull** $\mathrm{co}(V)$ of a subset V of a vector space is defined by

$$\mathrm{co}(V) := \{\lambda v + (1-\lambda)w \mid v, w \in V, \lambda \in [0,1]\}. \tag{2.138}$$

Theorem 2.12.10 *A compact convex set K embedded in a locally convex vector space is the closure of the convex hull of its extreme points. In other words, $K = \overline{\mathrm{co}}(\partial_e K)$.*

It follows that arbitrary states on a C^*-algebra may be approximated by finite convex sums of pure states. This is a spectacular result: for example, applied to $C(X)$ it shows that arbitrary probability measures on X may be approximated by finite convex sums of point (Dirac) measures. In general, it guarantees that a C^*-algebra has lots of pure states. For example, we may now refine Lemma 2.8.8 as follows

Theorem 2.12.11 *For every $A \in \mathfrak{A}_\mathbb{R}$ and $a \in \sigma(A)$ there is a pure state ω_a on $\mathfrak{A}$ for which $\omega_a(A) = a$. There exists a pure state ω such that $|\omega(A)| = \| A \|$.*

We extend the state in the proof of 2.8.8 to $C^*(A, \mathbb{I})$ by multiplicativity and continuity, that is, we put $\tilde{\omega}_a(A^n) = a^n$ etc. It follows from 2.12.9 that this extension is pure. One easily checks that the set of all extensions of $\tilde{\omega}_a$ to $\mathfrak{A}$ (which extensions we know to be states; see the proof of 2.8.8) is a closed convex subset K_a of $\mathcal{S}(\mathfrak{A})$; hence it is a compact convex set. By the Krein-Milman theorem 2.12.10 it has at least one extreme point ω_a. If ω_a were not an extreme point in $\mathcal{S}(\mathfrak{A})$, it would be decomposable as in (2.132). But it is clear that, in that case, ω_1 and ω_2 would coincide on $C^*(A, \mathbb{I})$, so that ω_a cannot be an extreme point of K_a. ∎

We may now replace the use of 2.8.8 by 2.12.11 in the proof of the Gel'fand-Neumark Theorem 2.10.1, concluding that the universal representation π_{u} may be replaced by $\pi_{\mathrm{r}} := \oplus_{\omega \in \mathcal{P}(\mathfrak{A})} \pi_\omega$. We may further restrict this direct sum by defining two states to be **equivalent** if the corresponding GNS-representations are equivalent, and taking only one pure state in each equivalence class. Let us refer to the ensuing set of pure states as $[\mathcal{P}(\mathfrak{A})]$. We then have

$$\mathfrak{A} \simeq \pi_{\mathrm{r}}(\mathfrak{A}) := \oplus_{\omega \in [\mathcal{P}(\mathfrak{A})]} \pi_\omega(\mathfrak{A}). \tag{2.139}$$

It is obvious that the proof of 2.10.1 still goes through.

The simplest application of this refinement is

Proposition 2.12.12 *Every finite-dimensional C^*-algebra is a direct sum of matrix algebras.*

For any morphism φ, hence certainly for any representation $\varphi = \pi$, one has the isomorphism $\varphi(\mathfrak{A}) \simeq \mathfrak{A}/\ker(\varphi)$. Since $\mathfrak{A}/\ker(\pi)$ is finite-dimensional, it must be that $\pi(\mathfrak{A})$ is isomorphic to an algebra acting on a finite-dimensional vector space. Furthermore, it follows from Theorem 2.14.3 below that $\pi(\mathfrak{A})'' = \pi(\mathfrak{A})$ in every finite-dimensional representation of $\mathfrak{A}$, upon which 2.12.3.1 implies that $\pi(\mathfrak{A})$ must be a matrix algebra (as $\mathfrak{B}(\mathcal{H})$ is the algebra of $n \times n$ matrices for $\mathcal{H} = \mathbb{C}^n$). Then apply the isomorphism 2.139. ∎

2.13 The C^*-algebra of compact operators

It would appear that the appropriate generalization of the C^*-algebra $\mathfrak{M}^n(\mathbb{C})$ of $n \times n$ matrices to infinite-dimensional Hilbert spaces $\mathcal{H}$ is the C^*-algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. This is not the case. For one thing, unlike $\mathfrak{M}^n(\mathbb{C})$ (which, as will follow from this section, has only one irreducible representation up to equivalence), $\mathfrak{B}(\mathcal{H})$ has a huge number of inequivalent representations; even when $\mathcal{H}$ is separable, most of these are realized on non-separable Hilbert spaces.

For example, it follows from 2.12.6 that any vector state ψ on $\mathfrak{B}(\mathcal{H})$ defines an irreducible representation of $\mathfrak{B}(\mathcal{H})$ which is equivalent to the defining representation. On the other hand, we know from 2.12.11 and the existence of bounded self-adjoint operators with continuous spectrum (such as any multiplication operator on $L^2(X)$, where X is connected), that there are many other pure states whose GNS-representation is not equivalent to the defining representation π. Namely, when $A \in \mathfrak{B}(\mathcal{H})$ and $a \in \sigma(A)$, but a is not in the discrete spectrum of A as an operator on $\mathcal{H}$ (i.e., there is no eigenvector $\Psi_a \in \mathcal{H}$ for which $A\Psi_a = a\Psi_a$), then π_{ω_a} cannot be equivalent to π. For it is easy to show from (2.116) that $\Omega_{\omega_a} \in \mathcal{H}_{\omega_a}$ is an eigenvector of $\pi_{\omega_a}(A)$ with eigenvalue a. In other words, a is in the continuous spectrum of $A = \pi(A)$ but in the discrete spectrum of $\pi_{\omega_a}(A)$, which excludes the possibility that $\pi(\mathfrak{A})$ and $\pi_{\omega_a}(\mathfrak{A})$ are equivalent (as the spectrum is invariant under unitary transformations).

Another argument against $\mathfrak{B}(\mathcal{H})$ is that it is non-separable in the nom-topology even when $\mathcal{H}$ is separable. The appropriate generalization of $\mathfrak{M}^n(\mathbb{C})$ to an infinite-dimensional Hilbert space $\mathcal{H}$ turns out to be the C^*-algebra $\mathfrak{B}_0(\mathcal{H})$ of compact operator on $\mathcal{H}$. In non-commutative geometry elements of this C^*-algebra play the role of infinitesimals; in general, $\mathfrak{B}_0(\mathcal{H})$ is a basic building block in the theory of C^*-algebras. This section is devoted to an exhaustive study of this C^*-algebra.

Definition 2.13.1 *Let $\mathcal{H}$ be a Hilbert space. The *-algebra $\mathfrak{B}_f(\mathcal{H})$ of* **finite-rank operators on** *$\mathcal{H}$ is the (finite) linear span of all finite-dimensional projections on $\mathcal{H}$. In other words, an operator $A \in \mathfrak{B}(\mathcal{H})$ lies in $\mathfrak{B}_f(\mathcal{H})$ when $A\mathcal{H} := \{A\Psi \mid \Psi \in \mathcal{H}\}$ is finite-dimensional.*

The C^***-algebra** $\mathfrak{B}_0(\mathcal{H})$ **of compact operators** *on* $\mathcal{H}$ *is the norm-closure of* $\mathfrak{B}_f(\mathcal{H})$ *in* $\mathfrak{B}(\mathcal{H})$*; in other words, it is the smallest* C^**-algebra of* $\mathfrak{B}(\mathcal{H})$ *containing* $\mathfrak{B}_f(\mathcal{H})$*. In particular, the norm in* $\mathfrak{B}_0(\mathcal{H})$ *is the operator norm (2.3). An operator* $A \in \mathfrak{B}(\mathcal{H})$ *lies in* $\mathfrak{B}_0(\mathcal{H})$ *when it can be approximated in norm by finite-rank operators.*

It is clear that $\mathfrak{B}_f(\mathcal{H})$ is a *-algebra, since $p^* = p$ for any projection p. The third item in the next proposition explains the use of the word 'compact' in the present context.

Proposition 2.13.2 *1. The unit operator* $\mathbb{I}$ *lies in* $\mathfrak{B}_0(\mathcal{H})$ *iff* $\mathcal{H}$ *is finite-dimensional.*

2. *The* C^**-algebra* $\mathfrak{B}_0(\mathcal{H})$ *is an ideal in* $\mathfrak{B}(\mathcal{H})$.

3. *If* $A \in \mathfrak{B}_0(\mathcal{H})$ *then* $A\mathcal{B}_1$ *is compact in* $\mathcal{H}$ *(with the norm-topology). Here* $\mathcal{B}_1$ *is the unit ball in* $\mathcal{H}$*, i.e., the set of all* $\Psi \in \mathcal{H}$ *with* $\| \Psi \| \leq 1$.

Firstly, for any sequence (or net) $A_n \in \mathfrak{B}_f(\mathcal{H})$ we may choose a unit vector $\Psi_n \in (A_n\mathcal{H})^\perp$. Then $(A_n - \mathbb{I})\Psi = -\Psi$, so that $\| (A_n - \mathbb{I})\Psi \| = 1$. Hence $\sup_{\|\Psi\|=1} \| (A_n - \mathbb{I})\Psi \| \geq 1$, hence $\| A_n - \mathbb{I} \| \to 0$ is impossible by definition of the norm (2.3) in $\mathfrak{B}(\mathcal{H})$ (hence in $\mathfrak{B}_0(\mathcal{H})$).

Secondly, when $A \in \mathfrak{B}_f(\mathcal{H})$ and $B \in \mathfrak{B}(\mathcal{H})$ then $AB \in \mathfrak{B}_f(\mathcal{H})$, since $AB\mathcal{H} = A\mathcal{H}$. But since $BA = (A^*B^*)^*$, and $\mathfrak{B}_f(\mathcal{H})$ is a *-algebra, one has $A^*B^* \in \mathfrak{B}_f(\mathcal{H})$ and hence $BA \in \mathfrak{B}_f(\mathcal{H})$. Hence $\mathfrak{B}_f(\mathcal{H})$ is an ideal in $\mathfrak{B}(\mathcal{H})$, save for the fact that it is not norm-closed (unless $\mathcal{H}$ has finite dimension). Now if $A_n \to A$ then $A_nB \to AB$ and $BA_n \to BA$ by continuity of multiplication in $\mathfrak{B}(\mathcal{H})$. Hence $\mathfrak{B}_0(\mathcal{H})$ is an ideal by virtue of its definition.

Thirdly, note that the weak topology on $\mathcal{H}$ (in which $\Psi_n \to \Psi$ iff $(\Phi, \Psi_n) \to (\Phi, \Psi)$ for all $\Phi \in \mathcal{H}$) is actually the w^*-topology under the duality of $\mathcal{H}$ with itself given by the Riesz-Fischer theorem. Hence the unit ball $\mathcal{B}_1$ is compact in the weak topology by the Banach-Alaoglu theorem. So if we can show that $A \in \mathfrak{B}_0(\mathcal{H})$ maps weakly convergent sequences to norm-convergent sequences, then A is continuous from $\mathcal{H}$ with the weak topology to $\mathcal{H}$ with the norm-topology; since compactness is preserved under continuous maps, it follows that $A\mathcal{B}_1$ is compact.

Indeed, let $\Psi_n \to \Psi$ in the weak topology, with $\| \Psi_n \| = 1$ for all n. Since

$$\| \Psi \|^2 = (\Psi, \Psi) = \lim_n (\Psi, \Psi_n) \leq \| \Psi \| \, \| \Psi_n \| = \Psi,$$

one has $\| \Psi \| \leq 1$. Given $\epsilon > 0$, choose $A_f \in \mathfrak{B}_f(\mathcal{H})$ such that $\| A - A_f \| < \epsilon/3$, and put $p := [A_f\mathcal{H}]$, the finite-dimensional projection onto the image of A_f. Then

$$\| A\Psi_n - A\Psi \| = \| (A - A_f)\Psi_n + (A - A_f)\Psi + A_f(\Psi_n - \Psi) \| \leq \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \| A_f \| \, \| p(\Psi_n - \Psi) \| .$$

Since the weak and the norm topology on a finite-dimensional Hilbert space coincide, they coincide on $p\mathcal{H}$, so that we can find N such that $\| p(\Psi_n - \Psi) \| < \epsilon/3$ for all $n > N$. Hence $\| A\Psi_n - A\Psi \| < \epsilon$. ∎

Corollary 2.13.3 *A self-adjoint operator* $A \in \mathfrak{B}_0(\mathcal{H})$ *has an eigenvector* Ψ_a *with eigenvalue* a *such that* $|a| = \| A \|$.

Define $f_A : \mathcal{B}_1 \to \mathbb{R}$ by $f_A(\Psi) := \| A\Psi \|^2$. When $\Psi_n \to \Psi$ weakly with $\| \Psi_n \| = 1$, then

$$|f_A(\Psi_n) - f_A(\Psi)| = |(\Psi_n, A^*A(\Psi_n - \Psi)) - (\Psi - \Psi_n, A^*A\Psi)| \leq \| A^*A(\Psi_n - \Psi) \| + |(\Psi - \Psi_n, A^*A\Psi)|.$$

The first term goes to zero by the proof of 2.13.2.3 (noting that $A^*A \in \mathfrak{B}_0(\mathcal{H})$), and the second goes to zero by definition of weak convergence. Hence f_A is continuous. Since $\mathcal{B}_1$ is weakly compact, f_A assumes its maximum at some Ψ_a. This maximum is $\| A \|^2$ by (2.3). Now the Cauchy-Schwarz inequality with $\Psi = 1$ gives $\| A\Psi \|^2 = (\Psi, A^*A\Psi) \leq \| A^*A\Psi \|$, with equality iff $A^*A\Psi$ is proportional to Ψ. Hence when $A^* = A$ the property $\| A \|^2 = \| A\Psi_a \|^2$ with $\| \Psi_a \| = 1$ implies $A^2\Psi_a = a^2\Psi_a$, where $a^2 = \| A \|^2$. The spectral theorem or the continuous functional calculus with $f(A^2) = \sqrt{A^2} = A$ implies $A\Psi_a = a\Psi_a$. Clearly $|a| = \| A \|$. ∎

Theorem 2.13.4 *A self-adjoint operator $A \in \mathfrak{B}(\mathcal{H})$ is compact iff $A = \sum_i a_i[\Psi_i]$ (norm-convergent sum), where each eigenvalue a_i has finite multiplicity. Ordering the eigenvalues so that $a_i \le a_j$ when $i > j$, one has $\lim_{i\to\infty} |a_i| = 0$. In other words, the set of eigenvalues is discrete, and can only have 0 as a possible accumulation point.*

This ordering is possible because by 2.13.3 there is a largest eigenvalue.

Let $A \in \mathfrak{B}_0(\mathcal{H})$ be self-adjoint, and let p be the projection onto the closure of the linear span of all eigenvectors of A. As in Lemma 2.9.3 one sees that $[A,p] = 0$, so that $(pA)^* = pA$. Hence $p^\perp A = (\mathbb{I} - p)A$ is self-adjoint, and compact by 2.13.2.2. By 2.13.3 the compact self-adjoint operator $p^\perp A$ has an eigenvector, which must lie in $p^\perp\mathcal{H}$, and must therefore be an eigenvector of A in $p^\perp\mathcal{H}$. By assumption this eigenvector can only be zero. Hence $\| p^\perp A \| = 0$ 2.13.3, which implies that A restricted to $p^\perp\mathcal{H}$ is zero, which implies that all vectors in $p^\perp\mathcal{H}$ are eigenvectors with eigenvalue zero. This contradicts the definition of $p^\perp\mathcal{H}$ unless $p^\perp\mathcal{H} = 0$. This proves "A compact and self-adjoint $\Rightarrow$ A diagonalizable".

Let A be compact and self-adjoint, hence diagonalizable. Normalize the eigenvectors $\Psi_i := \Psi_{a_i}$ to unit length. Then $\lim_{i\to\infty}(\Psi, \Psi_i) = 0$ for all $\Psi \in \mathcal{H}$, since the Ψ_i form a basis, so that

$$(\Psi, \Psi) = \sum_i |(\Psi, \Psi_i)|^2, \tag{2.140}$$

which clearly converges. Hence $\Psi_i \to 0$ weakly, so $\| A\Psi_i \| = |a_i| \to 0$ by (the proof of) 2.13.2.3. Hence $\lim_{i\to\infty} |a_i| = 0$. This proves "$A$ compact and self-adjoint $\Rightarrow$ A diagonalizable with $\lim_{i\to\infty} |a_i| = 0$".

Let now A be self-adjoint and diagonalizable, with $\lim_{i\to\infty} |a_i| = 0$. For $N < \infty$ and $\Psi \in \mathcal{H}$ one then has

$$\| (A - \sum_{i=1}^N a_i[\Psi_i])\Psi \|^2 = \| \sum_{i=N+1}^\infty a_i(\Psi_i, \Psi)\Psi_i \|^2 \le \sum_{i=N+1}^\infty |a_i|^2 \, |(\Psi, \Psi_i)|^2 \le |a_N|^2 \sum_{i=N+1}^\infty |(\Psi, \Psi_i)|^2.$$

Using (2.140), this is $\le |a_N|^2(\Psi, \Psi)$, so that $\lim_{N\to\infty} \| A - \sum_{i=1}^N a_i[\Psi_i] \| = 0$, because $\lim_{N\to\infty} |a_N| = 0$. Since the operator $\sum_{i=1}^N a_i[\Psi_i]$ is clearly of finite rank, this proves that A is compact. Hence "A self-adjoint and diagonalizable with $\lim_{i\to\infty} |a_i| = 0 \Rightarrow A$ compact".

Finally, when A is compact its restriction to any closed subspace of $\mathcal{H}$ is compact, which by 2.13.2.1 proves the claim about the multiplicity of the eigenvalues. ■

We now wish to compute the state space of $\mathfrak{B}_0(\mathcal{H})$. This involves the study of a number of subspaces of $\mathfrak{B}(\mathcal{H})$ which are not C^*-algebras, but which are ideals of $\mathfrak{B}(\mathcal{H})$, except for the fact that they are not closed.

Definition 2.13.5 *The* **Hilbert-Schmidt norm** $\| A \|_2$ *of* $A \in \mathfrak{B}(\mathcal{H})$ *is defined by*

$$\| A \|_2^2 := \sum_i \| A\mathbf{e}_i \|^2, \tag{2.141}$$

where $\{\mathbf{e}_i\}_i$ *is an arbitrary basis of* $\mathcal{H}$*; the right-hand side is independent of the choice of the basis. The* **Hilbert-Schmidt class** $\mathfrak{B}_2(\mathcal{H})$ *consists of all* $A \in \mathfrak{B}(\mathcal{H})$ *for which* $\| A \|_2 < \infty$.

The **trace norm** $\| A \|_1$ *of* $A \in \mathfrak{B}(\mathcal{H})$ *is defined by*

$$\| A \|_1 := \| (A^*A)^{\frac{1}{4}} \|_2^2, \tag{2.142}$$

where $(A^*A)^{\frac{1}{4}}$ *is defined by the continuous functional calculus. The* **trace class** $\mathfrak{B}_1(\mathcal{H})$ *consists of all* $A \in \mathfrak{B}(\mathcal{H})$ *for which* $\| A \|_2 < \infty$.

To show that (2.141) is independent of the basis, we take a second basis $\{\mathbf{u}_i\}_i$, with corresponding resolution of the identity $\mathbb{I} = \sum_i [\mathbf{u}_i]$ (weakly). Aince $\mathbb{I} = \sum_i [\mathbf{e}_i]$ we then have

$$\| A \|_2^2 := \sum_{i,j} (\mathbf{e}_j, \mathbf{u}_i)(\mathbf{u}_i, A^*A\mathbf{e}_j) = \sum_{i,j} (A^*A\mathbf{u}_i, \mathbf{e}_j)(\mathbf{e}_j, \mathbf{u}_i) = \sum_i \| A\mathbf{u}_i \|^2 .$$

If $A \in \mathfrak{B}_1(\mathcal{H})$ then

$$\mathrm{Tr}\, A := \sum_i (\mathbf{e}_i, A\mathbf{e}_i) \tag{2.143}$$

is finite and independent of the basis (when $A \notin \mathfrak{B}_1(\mathcal{H})$, it may happen that $\mathrm{Tr}\, A$ depends on the basis; it may even be finite in one basis and infinite in another). Conversely, it can be shown that $A \in \mathfrak{B}_1(\mathcal{H})$ when $\mathrm{Tr}_+ A < \infty$, where Tr_+ is defined in terms of the decomposition (2.107) by $\mathrm{Tr}_+ A := \mathrm{Tr}\, A'_+ - \mathrm{Tr}\, A'_- + i\mathrm{Tr}\, A''_+ - i\mathrm{Tr}\, A''_-$. For $A \in \mathfrak{B}_1(\mathcal{H})$ one has $\mathrm{Tr}_+ A = \mathrm{Tr}\, A$. One always has the equalities

$$\| A \|_1 = \mathrm{Tr}\, |A|; \tag{2.144}$$

$$\| A \|_2 = \mathrm{Tr}\, |A|^2 = \mathrm{Tr}\, A^* A, \tag{2.145}$$

where

$$|A| := \sqrt{A^* A}. \tag{2.146}$$

In particular, when $A \geq 0$ one simply has $\| A \|_1 = \mathrm{Tr}\, A$, which does not depend on the basis, whether or not $A \in \mathfrak{B}_1(\mathcal{H})$. The properties

$$\mathrm{Tr}\, A^* A = \mathrm{Tr}\, A A^* \tag{2.147}$$

for all $A \in \mathfrak{B}(\mathcal{H})$, and

$$\mathrm{Tr}\, U A U^* = \mathrm{Tr}\, A \tag{2.148}$$

for all positive $A \in \mathfrak{B}(\mathcal{H})$ and all unitaries U, follow from (2.143) by manipulations similar to those establishing the basis-independence of (2.141). Also, the linearity property

$$\mathrm{Tr}\, (A + B) = \mathrm{Tr}\, A + \mathrm{Tr}\, B \tag{2.149}$$

for all $A, B \in \mathfrak{B}_1(\mathcal{H})$ is immediate from (2.143).

It is easy to see that the Hilbert-Schmidt norm is indeed a norm, and that $\mathfrak{B}_2(\mathcal{H})$ is complete in this norm. The corresponding properties for the trace norm are nontrivial (but true), and will not be needed. In any case, for all $A \in \mathfrak{B}(\mathcal{H})$ one has

$$\| A \| \leq \| A \|_1; \tag{2.150}$$

$$\| A \| \leq \| A \|_2 . \tag{2.151}$$

To prove this, we use our old trick: although $\| B \| \geq \| B\Psi \|$ for all unit vectors Ψ, for every $\epsilon > 0$ there is a $\Psi_\epsilon \in \mathcal{H}$ of norm 1 such that $\| B \|^2 \leq \| B\Psi_\epsilon \|^2 + \epsilon$. Put $B = (A^* A)^{\frac{1}{4}}$, and note that $\| (A^* A)^{\frac{1}{4}} \|^2 = \| A \|$ by (2.16). Completing Ψ_ϵ to a basis $\{\mathbf{e}_i\}_i$, we have

$$\| A \| = \| (A^* A)^{\frac{1}{4}} \|^2 \leq \| (A^* A)^{\frac{1}{4}} \Psi_\epsilon \|^2 + \epsilon \leq \sum_i \| (A^* A)^{\frac{1}{4}} \mathbf{e}_i \|^2 + \epsilon = \| A \|_1 + \epsilon.$$

Letting $\epsilon \to 0$ then proves (2.150). The same trick with $\| A \| \leq \| A\Psi_\epsilon \| + \epsilon$ establishes (2.151).

The following decomposition will often be used.

Lemma 2.13.6 *Every operator $A \in \mathfrak{B}(\mathcal{H})$ has a* **polar decomposition**

$$A = U|A|, \tag{2.152}$$

where $|A| = \sqrt{A^* A}$ (cf. (2.146)) and U is a partial isometry with the same kernel as A.

First define U on the range of $|A|$ by $U|A|\Psi := A\Psi$. Then compute

$$(U|A|\Psi, U|A|\Phi) = (A\Psi, A\Phi) = (\Psi, A^* A\Phi) = (\Psi, |A|^2 \Phi) = (|A|\Psi, |A|\Phi).$$

Hence U is an isometry on $\mathrm{ran}(|A|)$. In particular, U is well defined, for this property implies that if $|A|\Psi_1 = |A|\Psi_2$ then $U|A|\Psi_1 = U|A|\Psi_2$. Then extend U to the closure of $\mathrm{ran}(|A|)$ by continuity, and put $U = 0$ on $\mathrm{ran}(|A|)^\perp$. One easily verifies that

$$|A| = U^* A, \tag{2.153}$$

and that $U^* U$ is the projection onto the closure of $\mathrm{ran}(|A|)$, whereas UU^* is the projection onto the closure of $\mathrm{ran}(A)$. ■

Proposition 2.13.7 *One has the inclusions*

$$\mathfrak{B}_f(\mathcal{H}) \subseteq \mathfrak{B}_1(\mathcal{H}) \subseteq \mathfrak{B}_2(\mathcal{H}) \subseteq \mathfrak{B}_0(\mathcal{H}) \subseteq \mathfrak{B}(\mathcal{H}), \tag{2.154}$$

with equalities iff $\mathcal{H}$ is finite-dimensional.

We first show that $\mathfrak{B}_1(\mathcal{H}) \subseteq \mathfrak{B}_0(\mathcal{H})$. Let $A \in \mathfrak{B}_1(\mathcal{H})$. Since $\sum_i (\mathbf{e}_i, |A|\mathbf{e}_i) < \infty$, for every $\epsilon > 0$ we can find $N(\epsilon)$ such that $\sum_{i>N(\epsilon)} (\mathbf{e}_i, |A|\mathbf{e}_i) < \epsilon$. Let $p_{N(\epsilon)}$ be the projection onto the linear span of all $\mathbf{e}_i$, $i > N(\epsilon)$. Using (2.16) and (2.150), we have

$$\| \ |A|^{\frac{1}{2}} p_{N(\epsilon)} \|^2 = \| p_{N(\epsilon)} |A| p_{N(\epsilon)} \| \leq \| p_{N(\epsilon)} |A| p_{N(\epsilon)} \|_1 < \epsilon,$$

so that $|A|^{\frac{1}{2}} p^{\perp}_{N(\epsilon)} \to |A|^{\frac{1}{2}}$ in the operator-norm topology. Since the star is norm-continuous by (2.17), this implies $p^{\perp}_{N(\epsilon)} |A|^{\frac{1}{2}} \to |A|^{\frac{1}{2}}$. Now $p^{\perp}_{N(\epsilon)} |A|^{\frac{1}{2}}$ obviously has finite rank for every $\epsilon > 0$, so that $|A|^{\frac{1}{2}}$ is compact by Definition 2.13.1. Since $A = U|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}$ by (2.152), Proposition 2.13.2.2 implies that $A \in \mathfrak{B}_0(\mathcal{H})$.

The proof that $\mathfrak{B}_2(\mathcal{H}) \subseteq \mathfrak{B}_0(\mathcal{H})$ is similar: this time we have

$$\| \ |A| p_{N(\epsilon)} \|^2 = \| p_{N(\epsilon)} |A|^2 p_{N(\epsilon)} \| \leq \| p_{N(\epsilon)} |A|^2 p_{N(\epsilon)} \|_2 < \epsilon,$$

so that $|A| p^{\perp}_{N(\epsilon)} \to |A|$, with the same conclusion.

Finally, we use Theorem 2.13.4 to rewrite (2.144) and (2.141) as

$$\begin{aligned} \| A \|_1 &= \sum_i a_i; \\ \| A \|_2 &= \sum_i a_i^2, \end{aligned} \tag{2.155}$$

where the a_i are the eigenvalues of $|A|$. This immediately gives

$$\| A \|_2 \leq \| A \|_1, \tag{2.156}$$

implying $\mathfrak{B}_1(\mathcal{H}) \subseteq \mathfrak{B}_2(\mathcal{H})$.

Finally, the claim about proper inclusions is trivially established by producing examples on the basis of 2.13.4 and (2.155). ■

The chain of inclusions (2.154) is sometimes seen as the non-commutative analogue of

$$\ell_c(X) \subseteq \ell^1(X) \subseteq \ell^2(X) \subseteq \ell_0(X) \subseteq \ell^\infty(X),$$

where X is an infinite discrete set. Since $\ell_1(X) = \ell_0(X)^*$ and $\ell^\infty(X) = \ell_1(X)^* = \ell_0(X)^{**}$, this analogy is strengthened by the following result.

Theorem 2.13.8 *One has $\mathfrak{B}_0(\mathcal{H})^* = \mathfrak{B}_1(\mathcal{H})$ and $\mathfrak{B}_1(\mathcal{H})^* = \mathfrak{B}_0(\mathcal{H})^{**} = \mathfrak{B}(\mathcal{H})$ under the pairing*

$$\hat{\rho}(A) = \mathrm{Tr}\, \rho A = \hat{A}(\rho). \tag{2.157}$$

Here $\hat{\rho} \in \mathfrak{B}_0(\mathcal{H})^*$ is identified with $\rho \in \mathfrak{B}_1(\mathcal{H})$, and $\hat{A} \in \mathfrak{B}_1(\mathcal{H})^*$ is identified with $A \in \mathfrak{B}(\mathcal{H})$.

The basic ingredient in the proof is the following lemma, whose proof is based on the fact that $\mathfrak{B}_2(\mathcal{H})$ is a Hilbert space in the inner product

$$(A, B) := \mathrm{Tr}\, A^* B. \tag{2.158}$$

To show that this is well defined, use (2.1) and (2.145).

Lemma 2.13.9 *For $\rho \in \mathfrak{B}_1(\mathcal{H})$ and $A \in \mathfrak{B}(\mathcal{H})$ one has*

$$|\mathrm{Tr}\, A\rho| \ \leq \| A \| \| \rho \|_1 \ . \tag{2.159}$$

Using (2.152) for ρ and (2.1) for the inner product (2.158), as well as (2.147) and (2.142), we estimate

$$|\mathrm{Tr}\, A\rho|^2 = |\mathrm{Tr}\, AU|\rho|^{\frac{1}{2}}|\rho|^{\frac{1}{2}}| = |((AU|\rho|^{\frac{1}{2}})^*, |\rho|^{\frac{1}{2}})|$$

$$\leq \parallel |\rho|^{\frac{1}{2}} \parallel_2^2 \parallel (AU|\rho|^{\frac{1}{2}})^* \parallel_2^2 = \parallel \rho \parallel_1 \mathrm{Tr}\, (|\rho|^{\frac{1}{2}} U^* A^* AU|\rho|^{\frac{1}{2}}).$$

Now observe that if $0 \leq A_1 \leq A_2$ then $\mathrm{Tr}\, A_2 \leq \mathrm{Tr}\, A_2$ for all $A_1, A_2 \in \mathfrak{B}_0(\mathcal{H})$, since on account of 2.13.4 one has $A_1 \leq A_2$ iff all eigenvalues of A_1 are $\leq$ all eigenvalues of A_2. Then use (2.155). From (2.91) we have $|\rho|^{\frac{1}{2}} U^* A^* AU|\rho|^{\frac{1}{2}} \leq \parallel AU \parallel^2 \rho$, so from the above insight we arrive at

$$\mathrm{Tr}\, (|\rho|^{\frac{1}{2}} U^* A^* AU|\rho|^{\frac{1}{2}}) \leq \parallel \rho \parallel_1 \parallel AU \parallel^2 \leq \parallel A \parallel^2,$$

since U is a partial isometry. Hence we have (2.159). ■

We now prove $\mathfrak{B}_0(\mathcal{H})^* = \mathfrak{B}_1(\mathcal{H})$. It is clear from 2.13.9 that

$$\mathfrak{B}_1(\mathcal{H}) \subseteq \mathfrak{B}_0(\mathcal{H})^*, \tag{2.160}$$

with

$$\parallel \hat{\rho} \parallel \leq \parallel \rho \parallel_1 . \tag{2.161}$$

To prove that $\mathfrak{B}_0(\mathcal{H})^* \subseteq \mathfrak{B}_1(\mathcal{H})$, we use (2.151). For $\hat{\rho} \in \mathfrak{B}_0(\mathcal{H})^*$ and $A \in \mathfrak{B}_2(\mathcal{H}) \subseteq \mathfrak{B}_0(\mathcal{H})$ we therefore have

$$|\hat{\rho}(A)| \leq \parallel \hat{\rho} \parallel \parallel A \parallel \leq \parallel \hat{\rho} \parallel \parallel A \parallel_2 .$$

Hence $\hat{\rho} \in \mathfrak{B}_2(\mathcal{H})^*$; since $\mathfrak{B}_2(\mathcal{H})$ is a Hilbert space, by Riesz-Fischer there is an operator $\rho \in \mathfrak{B}_2(\mathcal{H})$ such that $\hat{\rho}(A) = \mathrm{Tr}\, \rho A$ for all $A \in \mathfrak{B}_2(\mathcal{H})$. In view of (2.154), we need to sharpen $\rho \in \mathfrak{B}_2(\mathcal{H})$ to $\rho \in \mathfrak{B}_1(\mathcal{H})$. To do so, choose a finite-dimensional projection p, and note that $p|\rho| \in \mathfrak{B}_f(\mathcal{H}) \subseteq \mathfrak{B}_1(\mathcal{H})$; the presence of p even causes the sum in (2.143) to be finite in a suitable basis. Now use the polar decomposition $\rho = U|\rho|$ with (2.153) to write

$$\mathrm{Tr}\, p|\rho| = \mathrm{Tr}\, pU^*\rho = \mathrm{Tr}\, \rho p U^* = \hat{\rho}(pU^*);$$

changing the order inside the trace is justified by naive arguments, since the sum in (2.143) is finite. Using the original assumption $\hat{\rho} \in \mathfrak{B}_0(\mathcal{H})^*$, we have

$$|\mathrm{Tr}\, p|\rho|\,| \leq \parallel \hat{\rho} \parallel \parallel pU^* \parallel \leq \parallel \hat{\rho} \parallel \parallel p \parallel = \parallel \hat{\rho} \parallel \tag{2.162}$$

since U is a partial isometry, whereas $\parallel p \parallel = 1$ in view of (2.16) and $p = p^2 = p^*$. Now choose a basis of $\mathcal{H}$, and take p to be the projection onto the subspace spanned by the first N elements; from (2.143) and (2.162) we then have

$$|\mathrm{Tr}\, p|\rho|\,| = |\sum_{i=1}^{N} (\mathbf{e}_i, |\rho|\mathbf{e}_i)| \leq \parallel \hat{\rho} \parallel .$$

It follows that the sequence $s_N := |\sum_{i=1}^{N} (\mathbf{e}_i, |\rho|\mathbf{e}_i)|$ is bounded, and since it is positive it must have a limit. By (2.162) and (2.144) this means that $\parallel \rho \parallel_1 \leq \parallel \hat{\rho} \parallel$, so that $\rho \in \mathfrak{B}_1(\mathcal{H})$, hence $\mathfrak{B}_0(\mathcal{H})^* \subseteq \mathfrak{B}_1(\mathcal{H})$. Combining this with (2.160) and (2.161), we conclude that $\mathfrak{B}_0(\mathcal{H})^* = \mathfrak{B}_1(\mathcal{H})$ and $\parallel \rho \parallel_1 = \parallel \hat{\rho} \parallel$.

We turn to the proof of $\mathfrak{B}_1(\mathcal{H})^* = \mathfrak{B}(\mathcal{H})$. It is clear from 2.13.9 that $\mathfrak{B}(\mathcal{H}) \subseteq \mathfrak{B}_1(\mathcal{H})^*$, with

$$\parallel \hat{A} \parallel \leq \parallel A \parallel . \tag{2.163}$$

To establish the converse, pick $\hat{A} \in \mathfrak{B}_1(\mathcal{H})^*$ and $\Psi, \Phi \in \mathcal{H}$, and define a quadratic form Q_A on $\mathcal{H}$ by

$$Q_A(\Psi, \Phi) := \hat{A}(|\Phi><\Psi|). \tag{2.164}$$

Here the operator $|\Phi >< \Psi|$ is defined by $|\Phi >< \Psi|\Omega := (\Psi, \Omega)\Phi$. For example, when Ψ has unit length, $|\Psi >< \Psi|$ is the projection $[\Psi]$, and in general $|\Psi >< \Psi| = \| \Psi \|^2 [\Psi]$. Note that $(|\Phi >< \Psi|)^* = |\Psi >< \Phi|$, so that

$$| \, |\Phi >< \Psi| \, | = \sqrt{(|\Phi >< \Psi|)^* |\Phi >< \Psi|} = \sqrt{(\Phi, \Phi) |\Psi >< \Psi|} = \| \Phi \| \, \| \Psi \| \, [\Psi].$$

Since, for any projection p, the number $\mathrm{Tr}\, p$ is the dimension of $p\mathcal{H}$ (take a basis whose elements lie either in $p\mathcal{H}$ or in $p^\perp\mathcal{H}$), we have $\mathrm{Tr}\,[\Psi] = 1$. Hence from (2.144) we obtain

$$\| \, |\Phi >< \Psi| \, \|_1 = \| \Phi \| \, \| \Psi \| \, . \tag{2.165}$$

Since $\hat{A} \in \mathfrak{B}_1(\mathcal{H})^*$ by assumption, one has

$$|\hat{A}(|\Phi >< \Psi|)| \, \leq \| \hat{A} \| \, \| \, |\Phi >< \Psi| \, \|_1 \, . \tag{2.166}$$

Combining (2.166), (2.165), and (2.164), we have

$$|Q_A(\Psi, \Phi)| \, \leq \| \hat{A} \| \, \| \Phi \| \, \| \Psi \| \, . \tag{2.167}$$

Hence by Lemma 2.12.5 and (2.164) there is an operator A, with

$$\| A \| \leq \| \hat{A} \|, \tag{2.168}$$

such that $\hat{A}(|\Phi >< \Psi|) = (\Psi, A\Phi)$. Now note that $(\Psi, A\Phi) = \mathrm{Tr}\, |\Phi >< \Psi| A$; this follows by evaluating (2.144) over a basis containing $\| \Phi \|^{-1} \, |\Phi >$. Hence $\hat{A}(|\Phi >< \Psi|) = \mathrm{Tr}\, |\Phi >< \Psi| A$. Extending this equation by linearity to the span $\mathfrak{B}_f(\mathcal{H})$ of all $|\Phi >< \Psi|$, and subsequently by continuity to $\mathfrak{B}_1(\mathcal{H})$, we obtain $\hat{A}\rho = \mathrm{Tr}\, \rho A$. Hence $\mathfrak{B}_1(\mathcal{H})^* \subseteq \mathfrak{B}(\mathcal{H})$, so that, with (2.160), we obtain $\mathfrak{B}_1(\mathcal{H})^* = \mathfrak{B}(\mathcal{H})$. Combining (2.163) and (2.168), we find $\| A \| = \| \hat{A} \|$, so that the identification of $\mathfrak{B}_1(\mathcal{H})^*$ with $\mathfrak{B}(\mathcal{H})$ is isometric. ■

Corollary 2.13.10 *1. The state space of the C^*-algebra $\mathfrak{B}_0(\mathcal{H})$ of all compact operators on some Hilbert space $\mathcal{H}$ consists of all density matrices, where a* **density matrix** *is an element $\rho \in \mathfrak{B}_1(\mathcal{H})$ which is positive ($\rho \geq 0$) and has unit trace ($\mathrm{Tr}\, \rho = 1$).*

2. The pure state space of $\mathfrak{B}_0(\mathcal{H})$ consists of all one-dimensional projections.

3. The C^*-algebra $\mathfrak{B}_0(\mathcal{H})$ possesses only one irreducible representation, up to unitary equivalence, namely the defining one.

Diagonalize $\rho = \sum_i p_i [\Psi_i]$; cf. 2.13.4 and 2.154. Using $A = [\Psi_i]$, which is positive, the condition $\hat{\rho}(A) \geq 0$ yields $p_i \geq 0$. Conversely, when all $p_i \geq 0$ the operator ρ is positive. The normalization condition $\| \hat{\rho} \| = \| \rho \|_1 = \sum p_i = 1$ (see 2.8.6) and (2.155)) yields 2.13.10.1.

The next item 2.13.10.2 is then obvious from 2.12.1.

Finally, 2.13.10.3 follows from 2.13.10.2 and Corollaries 2.12.7 and 2.12.6. ■

Corollary 2.13.10.3 is one of the most important results in the theory of C^*-algebras. Applied to the finite-dimensional case, it shows that the C^*-algebra $\mathfrak{M}^n(\mathbb{C})$ of $n \times n$ matrices has only one irreducible representation.

The opposite extreme to a pure state on $\mathfrak{B}_0(\mathcal{H})$ is a **faithful state** $\hat{\rho}$, for which by definition the left-ideal $\mathcal{N}_\rho$ defined in (2.112) is zero. In other words, one has $\mathrm{Tr}\, A^*A > 0$ for all $A \neq 0$.

Proposition 2.13.11 *The GNS-representation π_ρ corresponding to a faithful state $\hat{\rho}$ on $\mathfrak{B}_0(\mathcal{H})$ is unitarily equivalent to the representation $\hat{\pi}_\rho(\mathfrak{B}_0(\mathcal{H}))$ on the Hilbert space $\mathfrak{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators given by left-multiplication, i.e.,*

$$\hat{\pi}_\rho(A)B := AB. \tag{2.169}$$

It is obvious from (2.141) that for $A \in \mathfrak{B}(\mathcal{H})$ and $B \in \mathfrak{B}_2(\mathcal{H})$ one has

$$\| AB \|_2 \leq \| A \| \| B \|_2, \tag{2.170}$$

so that the representation (2.169) is well-defined (even for $A \in \mathfrak{B}(\mathcal{H})$ rather than merely $A \in \mathfrak{B}_0(\mathcal{H})$). Moreover, when $A, B \in \mathfrak{B}_2(\mathcal{H})$ one has

$$\mathrm{Tr}\, AB = \mathrm{Tr}\, BA. \tag{2.171}$$

This follows from (2.147) and the identity

$$AB = \tfrac{1}{4}\sum_{n=0}^{3} i^n (B + i^n A^*)^*(B + i^n A^*). \tag{2.172}$$

When $\rho \in \mathfrak{B}_1(\mathcal{H})$ and $\rho \geq 0$ then $\rho^{1/2} \in \mathfrak{B}_2(\mathcal{H})$; see (2.144) and (2.145). It is easily seen that $\rho^{1/2}$ is cyclic for $\hat{\pi}_\rho(\mathfrak{B}_0(\mathcal{H}))$ when $\hat{\rho}$ is faithful. Using (2.158) and (2.171) we compute

$$(\rho^{1/2}, \hat{\pi}_\rho(A)\rho^{1/2}) = \mathrm{Tr}\, \rho^{1/2}\hat{\pi}_\rho(A)\rho^{1/2} = \mathrm{Tr}\, \rho A = \hat{\rho}(A).$$

The equivalence between π_ρ and $\hat{\pi}_\rho$ now follows from 2.9.6 or 2.9.5. ■

For an alternative proof, use the GNS construction itself. The map $A \to A\rho^{1/2}$, with $\rho \in \mathfrak{B}_1(\mathcal{H})$, maps $\mathfrak{B}_0(\mathcal{H})$ into $\mathfrak{B}_2(\mathcal{H})$, and if $\hat{\rho}$ is faithful the closure (in norm derived from the inner product (2.158)) of the image of this map is $\mathfrak{B}_2(\mathcal{H})$.

2.14 The double commutant theorem

The so-called double commutant theorem was proved by von Neumann in 1929, and remains a central result in operator algebra theory. For example, although it is a statement about von Neumann algebras, it controls the (ir)reducibility of representations. Recall that the commutant $\mathfrak{M}'$ of a collection $\mathfrak{M}$ of bounded operators consists of all bounded operators which commute with all elements of $\mathfrak{M}$; the bicommutant $\mathfrak{M}''$ is $(\mathfrak{M}')'$.

We first give the finite-dimensional version of the theorem; this is already nontrivial, and its proof contains the main idea of the proof of the infinite-dimensional case as well.

Proposition 2.14.1 *Let $\mathfrak{M}$ be a *-algebra (and hence a C^*-algebra) in $\mathfrak{M}^n(\mathbb{C})$ containing $\mathbb{I}$ (here $n < \infty$). Then $\mathfrak{M}'' = \mathfrak{M}$.*

The idea of the proof is to take n arbitrary vectors $\Psi_1, \ldots, \Psi_n$ in $\mathbb{C}^n$, and, given $A \in \mathfrak{M}''$, construct a matrix $A_0 \in \mathfrak{M}$ such that $A\Psi_i = A_0\Psi_i$ for all $i = 1, \ldots, n$. Hence $A = A_0 \in \mathfrak{M}$. We will write $\mathcal{H}$ for $\mathbb{C}^n$.

Choose some $\Psi = \Psi_1 \in \mathcal{H}$, and form the linear subspace $\mathfrak{M}\Psi$ of $\mathcal{H}$. Since $\mathcal{H}$ is finite-dimensional, this subspace is closed, and we may consider the projection $p = [\mathfrak{M}\Psi]$ onto this subspace. By Lemma 2.9.3 one has $p \in \mathfrak{M}'$. Hence $A \in \mathfrak{M}''$ commutes with p. Since $\mathbb{I} \in \mathfrak{M}$, we therefore have $\Psi = \mathbb{I}\Psi \in \mathfrak{M}\Psi$, so $\Psi = p\Psi$, and $A\Psi = Ap\Psi = pA\Psi \in \mathfrak{M}\Psi$. Hence $A\Psi = A_0\Psi$ for some $A_0 \in \mathfrak{M}$.

Now choose $\Psi_1, \ldots, \Psi_n \in \mathcal{H}$, and regard $\Psi_1 \dot{+} \ldots \dot{+} \Psi_n$ as an element of $\mathcal{H}^n := \oplus^n \mathcal{H} \simeq \mathcal{H} \otimes \mathbb{C}^n$ (the direct sum of n copies of $\mathcal{H}$), where Ψ_i lies in the i'th copy. Furthermore, embed $\mathfrak{M}$ in $\mathfrak{B}(\mathcal{H}^n) \simeq \mathfrak{M}^n(\mathfrak{B}(\mathcal{H}))$ by $A \to \delta(A) := A\mathbb{I}_n^{\otimes}$ (where $\mathbb{I}_n^{\otimes}$ is the unit in $\mathfrak{M}^n(\mathfrak{B}(\mathcal{H}))$); this is the diagonal matrix in $\mathfrak{M}^n(\mathfrak{B}(\mathcal{H}))$ in which all diagonal entries are A.

Now use the first part of the proof, with the substitutions $\mathcal{H} \to \mathcal{H}^n$, $\mathfrak{M} \to \delta(\mathfrak{M})$, $A \to \mathbb{A} := \delta(A)$, and $\Psi \to \Psi_1 \dot{+} \ldots \dot{+} \Psi_n$. Hence given $\Psi_1 \dot{+} \ldots \dot{+} \Psi_n$ and $\delta(A) \in \delta(\mathfrak{M})$ there exists $\mathbb{A}_0 \in \delta(\mathfrak{M})''$ such that

$$\delta(A)(\Psi_1 \dot{+} \ldots \dot{+} \Psi_n) = \mathbb{A}_0(\Psi_1 \dot{+} \ldots \dot{+} \Psi_n). \tag{2.173}$$

For arbitrary $\mathbb{B} \in \mathfrak{M}^n(\mathfrak{B}(\mathcal{H}))$, compute $([\mathbb{B}, \delta(A)])_{ij} = [B_{ij}, A]$. Hence $\delta(\mathfrak{M})' = \mathfrak{M}^n(\mathfrak{M}')$. It is easy to see that $\mathfrak{M}^n(\mathfrak{M}')' = \mathfrak{M}_n(\mathfrak{M}'')$, so that

$$\delta(\mathfrak{M})'' = \delta(\mathfrak{M}''). \tag{2.174}$$

Therefore, $\mathbb{A}_0 = \delta(A)_0$ for some $A_0 \in \mathfrak{M}$. Hence (2.173) reads $A\Psi_i = A_0\Psi_i$ for all $i = 1, \ldots, n$. ■

As it stands, Proposition 2.14.1 is not valid when $\mathfrak{M}^n(\mathbb{C})$ is replaced by $\mathfrak{B}(\mathcal{H})$, where $\dim(\mathcal{H}) = \infty$. To describe the appropriate refinement, we define two topologies on $\mathfrak{B}(\mathcal{H})$ which are weaker than the norm-topology we have used so far (and whose definition we repeat for convenience).

Definition 2.14.2 • *The* **norm-topology** *on* $\mathfrak{B}(\mathcal{H})$ *is defined by the criterion for convergence* $A_\lambda \to A$ *iff* $\| A_\lambda - A \| \to 0$. *A basis for the norm-topology is given by all sets of the form*

$$\mathcal{O}^n_\epsilon(A) := \{B \in \mathfrak{B}(\mathcal{H}) | \; \| B - A \| < \epsilon\}, \tag{2.175}$$

where $A \in \mathfrak{B}(\mathcal{H})$ *and* $\epsilon > 0$.

- *The* **strong topology** *on* $\mathfrak{B}(\mathcal{H})$ *is defined by the convergence* $A_\lambda \to A$ *iff* $\| (A_\lambda - A)\Psi \| \to 0$ *for all* $\Psi \in \mathcal{H}$. *A basis for the strong topology is given by all sets of the form*

$$\mathcal{O}^s_\epsilon(A, \Psi_1, \ldots, \Psi_n) := \{B \in \mathfrak{B}(\mathcal{H}) | \; \| (B - A)\Psi_i \| < \epsilon \; \forall i = 1, \ldots, n\}, \tag{2.176}$$

where $A \in \mathfrak{B}(\mathcal{H})$, $\Psi_1, \ldots, \Psi_n \in \mathcal{H}$, *and* $\epsilon > 0$.

- *The* **weak topology** *on* $\mathfrak{B}(\mathcal{H})$ *is defined by the convergence* $A_\lambda \to A$ *iff* $|(\Psi, (A_\lambda - A)\Psi)| \to 0$ *for all* $\Psi \in \mathcal{H}$. *A basis for the weak topology is given by all sets of the form*

$$\mathcal{O}^w_\epsilon(A, \Psi_1, \ldots, \Psi_n, \Phi_1, \ldots, \Phi_n) := \{B \in \mathfrak{B}(\mathcal{H}) | \, |(\Phi_i, (A_\lambda - A)\Psi_i)| \; < \epsilon \; \forall i = 1, \ldots, n\}, \tag{2.177}$$

where $A \in \mathfrak{B}(\mathcal{H})$, $\Psi_1, \ldots, \Psi_n, \Phi_1, \ldots, \Phi_n \in \mathcal{H}$, *and* $\epsilon > 0$.

These topologies should all be seen in the light of the general theory of locally convex topological vector spaces. These are vector spaces whose topology is defined by a family $\{p_\alpha\}$ of semi-norms; recall that a semi-norm on a vector space $\mathcal{V}$ is a function $p : \mathcal{V} \to \mathbb{R}$ satisfying 2.1.1.1, 3, and 4. A net $\{v_\lambda\}$ in $\mathcal{V}$ converges to v in the topology generated by a given iff $p_\alpha(v_\lambda - v) \to 0$ for all α.

The norm-topology is defined by a single semi-norm, namely the operator norm, which is even a norm. Its open sets are generated by ϵ-balls in the operator norm, whereas the strong and the weak topologies are generated by finite intersections of ϵ-balls defined by semi-norms of the form $p^s_\Psi(A) :=\| A\Psi \|$ and $p^w_{\Psi,\Phi}(A) := |(\Phi, A\Psi)|$, respectively. The equivalence between the definitions of convergence stated in 2.14.2 and the topologies defined by the open sets in question is given in theory of locally convex topological vector spaces.

The estimate (2.4) shows that norm-convergence implies strong convergence. Using the Cauch-Schwarz inequality (2.1) one sees that strong convergence implies weak convergence. In other words, the norm topology is stronger than the strong topology, which in turn is stronger than the weak topology.

Theorem 2.14.3 *Let* $\mathfrak{M}$ *be a* **-algebra in* $\mathfrak{B}(\mathcal{H})$, *containing* $\mathbb{I}$. *The following are equivalent:*

1. $\mathfrak{M}'' = \mathfrak{M}$;
2. $\mathfrak{M}$ *is closed in the weak operator topology;*
3. $\mathfrak{M}$ *is closed in the strong operator topology.*

It is easily verified from the definition of weak convergence that the commutant $\mathfrak{N}'$ of a *-algebra $\mathfrak{N}$ is always weakly closed: for if $A_\lambda \to A$ weakly with all $A_\alpha \in \mathfrak{N}$, and $B \in \mathfrak{N}$, then

$$(\Phi, [A, B]\Psi) = (\Phi, AB\Psi) - (B^*\Phi, A\Psi) = \lim_\alpha (\Phi, A_\lambda B\Psi) - (B^*\Phi, A_\lambda\Psi) = \lim_\alpha (\Phi, [A_\lambda, B]\Psi) = 0.$$

If $\mathfrak{M}'' = \mathfrak{M}$ then $\mathfrak{M} = \mathfrak{N}'$ for $\mathfrak{N} = \mathfrak{M}'$, so that $\mathfrak{M}$ is weakly closed. Hence "1 ⇒ 2".

Since the weak topology is weaker than the strong topology, "2 ⇒ 3" is trivial.

To prove "3 ⇒ 1", we adapt the proof of 2.14.1 to the infinite-dimensional situation. Instead of $\mathfrak{M}\Psi$, which may not be closed, we consider its closure $\overline{\mathfrak{M}\Psi}$, so that $p = [\overline{\mathfrak{M}\Psi}]$. Hence $A \in \mathfrak{M}''$

implies $A \in \overline{\mathfrak{M}\Psi}$; in other words, for every $\epsilon > 0$ there is an $A_\epsilon \in \mathfrak{M}$ such that $\| (A - A_\epsilon)\Psi \| < \epsilon$. For $\mathcal{H}^n$ this means that

$$\| \delta(A - A_\epsilon)(\Psi_1 \dot{+} \ldots \dot{+} \Psi_n) \|^2 = \sum_{i=1}^{n} \| (A - A_\epsilon)\Psi_i \|^2 < \epsilon^2.$$

Noting the inclusion

$$\{\sum_{i=1}^{n} \| (A - B)\Psi_i \|^2 < \epsilon^2\} \subseteq \mathcal{O}^s_\epsilon(A, \Psi_1, \ldots, \Psi_n)$$

(cf. (2.176)), it follows that $A_\epsilon \to A$ for $\epsilon \to 0$. Since all $A_\epsilon \in \mathfrak{M}$ and $\mathfrak{M}$ is strongly closed, this implies that $A \in \mathfrak{M}$, so that $\mathfrak{M}'' \subseteq \mathfrak{M}$. With the trivial inclusion $\mathfrak{M} \subseteq \mathfrak{M}''$, this proves that $\mathfrak{M}'' = \mathfrak{M}$. ■

3 Hilbert C^*-modules and induced representations

3.1 Vector bundles

This chapter is concerned with the 'non-commutative analogue' of a vector bundle. Let us first recall the notion of an ordinary vector bundle; this is a special case of the following

Definition 3.1.1 *A* **bundle** $\mathsf{B}(X, F, \tau)$ *consists of topological spaces* B *(the* **total space***),* X *(the* **base***),* F *(the* **typical fiber***), and a continuous surjection* $\tau : \mathsf{P} \to X$ *with the following property: each* $x \in X$ *has a neighbourhood* $\mathcal{N}_\alpha$ *such that there is a homeomorphism* $\psi_\alpha : \tau^{-1}(\mathcal{N}_\alpha) \to \mathcal{N}_\alpha \times F \subset X \times F$ *for which* $\tau = \tau_X \circ \psi_\alpha$ *(where* $\tau_X : X \times F \to X$ *is the projection onto the first factor).*

The maps ψ_α are called **local trivializations**. We factorize $\psi_\alpha = (\tau, \psi^F_\alpha)$, so that ψ^F_α restricted to $\tau^{-1}(x)$ provides a homeomorphism between the latter and the typical fiber F. Each subset $\tau^{-1}(x)$ is called a **fiber** of B. One may think of B as X with a copy of F attached at each point.

The simplest example of a bundle over a base X with typical fiber F is the **trivial bundle** $\mathsf{B} = X \times F$, with $\tau(x, f) := x$. According to the definition, any bundle is locally trivial in the specified sense.

Definition 3.1.2 *A* **vector bundle** *is a bundle in which*

1. *each fiber is a finite-dimensional vector space, such that the relative topology of each fiber coincides with its topology as a vector space;*

2. *each local trivialization* $\psi^F_\alpha : \tau^{-1}(x) \to F$ *(where* $x \in \mathcal{N}_\alpha$*) is linear.*

A **complex vector bundle** *is a vector bundle with typical fiber* $\mathbb{C}^m$*, for some* $m \in \mathbb{N}$*.*

We will generically denote vector bundles by the letter V, with typical fiber $F = V$. The simplest vector bundle over X with fiber $V = \mathbb{C}^n$ is the trivial bundle $\mathsf{V} = X \times \mathbb{C}^n$. This bundle leads to possibly nontrivial sub-bundles, as follows. Recall the definition of $\mathfrak{M}^n(\mathfrak{A})$ in 2.11, specialized to $\mathfrak{A} = C(X)$ in the proof of 2.11.4. If X is a compact Hausdorff space, then $\mathfrak{M}^n(C(X)) \simeq C(X, \mathfrak{M}^n(\mathbb{C}))$ is a C^*-algebra. Let X in addition be connected. One should verify that a matrix-valued function $p \in C(X, \mathfrak{M}^n(\mathbb{C}))$ is an idempotent (that is, $p^2 = p$) iff each $p(x)$ is an idempotent in $\mathfrak{M}^n(\mathbb{C})$. Such an idempotent p defines a vector bundle V_p, whose fiber above x is $\tau^{-1}(x) := p(x)\mathbb{C}^n$. The space V_p inherits a topology and a projection τ (onto the first co-ordinate) from $X \times \mathbb{C}^n$, relative to which all axioms for a vector bundle are satisfied. Note that the dimension of $p(x)$ is independent of x, because p is continuous and X is connected.

The converse is also true.

Proposition 3.1.3 *Let* V *be a complex vector bundle over a connected compact Hausdorff space* X*, with typical fiber* $\mathbb{C}^m$*. There is an integer* $n \geq m$ *and an idempotent* $p \in C(X, \mathfrak{M}^n(\mathbb{C}))$ *such that* $\mathsf{V} \subseteq X \times \mathbb{C}^n$*, with* $\tau^{-1}(x) = p(x)\mathbb{C}^n$*.*

The essence of the proof is the construction of a complex vector bundle V' such that $\mathsf{V} \oplus \mathsf{V}'$ is trivial (where the direct sum is defined fiberwise); this is the bundle $X \times \mathbb{C}^n$.

Following the philosophy of non-commutative geometry, we now try to describe vector bundles in terms of C^*-algebras. The first step is the notion of a **section** of V; this is a map $\Psi : X \to \mathsf{V}$ for which $\tau(\Psi(x)) = x$ for all $x \in X$. In other words, a section maps a point in the base space into the fiber above the point. Thus one defines the space $\Gamma(\mathsf{V})$ of all continuous sections of V. This is a vector space under pointwise addition and scalar multiplication (recall that each fiber of V is a vector space). Moreover, when X is a connected compact Hausdorff space, $\Gamma(\mathsf{V})$ is a right-module for the commutative C^*-algebra $C(X)$: one obtains a linear action π_R of $C(X)$ on $\Gamma(\mathsf{V})$ by

$$\pi_R(f)\Psi(x) := f(x)\Psi(x). \tag{3.1}$$

Since $C(X)$ is commutative, this is, of course, a left-action as well.

For example, in the trivial case one has the obvious isomorphisms

$$\Gamma(X \times \mathbb{C}^m) \simeq C(X, \mathbb{C}^m) \simeq C(X) \otimes \mathbb{C}^m \simeq \oplus^m C(X). \tag{3.2}$$

A fancy way of saying this is that $\Gamma(X \times \mathbb{C}^m)$ is a **finitely generated free module** for $C(X)$. Here a free (right-) module $\mathcal{E}$ for an algebra $\mathfrak{A}$ is a direct sum $\mathcal{E} = \oplus^n \mathfrak{A}$ of a number of copies of $\mathfrak{A}$ itself, on which $\mathfrak{A}$ acts by right-multiplication, i.e.,

$$\pi_R(B) A_1 \oplus \ldots \oplus A_n := A_1 B \oplus \ldots \oplus A_n B. \tag{3.3}$$

If this number is finite one says that the free module is finitely generated.

When V is non-trivial, one obtains $\Gamma(\mathsf{V})$ as a certain modification of a finitely generated free module for $C(X)$. For any algebra $\mathfrak{A}$, and idempotent $p \in \mathfrak{M}^n(\mathfrak{A})$, the action of p on $\oplus^n \mathfrak{A}$ commutes with the action by $\mathfrak{A}$ given by right-multiplication on each component. Hence the vector space $p \oplus^m \mathfrak{A}$ is a right- $\mathfrak{A}$-module, called **projective**. When $m < \infty$, one calls $p \oplus^m \mathfrak{A}$ a **finitely generated projective module** for $\mathfrak{A}$.

In particular, when $\mathsf{V} = X \times \mathbb{C}^n$ and V_p is the vector bundle described prior to 3.1.3, we see that

$$\Gamma(\mathsf{V}_p) = p \oplus^n C(X) \tag{3.4}$$

under the obvious (right-) action of $C(X)$.

This lead to the **Serre-Swan theorem**.

Theorem 3.1.4 *Let X be a connected compact Hausdorff space. There is a bijective correspondence between complex vector bundles V over X and finitely generated projective modules $\mathcal{E}(\mathsf{V}) = \Gamma(\mathsf{V})$ for $C(X)$.*

This is an immediate consequence of Proposition 3.1.3: any vector bundle is of the form V_p, leading to $\Gamma(\mathsf{V}_p)$ as a finitely generated projective $C(X)$-module by (3.4). Conversely, given such a module $p \oplus^n C(X)$, one has $p \in C(X, \mathfrak{M}^n(\mathbb{C}))$, and thereby a vector bundle V_p as described prior to 3.1.3. ■

Thus we have achieved our goal of describing vector bundles over X purely in terms of concepts pertinent to the C^*-algebra $C(X)$. Let us now add further structure.

Definition 3.1.5 *A* **Hermitian vector bundle** *is a complex vector bundle V with an inner product $(\, ,\,)_x$ defined on each fiber $\tau^{-1}(x)$, which continuously depends on x. More precisely, for all $\Psi, \Phi \in \Gamma(\mathsf{V})$ the function $x \to (\Psi(x), \Phi(x))_x$ lies in $C(X)$.*

Using the local triviality of V and the existence of a partition of unity, it is easily shown that any complex vector bundle over a paracompact space can be equipped with such a Hermitian structure. Describing the bundle as V_p, a Hermitian structure is simply given by restricting the natural inner product on each fiber $\mathbb{C}^n$ of $X \times \mathbb{C}^n$ to V_p. One may then choose the idempotent $p \in C(X, \mathbb{C}^n)$ so as to be a projection with respect to the usual involution on $C(X, \mathbb{C}^n)$ (i.e., one has $p^* = p$ in addition to $p^2 = p$). Any other Hermitian structure on V_p may be shown to be equivalent to this canonical one.

There is no reason to restrict the dimension of the fibers so as to be finite-dimensional. A **Hilbert bundle** is defined by replacing 'finite-dimensional vector space' in 3.1.2.1 by 'Hilbert space', still requiring that all fibers have the same dimension (which may be infinite). A Hilbert bundle with finite-dimensional fibers is evidently the same as a Hermitian vector bundle. The simplest example of a Hilbert bundle is a Hilbert space, seen as a bundle over the base space consisting of a single point.

The following class of Hilbert bundles will play a central role in the theory of induced group representations.

Proposition 3.1.6 *Let H be a closed subgroup of a locally compact group G, and take a unitary representation U_χ of H on a Hilbert space $\mathcal{H}_\chi$. Then H acts on $G \times \mathcal{H}_\chi$ by $h : (x, v) \to (xh^{-1}, U_\chi(h)v)$, and the quotient*

$$\mathsf{H}^\chi := G \times_H \mathcal{H}_\chi = (G \times \mathcal{H}_\chi)/H \tag{3.5}$$

by this action is a Hilbert bundle over $X = G/H$, with projection

$$\tau_\chi([x, v]_H) := [x]_H \tag{3.6}$$

and typical fiber $\mathcal{H}_\chi$.

Here $[x, v]_H$ is the equivalence class in $G \times_H \mathcal{H}_\chi$ of $(x, v) \in G \times \mathcal{H}_\chi$, and $[x]_H = xH$ is the equivalence class in G/H of $x \in G$. Note that the projection τ_χ is well defined.

The proof relies on the fact that G is a bundle over G/H with projection

$$\tau(x) = [x]_H \tag{3.7}$$

and typical fiber H. This fact, whose proof we omit, implies that every $q \in G/H$ has a neighbourhood $\mathcal{N}_\alpha$, so that $\psi_\alpha = (\tau, \psi^H_\alpha) : \tau^{-1}(\mathcal{N}_\alpha) \to \mathcal{N}_\alpha \times H$ is a diffeomorphism, which satisfies

$$\psi^H_\alpha(xh) = \psi^H_\alpha(x)h. \tag{3.8}$$

This leads to a map $\psi^\chi_\alpha : \tau_\chi^{-1}(\mathcal{N}_\alpha) \to \mathcal{N}_\alpha \times \mathcal{H}_\chi$, given by $\psi^\chi_\alpha([x, v]_H) := ([x]_H, U_\chi(\psi^H_\alpha(x))v)$. This map is well defined because of (3.8), and is a local trivialization of $G \times_H \mathcal{H}_\chi$. All required properties are easily checked. ■

3.2 Hilbert C^*-modules

What follows generalizes the notion of a Hilbert bundle in such a way that the commutative C^*-algebra $C(X)$ is replaced by an arbitrary C^*-algebra $\mathfrak{B}$. This is an example of the strategy of non-commutative geometry.

Definition 3.2.1 *A* **Hilbert C^*-module** *over a C^*-algebra $\mathfrak{B}$ consists of*

- *A complex linear space $\mathcal{E}$.*
- *A right-action π_{R} of $\mathfrak{B}$ on $\mathcal{E}$ (i.e., π_{R} maps $\mathfrak{B}$ linearly into the space of all linear operators on $\mathcal{E}$, and satisfies $\pi_{\mathrm{R}}(AB) = \pi_{\mathrm{R}}(B)\pi_{\mathrm{R}}(A)$), for which we shall write $\Psi B := \pi_{\mathrm{R}}(B)\Psi$, where $\Psi \in \mathcal{E}$ and $B \in \mathfrak{B}$.*
- *A sesquilinear map $\langle\, ,\, \rangle_{\mathfrak{B}} : \mathcal{E} \times \mathcal{E} \to \mathfrak{B}$, linear in the second and anti-linear in the first entry, satisfying*

$$\langle \Psi, \Phi \rangle^*_{\mathfrak{B}} = \langle \Phi, \Psi \rangle_{\mathfrak{B}}; \tag{3.9}$$
$$\langle \Psi, \Phi B \rangle_{\mathfrak{B}} = \langle \Psi, \Phi \rangle_{\mathfrak{B}} B; \tag{3.10}$$
$$\langle \Psi, \Psi \rangle_{\mathfrak{B}} \geq 0; \tag{3.11}$$
$$\langle \Psi, \Psi \rangle_{\mathfrak{B}} = 0 \Leftrightarrow \Psi = 0, \tag{3.12}$$

for all $\Psi, \Phi \in \mathcal{E}$ and $B \in \mathfrak{B}$.

The space $\mathcal{E}$ is complete in the norm

$$\| \Psi \| := \| \langle \Psi, \Psi \rangle_{\mathfrak{B}} \|^{\frac{1}{2}} . \tag{3.13}$$

We say that $\mathcal{E}$ is a **Hilbert $\mathfrak{B}$-module**, *and write $\mathcal{E} \rightleftharpoons \mathfrak{B}$.*

One checks that (3.13) is indeed a norm: $\| \Psi \|^2$ equals $\sup\{\omega(\langle \Psi, \Psi \rangle_{\mathfrak{B}})\}$, where the supremum is taken over all states ω on $\mathfrak{B}$. Since each map $\Psi \to \sqrt{\omega(\langle \Psi, \Psi \rangle_{\mathfrak{B}})}$ is a semi-norm (i.e., a norm except for positive definiteness) by (3.11), the supremum is a semi-norm, which is actually positive definite because of Lemma 2.8.8 and (3.12).

The $\mathfrak{B}$-action on $\mathcal{E}$ is automatically non-degenerate: the property $\Psi B = 0$ for all $B \in \mathfrak{B}$ implies that $\langle \Psi, \Psi \rangle_{\mathfrak{B}} B = 0$ for all B, hence $\langle \Psi, \Psi \rangle_{\mathfrak{B}} = 0$ (when $\mathfrak{B}$ is unital this is follows by taking $B = \mathbb{I}$; otherwise one uses an approximate unit in $\mathfrak{B}$), so that $\Psi = 0$ by (3.12).

When all conditions in 3.2.1 are met except (3.12), so that $\| \cdot \|$ defined by (3.13) is only a semi-norm, one simply takes the quotient of $\mathcal{E}$ by its subspace of all null vectors and completes, obtaining a Hilbert C^*-module in that way.

It is useful to note that (3.9) and (3.10) imply that

$$\langle \Psi B, \Phi \rangle_{\mathfrak{B}} = B^* \langle \Psi, \Phi \rangle_{\mathfrak{B}}. \tag{3.14}$$

Example 3.2.2 *1. Any C^*-algebra $\mathfrak{A}$ is a $\mathfrak{A}$-module $\mathfrak{A} \rightleftharpoons \mathfrak{A}$ over itself, with $\langle A, B \rangle_{\mathfrak{A}} := A^* B$. Note that the norm (3.13) coincides with the C^*-norm by (2.16).*

2. Any Hilbert space $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module $\mathcal{H} \rightleftharpoons \mathbb{C}$ in its inner product.

3. Let H be a Hilbert bundle H over a compact Hausdorff space X. The space of continuous sections $\mathcal{E} = \Gamma(\mathsf{H})$ of H is a Hilbert C^*-module $\Gamma(\mathsf{H}) \rightleftharpoons C(X)$ over $\mathfrak{B} = C(X)$; for $\Psi, \Phi \in \Gamma_0(\mathsf{H})$ the function $\langle \Psi, \Phi \rangle_{C(X)}$ is defined by

$$\langle \Psi, \Phi \rangle_{C(X)} : x \to (\Psi(x), \Phi(x))_x , \tag{3.15}$$

where the inner product is the one in the fiber $\tau^{-1}(x)$. The right-action of $C(X)$ on $\Gamma(\mathsf{H})$ is defined by (3.1).

In the third example the norm in $\Gamma(\mathsf{H})$ is $\| \Psi \| = \sup_x \| \Psi(x) \|$, where $\| \Psi(x) \| = (\Psi(x), \Psi(x))_x^{\frac{1}{2}}$, so that it is easily seen that $\mathcal{E}$ is complete.

Many Hilbert C^*-modules of interest will be constructed in the following way. Recall that a pre-C^*-algebra is a *-algebra satisfying all properties of a C^*-algebra except perhaps completeness. Given a pre-C^*-algebra $\tilde{\mathfrak{B}}$, define a **pre-Hilbert $\tilde{\mathfrak{B}}$-module** $\tilde{\mathcal{E}} \rightleftharpoons \tilde{\mathfrak{B}}$ as in Definition 3.2.1, except that the final completeness condition is omitted.

Proposition 3.2.3 *In a pre-Hilbert $\tilde{\mathfrak{B}}$-module (and hence in a Hilbert $\mathfrak{B}$-module) one has the inequalities*

$$\| \Psi B \| \leq \| \Psi \| \| B \|; \tag{3.16}$$

$$\langle \Psi, \Phi \rangle_{\mathfrak{B}} \langle \Phi, \Psi \rangle_{\mathfrak{B}} \leq \| \Phi \|^2 \langle \Psi, \Psi \rangle_{\mathfrak{B}}; \tag{3.17}$$

$$\| \langle \Psi, \Phi \rangle_{\mathfrak{B}} \| \leq \| \Psi \| \| \Phi \| . \tag{3.18}$$

To prove (3.16) one uses (3.14), (2.91), (2.86), and (2.16). For (3.17) we substitute $\Phi \langle \Phi, \Psi \rangle_{\mathfrak{B}} - \Psi$ for Ψ in the inequality $\langle \Psi, \Psi \rangle_{\mathfrak{B}} \geq 0$. Expanding, the first term equals $\langle \Psi, \Phi \rangle_{\mathfrak{B}} \langle \Phi, \Phi \rangle_{\mathfrak{B}} \langle \Phi, \Psi \rangle_{\mathfrak{B}}$. Then use (2.91), and replace Φ by $\Phi / \| \Phi \|$. The inequality (3.18) is immediate from (3.17). ■

Corollary 3.2.4 *A pre-Hilbert $\tilde{\mathfrak{B}}$-module $\tilde{\mathcal{E}} \rightleftharpoons \tilde{\mathfrak{B}}$ can be completed to a Hilbert $\mathfrak{B}$-module.*

One first completes $\tilde{\mathcal{E}}$ in the norm (3.13), obtaining $\mathcal{E}$. Using (3.16), the $\tilde{\mathfrak{B}}$-action on $\tilde{\mathcal{E}}$ extends to a $\mathfrak{B}$-action on $\mathcal{E}$. The completeness of $\mathfrak{B}$ and (3.18) then allow one to extend the $\tilde{\mathfrak{B}}$-valued sesquilinear form on $\tilde{\mathcal{E}}$ to a $\mathfrak{B}$-valued one on $\mathcal{E}$. It is easily checked that the required properties hold by continuity. ■

In Example 3.2.2, it is almost trivial to see that $\mathfrak{A}$ and $\mathcal{H}$ are the closures of $\tilde{\mathfrak{A}}$ (defined over $\tilde{\mathfrak{A}}$) and of a dense subspace $\mathcal{D}$, respectively.

A Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$ defines a certain C^*-algebra $C^*(\mathcal{E}, \mathfrak{B})$, which plays an important role in the induction theory in 3.5. A map $A : \mathcal{E} \to \mathcal{E}$ for which there exists a map $A^* : \mathcal{E} \to \mathcal{E}$ such that

$$\langle \Psi, A\Phi \rangle_{\mathfrak{B}} = \langle A^*\Psi, \Phi \rangle_{\mathfrak{B}} \tag{3.19}$$

for all $\Psi, \Phi \in \mathcal{E}$ is called **adjointable**.

Theorem 3.2.5 *An adjointable map is automatically $\mathbb{C}$-linear, $\mathfrak{B}$-linear (that is, $(A\Psi)B = A(\Psi B)$ for all $\Psi \in \mathcal{E}$ and $B \in \mathfrak{B}$), and bounded. The adjoint of an adjointable map is unique, and the map $A \to A^*$ defines an involution on the space $C^*(\mathcal{E}, \mathfrak{B})$ of all adjointable maps on $\mathcal{E}$.*

Equipped with this involution, and with the norm (2.2), defined with respect to the norm (3.13) on $\mathcal{E}$, the space $C^*(\mathcal{E}, \mathfrak{B})$ is a C^*-algebra.

Each element $A \in C^*(\mathcal{E}, \mathfrak{B})$ satisfies the bound

$$\langle A\Psi, A\Psi \rangle_{\mathfrak{B}} \leq \| A \|^2 \langle \Psi, \Psi \rangle_{\mathfrak{B}} \tag{3.20}$$

for all $\Psi \in \mathcal{E}$. The (defining) action of $C^*(\mathcal{E}, \mathfrak{B})$ on $\mathcal{E}$ is non-degenerate. We write $C^*(\mathcal{E}, \mathfrak{B}) \to \mathcal{E} \rightleftharpoons \mathfrak{B}$.

The property of $\mathbb{C}$-linearity is immediate. To establish $\mathfrak{B}$-linearity one uses (3.14); this also shows that $A^* \in C^*(\mathcal{E}, \mathfrak{B})$ when $A \in C^*(\mathcal{E}, \mathfrak{B})$.

To prove boundedness of a given adjointable map A, fix $\Psi \in \mathcal{E}$ and define $T_\Psi : \mathcal{E} \to \mathfrak{B}$ by $T_\Psi \Phi := \langle A^*A\Psi, \Phi \rangle_{\mathfrak{B}}$. It is clear from (3.18) that $\| T_\Psi \| \leq \| A^*A\Psi \|$, so that T_Ψ is bounded. On the other hand, since A is adjointable, one has $T_\Psi \Phi = \langle \Psi, A^*A\Phi \rangle_{\mathfrak{B}}$, so that, using (3.18) once again, one has $\| T_\Psi \Phi \| \leq \| A^*A\Phi \| \| \Psi \|$. Hence $\sup\{\| T_\Psi \| \mid \| \Psi \| = 1\} < \infty$ by the principle of uniform boundedness (here it is essential that $\mathcal{E}$ is complete). It then follows from (3.13) that $\| A \| < \infty$.

Uniqueness and involutivity of the adjoint are proved as for Hilbert spaces; the former follows from (3.12), the latter in addition requires (3.9).

The space $C^*(\mathcal{E}, \mathfrak{B})$ is norm-closed, as one easily verifies from (3.19) and (3.13) that if $A_n \to A$ then A_n^* converges to some element, which is precisely A^*. As a norm-closed space of linear maps on a Banach space, $C^*(\mathcal{E}, \mathfrak{B})$ is a Banach algebra, so that its satisfies (2.15). To check (2.16) one infers from (3.13) and the definition (3.19) of the adjoint that $\| A \|^2 \leq \| A^*A \|$; then use Lemma 2.1.11.

Finally, it follows from (3.11), (2.88), and (3.19) that for fixed $\Psi \in \mathcal{E}$ the map $A \to \langle \Psi, A\Psi \rangle_{\mathfrak{B}}$ from $C^*(\mathcal{E}, \mathfrak{B})$ to $\mathfrak{B}$ is positive. Replacing A by A^*A in (2.85) and using (2.16) and (3.19) then leads to (3.20).

To prove the final claim, we note that, for fixed $\Psi, \Phi \in \mathcal{E}$, the map $Z \to \Psi\langle \Phi, Z \rangle_{\mathfrak{B}}$ is in $C^*(\mathcal{E}, \mathfrak{B})$. When the right-hand side vanishes for all Ψ, Φ it must be that $\langle \Phi, Z \rangle_{\mathfrak{B}} = 0$ for all Φ, hence for $\Phi = Z$, so that $Z = 0$. Here we used the fact that $\Psi B = 0$ for all Ψ and B in the linear span of $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathfrak{B}}$ implies $B = 0$, for by (3.10) it implies that $\langle \Psi, \Psi \rangle_{\mathfrak{B}} B = 0$. ■

Under a further assumption (which is by no means always met in our examples) one can completely characterize $C^*(\mathcal{E}, \mathfrak{B})$. A Hilbert C^*-module over $\mathfrak{B}$ is called **self-dual** when every bounded $\mathfrak{B}$-linear map $\varphi : \mathcal{E} \to \mathfrak{B}$ is of the form $\varphi(\Psi) = \langle \Phi, \Psi \rangle_{\mathfrak{B}}$ for some $\Phi \in \mathcal{E}$.

Proposition 3.2.6 *In a self-dual Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$ the C^*-algebra $C^*(\mathcal{E}, \mathfrak{B})$ coincides with the space $\mathcal{L}(\mathcal{E})^{\mathfrak{B}}$ of all bounded $\mathbb{C}$-linear and $\mathfrak{B}$-linear maps on $\mathcal{E}$.*

In view of Theorem 3.2.5 we only need to show that a given map $A \in \mathcal{L}(\mathcal{E})^{\mathfrak{B}}$ is adjointable. Indeed, for fixed $\Psi \in \mathcal{E}$ define $\varphi_{A,\Psi} : \mathcal{E} \to \mathfrak{B}$ by $\varphi_{A,\Psi}(Z) := \langle \Psi, AZ \rangle_{\mathfrak{B}}$. By self-duality this must equal $\langle \Phi, Z \rangle_{\mathfrak{B}}$ for some Φ, which by definition is $A^*\Psi$. ■

In the context of Example 3.2.2.1, one may wonder what $C^*(\mathfrak{A}, \mathfrak{A})$ is. The map $\rho : \mathfrak{A} \to \mathfrak{B}(\mathfrak{A})$ given by (2.72) is easily seen to map $\mathfrak{A}$ into $C^*(\mathfrak{A}, \mathfrak{A})$. This map is isometric (hence injective).

Using (3.19), one infers that $A\rho(B) = \rho(AB)$ for all $A, B \in \mathfrak{A}$. Hence $\rho(\mathfrak{A})$ is an ideal in $C^*(\mathfrak{A},\mathfrak{A})$. When $\mathfrak{A}$ has a unit, one therefore has $C^*(\mathfrak{A},\mathfrak{A}) = \rho(\mathfrak{A}) \simeq \mathfrak{A}$; cf. the proof of 2.4.5.

When $\mathfrak{A}$ has no unit, $C^*(\mathfrak{A},\mathfrak{A})$ is the so-called **multiplier algebra** of $\mathfrak{A}$. One may compute this object by taking a faithful non-degenerate representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$; it can be shown that $C^*(\mathfrak{A},\mathfrak{A})$ is isomorphic to the idealizer of $\pi(\mathfrak{A})$ in $\mathfrak{B}(\mathcal{H})$ (this is the set of all $B \in \mathfrak{B}(\mathcal{H})$ for which $B\pi(A) \in \pi(\mathfrak{A})$ for all $A \in \mathfrak{A}$). One thus obtains

$$C^*(C_0(X), C_0(X)) = C_b(X); \tag{3.21}$$

$$C^*(\mathfrak{B}_0(\mathcal{H}), \mathfrak{B}_0(\mathcal{H})) = \mathfrak{B}(\mathcal{H}). \tag{3.22}$$

Eq. (3.21) follows by taking $\pi(C_0(X))$ to be the representation on $L^2(X)$ by multiplication operators (where L^2 is defined by a measure with support X), and (3.22) is obtained by taking $\pi(\mathfrak{B}_0(\mathcal{H}))$ to be the defining representation; see the paragraph following 2.13.1.

In Example 3.2.2.2 the C^*-algebra $C^*(\mathcal{H},\mathbb{C})$ coincides with $\mathfrak{B}(\mathcal{H})$, because every bounded operator has an adjoint. Its subalgebra $\mathfrak{B}_0(\mathcal{H})$ of compact operators has an analogue in the general setting of Hilbert C^*-modules as well.

3.3 The C^*-algebra of a Hilbert C^*-module

In preparation for the imprimitivity theorem, and also as a matter of independent interest, we introduce the analogue for Hilbert C^*-modules of the C^*-algebra $\mathfrak{B}_0(\mathcal{H})$ of compact operators on a Hilbert space. This is the C^*-algebra most canonically associated to a Hilbert C^*-module.

Definition 3.3.1 *The C^*-algebra $C_0^*(\mathcal{E},\mathfrak{B})$ of "compact" operators on a Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$ is the C^*-subalgebra of $C^*(\mathcal{E},\mathfrak{B})$ generated by the adjointable maps of the type $T^{\mathfrak{B}}_{\Psi,\Phi}$, where $\Psi, \Phi \in \mathcal{E}$, and*

$$T^{\mathfrak{B}}_{\Psi,\Phi} Z := \Psi \langle \Phi, Z \rangle_{\mathfrak{B}}. \tag{3.23}$$

We write $C_0^*(\mathcal{E},\mathfrak{B}) \rightleftharpoons \mathcal{E} \rightleftharpoons \mathfrak{B}$, and call this a **dual pair**.

The word "compact" appears between quotation marks because in general elements of $C_0^*(\mathcal{E},\mathfrak{B})$ need not be compact operators. The significance of the notation introduced at the end of the definition will emerge from Theorem 3.3.3 below. Using the (trivially proved) properties

$$(T^{\mathfrak{B}}_{\Psi,\Phi})^* = T^{\mathfrak{B}}_{\Phi,\Psi}; \tag{3.24}$$

$$AT^{\mathfrak{B}}_{\Psi,\Phi} = T^{\mathfrak{B}}_{A\Psi,\Phi}; \tag{3.25}$$

$$T^{\mathfrak{B}}_{\Psi,\Phi} A = T^{\mathfrak{B}}_{\Psi,A^*\Phi}, \tag{3.26}$$

where $A \in C^*(\mathcal{E},\mathfrak{B})$, one verifies without difficulty that $C_0^*(\mathcal{E},\mathfrak{B})$ is a (closed 2-sided) ideal in $C^*(\mathcal{E},\mathfrak{B})$, so that it is a C^*-algebra by Theorem 3.2.5. From (3.16) and (3.18) one finds the bound

$$\parallel T^{\mathfrak{B}}_{\Psi,\Phi} \parallel \leq \parallel \Psi \parallel \parallel \Phi \parallel . \tag{3.27}$$

One sees from the final part of the proof of Theorem 3.2.5 that $C_0^*(\mathcal{E},\mathfrak{B})$ acts non-degenerately on $\mathcal{E}$.

When $C_0^*(\mathcal{E},\mathfrak{B})$ has a unit it must coincide with $C^*(\mathcal{E},\mathfrak{B})$.

Proposition 3.3.2 1. *When $\mathcal{E} = \mathfrak{B} = \mathfrak{A}$ (see Example 3.2.2.1) one has*

$$C_0^*(\mathfrak{A},\mathfrak{A}) \simeq \mathfrak{A}. \tag{3.28}$$

This leads to the dual pair $\mathfrak{A} \rightleftharpoons \mathfrak{A} \rightleftharpoons \mathfrak{A}$.

2. *For $\mathcal{E} = \mathcal{H}$ and $\mathfrak{B} = \mathbb{C}$ (see Example 3.2.2.2) one obtains*

$$C_0^*(\mathcal{H},\mathbb{C}) = \mathfrak{B}_0(\mathcal{H}), \tag{3.29}$$

whence the dual pair $\mathfrak{B}_0(\mathcal{H}) \rightleftharpoons \mathcal{H} \rightleftharpoons \mathbb{C}$.

One has $T^{\mathfrak{A}}_{\Psi,\Phi} = \rho(\Psi\Phi^*)$; see (2.72). Since $\rho : \mathfrak{A} \to \mathfrak{B}(\mathfrak{A})$ is an isometric morphism, the map φ from the linear span of all $T^{\mathfrak{A}}_{\Psi,\Phi}$ to $\mathfrak{A}$, defined by linear extension of $\varphi(T^{\mathfrak{A}}_{\Psi,\Phi}) = \Psi\Phi^*$, is an isometric morphism as well. It is, in particular, injective. When $\mathfrak{A}$ has a unit it is obvious that φ is surjective; in the non-unital case the existence of an approximate unit implies that the linear span of all $\Psi\Phi^*$ is dense in $\mathfrak{A}$. Extending φ to $C^*_0(\mathfrak{A},\mathfrak{A})$ by continuity, one sees from Corollary 2.7.7 that $\varphi(C^*_0(\mathfrak{A},\mathfrak{A})) = \mathfrak{A}$.

Eq. (3.29) follows from Definition 2.13.1 and the fact that the linear span of all $T^{\mathbb{C}}_{\Psi,\Phi}$ is $\mathfrak{B}_f(\mathcal{H})$. ■

A Hilbert C^*-module $\mathcal{E}$ over $\mathfrak{B}$ is called **full** when the collection $\{\langle\Psi,\Phi\rangle_{\mathfrak{B}}\}$, where Ψ,Φ run over $\mathcal{E}$, is dense in $\mathfrak{B}$. A similar definition applies to pre-Hilbert C^*-modules.

Given a complex linear space $\mathcal{E}$, the conjugate space $\overline{\mathcal{E}}$ is equal to $\mathcal{E}$ as a real vector space, but has the conjugate action of complex scalars.

Theorem 3.3.3 *Let $\mathcal{E}$ be a full Hilbert $\mathfrak{B}$-module. The expression*

$$\langle\Psi,\Phi\rangle_{C^*_0(\mathcal{E},\mathfrak{B})} := T^{\mathfrak{B}}_{\Psi,\Phi} \tag{3.30}$$

in combination with the right-action $\pi_{\mathrm{R}}(A)\Psi := A^*\Psi$, where $A \in C^*_0(\mathcal{E},\mathfrak{B})$, defines $\overline{\mathcal{E}}$ as a full Hilbert C^*-module over $C^*_0(\mathcal{E},\mathfrak{B})$. In other words, from $\mathcal{E} \rightleftharpoons \mathfrak{B}$ one obtains $\overline{\mathcal{E}} \rightleftharpoons C^*_0(\mathcal{E},\mathfrak{B})$. The left-action $\pi_{\mathrm{L}}(B)\Psi := \Psi B^*$ of $\mathfrak{B}$ on $\overline{\mathcal{E}}$ implements the isomorphism

$$C^*_0(\overline{\mathcal{E}}, C^*_0(\mathcal{E},\mathfrak{B})) \simeq \mathfrak{B}. \tag{3.31}$$

We call $\mathfrak{A} := C^*_0(\mathcal{E},\mathfrak{B})$; in the references to (3.9) etc. below one should substitute $\mathfrak{A}$ for $\mathfrak{B}$ when appropriate. The properties (3.9), (3.10), and (3.11) follow from (3.24), (3.26), and Lemma 3.5.2, respectively.

To prove (3.12), we use (3.30) with $\Phi = \Psi$, (3.23) with $Z = \Psi$, (3.10), (3.14), and (3.13) to show that $\langle\Psi,\Psi\rangle_{\mathfrak{A}} = 0$ implies $\| \langle\Psi,\Psi\rangle^3_{\mathfrak{B}} \| = 0$. Since $\langle\Psi,\Psi\rangle_{\mathfrak{B}}$ is positive by (3.11), this implies $\langle\Psi,\Psi\rangle_{\mathfrak{B}} = 0$, hence $\Psi = 0$ by (3.12).

It follows from (3.14) and (3.26) that each $\pi_{\mathrm{L}}(B)$ is adjointable with respect to $\langle\, ,\, \rangle_{\mathfrak{A}}$. Moreover, applying (3.13), (3.30), (3.27), and (3.16) one finds that $\pi_{\mathrm{L}}(B)$ is a bounded operator on $\overline{\mathcal{E}}$ with respect to $\| \cdot \|_{\mathfrak{A}}$, whose norm is majorized by the norm of B in $\mathfrak{B}$. The map π_{L} is injective because $\mathcal{E}$ is non-degenerate as a right-$\mathfrak{B}$-module.

Let $\overline{\mathcal{E}}_c$ be the completion of $\overline{\mathcal{E}}$ in $\| \cdot \|_{\mathfrak{A}}$; we will shortly prove that $\overline{\mathcal{E}}_c = \overline{\mathcal{E}}$. It follows from the previous paragraph that $\pi_{\mathrm{L}}(B)$ extends to an operator on $\overline{\mathcal{E}}_c$ (denoted by the same symbol), and that π_{L} maps $\mathfrak{B}$ into $C^*(\overline{\mathcal{E}}_c,\mathfrak{A})$. It is trivial from its definition that π_{L} is a morphism. Now observe that

$$\pi_{\mathrm{L}}(\langle\Psi,\Phi\rangle_{\mathfrak{B}}) = T^{\mathfrak{A}}_{\Psi,\Phi}, \tag{3.32}$$

for the definitions in question imply that

$$T^{\mathfrak{A}}_{\Psi,\Phi}Z = \Psi\langle\Phi,Z\rangle_{\mathfrak{A}} = T^{\mathfrak{B}}_{Z,\Phi}\Psi = Z\langle\Phi,\Psi\rangle_{\mathfrak{B}}. \tag{3.33}$$

The fullness of $\mathcal{E} \rightleftharpoons \mathfrak{B}$ and the definition of $C^*_0(\overline{\mathcal{E}}_c,\mathfrak{A})$ imply that $\pi_{\mathrm{L}} : \mathfrak{B} \to C^*_0(\overline{\mathcal{E}}_c,\mathfrak{A})$ is an isomorphism. In particular, it is norm-preserving by Lemma 2.7.6.

The space $\mathcal{E}$ is equipped with two norms by applying (3.13) with $\mathfrak{B}$ or with $\mathfrak{A}$; we write $\| \cdot \|_{\mathfrak{B}}$ and $\| \cdot \|_{\mathfrak{A}}$. From (3.30) and (3.27) one derives

$$\| \Psi \|_{\mathfrak{A}} \leq \| \Psi \|_{\mathfrak{B}} . \tag{3.34}$$

For $\Psi \in \mathcal{E}$ we now use (3.13), the isometric nature of π_{L}, and (3.32) to find that

$$\| \Psi \|_{\mathfrak{B}} = \| T^{\mathfrak{A}}_{\Psi,\Psi} \|^{\frac{1}{2}} . \tag{3.35}$$

From (3.27) with $\mathfrak{B} \to \mathfrak{A}$ one then derives the converse inequality to (3.34), so that $\| \Psi \|_{\mathfrak{A}} = \| \Psi \|_{\mathfrak{B}}$. Hence $\overline{\mathcal{E}}_c = \overline{\mathcal{E}}$, as $\mathcal{E}$ is complete in $\| \cdot \|_{\mathfrak{B}}$ by assumption. The completeness of $\mathcal{E}$ as a Hilbert $\mathfrak{B}$-module is equivalent to the completeness of $\overline{\mathcal{E}}$ as a Hilbert $\mathfrak{A}$-module.

We have now proved (3.31). Finally noticing that as a Hilbert C^*-module over $\mathfrak{A}$ the space $\overline{\mathcal{E}}$ is full by definition of $C_0^*(\mathcal{E}, \mathfrak{B})$, the proof of Theorem 3.3.3 is ready. ■

For later reference we record the remarkable identity

$$\langle \Psi, \Phi \rangle_{C_0^*(\mathcal{E},\mathfrak{B})} Z = \Psi \langle \Phi, Z \rangle_{\mathfrak{B}}, \tag{3.36}$$

which is a restatement of (3.33).

3.4 Morita equivalence

The imprimitivity theorem establishes an isomorphism between the respective representation theories of two C^*-algebras that stand in a certain equivalence relation to each other.

Definition 3.4.1 *Two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are* **Morita-equivalent** *when there exists a full Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$ under which $\mathfrak{A} \simeq C_0^*(\mathcal{E}, \mathfrak{B})$. We write $\mathfrak{A} \overset{M}{\sim} \mathfrak{B}$ and $\mathfrak{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathfrak{B}$.*

Proposition 3.4.2 *Morita equivalence is an equivalence relation in the class of all C^*-algebras.*

The reflexivity property $\mathfrak{B} \overset{M}{\sim} \mathfrak{B}$ follows from (3.28), which establishes the dual pair $\mathfrak{B} \rightleftharpoons \mathfrak{B} \rightleftharpoons \mathfrak{B}$. Symmetry is implied by (3.31), proving that $\mathfrak{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathfrak{B}$ implies $\mathfrak{B} \rightleftharpoons \overline{\mathcal{E}} \rightleftharpoons \mathfrak{A}$.

The proof of transitivity is more involved. When $\mathfrak{A} \overset{M}{\sim} \mathfrak{B}$ and $\mathfrak{B} \overset{M}{\sim} \mathfrak{C}$ we have the chain of dual pairs

$$\mathfrak{A} \rightleftharpoons \mathcal{E}_1 \rightleftharpoons \mathfrak{B} \rightleftharpoons \mathcal{E}_2 \rightleftharpoons \mathfrak{C}.$$

We then form the linear space $\mathcal{E}_1 \otimes_{\mathfrak{B}} \mathcal{E}_2$ (which is the quotient of $\mathcal{E}_1 \otimes \mathcal{E}_2$ by the ideal $\mathcal{I}_{\mathfrak{B}}$ generated by all vectors of the form $\Psi_1 B \otimes \Psi_2 - \Psi_1 \otimes B\Psi_2$), which carries a right-action $\pi_{\mathrm{R}}^{\otimes}(\mathfrak{C})$ given by

$$\pi_{\mathrm{R}}^{\otimes}(C)(\Psi_1 \otimes_{\mathfrak{B}} \Psi_2) := \Psi_1 \otimes_{\mathfrak{B}} (\Psi_2 C). \tag{3.37}$$

Moreover, we can define a sesquilinear map $\langle \, , \, \rangle_{\mathfrak{C}}^{\otimes}$ on $\mathcal{E}_1 \otimes_{\mathfrak{B}} \mathcal{E}_2$ by

$$\langle \Psi_1 \otimes_{\mathfrak{B}} \Psi_2, \Phi_1 \otimes_{\mathfrak{B}} \Phi_2 \rangle_{\mathfrak{C}}^{\otimes} := \langle \Psi_2, \langle \Psi_1, \Phi_1 \rangle_{\mathfrak{B}} \Phi_2 \rangle_{\mathfrak{C}}. \tag{3.38}$$

With (3.37) this satisfies (3.9) and (3.10); as explained prior to (3.14), one may therefore construct a Hilbert C^*-module, denoted by $\mathcal{E}_{\otimes} \rightleftharpoons \mathfrak{C}$. (Remarkably, if one looks at (3.38) as defined on $\mathcal{E}_1 \otimes \mathcal{E}_2$, the null space of (3.13) is easily seen to contain $\mathcal{I}_{\mathfrak{B}}$, but in fact coincides with it, so that in constructing $\mathcal{E}_{\otimes}$ one only needs to complete $\mathcal{E}_1 \otimes_{\mathfrak{B}} \mathcal{E}_2$.)

Apart from the right-action $\pi_{\mathrm{R}}^{\otimes}(\mathfrak{C})$, the space $\mathcal{E}_{\otimes}$ carries a left-action $\pi_L^{\otimes}(\mathfrak{A})$: the operator

$$\pi_{\mathrm{L}}^{\otimes}(A)(\Psi_1 \otimes_{\mathfrak{B}} \Psi_2) := (A\Psi_1) \otimes_{\mathfrak{B}} \Psi_2 \tag{3.39}$$

is bounded on $\mathcal{E}_1 \otimes_{\mathfrak{B}} \mathcal{E}_2$ and extends to $\mathcal{E}_{\otimes}$. We now claim that

$$C_0^*(\mathcal{E}_{\otimes}, \mathfrak{C}) = \pi_L^{\otimes}(\mathfrak{A}). \tag{3.40}$$

Using (3.23), the definition of $\otimes_{\mathfrak{B}}$, and (3.10), it is easily shown that

$$\pi_{\mathrm{L}}^{\otimes}(T^{\mathfrak{B}}_{\Psi_1 \langle \Psi_2, \Phi_2 \rangle_{\mathfrak{B}}, \Phi_1}) \Omega_1 \otimes_{\mathfrak{B}} \Omega_2 = \Psi_1 \otimes_{\mathfrak{B}} \langle \Psi_2, \Phi_2 \langle \Phi_1, \Omega_1 \rangle_{\mathfrak{B}} \rangle_{\mathfrak{B}} \Omega_2. \tag{3.41}$$

Now use the assumption $C_0^*(\mathcal{E}_2, \mathfrak{C}) = \mathfrak{B}$; as in (3.30), with $\mathfrak{B} \to \mathfrak{C}$, and $\mathcal{E} \to \mathcal{E}_2$, this yields $\langle \Psi, \Phi \rangle_{\mathfrak{B}} = T^{\mathfrak{C}}_{\Psi, \Phi}$. Substituting this in the right-hand side of (3.41), and using (3.23) with $\mathfrak{B} \to \mathfrak{C}$, the right-hand side of (3.41) becomes $\Psi_1 \otimes_{\mathfrak{B}} \Psi_2 \langle \Phi_2 \langle \Phi_1, \Omega_1 \rangle_{\mathfrak{B}}, \Omega_2 \rangle_{\mathfrak{C}}$. Using $\Psi B^* = \pi_L(B)\Psi$ (see 3.3.3), (3.19) with $\mathfrak{B} \to \mathfrak{C}$, (3.38), and (3.23) with $\mathfrak{B} \to \mathfrak{C}$, we eventually obtain

$$T^{\mathfrak{C}}_{\Psi_1 \otimes_{\mathfrak{B}} \Psi_2, \Phi_1 \otimes_{\mathfrak{B}} \Phi_2} = \pi_{\mathrm{L}}^{\otimes}(T^{\mathfrak{B}}_{\Psi_1 \langle \Psi_2, \Phi_2 \rangle_{\mathfrak{B}}, \Phi_1}). \tag{3.42}$$

This leads to the inclusion $C_0^*(\mathcal{E}_{\otimes}, \mathfrak{C}) \subseteq \pi_L^{\otimes}(\mathfrak{A})$. To prove the opposite inclusion, one picks a double sequence $\{\Psi_2^i, \Phi_2^i\}$ such that $\sum_i^N T^{\mathfrak{C}}_{\Psi_2^i, \Phi_2^i}$ is an approximate unit in $\mathfrak{B} = C_0^*(\mathcal{E}_2, \mathfrak{C})$. One has $\lim_N \sum_i^N \Psi_2^i\langle\Phi_2^i, Z\rangle_{\mathfrak{C}} = Z$ from (3.23), and a short computation using (3.23) with (3.38) then yields

$$\lim_N \sum_i^N T^{\mathfrak{C}}_{\Psi_1\otimes_{\mathfrak{B}}\Psi_2^i,\Phi_1\otimes_{\mathfrak{B}}\Phi_2^i} = \pi_{\mathrm{L}}^{\otimes}(T^{\mathfrak{B}}_{\Psi_1,\Phi_1}).$$

Hence $\pi_L^{\otimes}(\mathfrak{A}) \subseteq C_0^*(\mathcal{E}_{\otimes}, \mathfrak{C})$, and combining both inclusions one finds (3.42).

Therefore, one has the dual pair $\mathfrak{A} \rightleftharpoons \mathcal{E}_{\otimes} \rightleftharpoons \mathfrak{C}$, implying that $\mathfrak{A} \overset{M}{\sim} \mathfrak{C}$. This proves transitivity. ■

Here is a simple example of this concept.

Proposition 3.4.3 *The C^*-algebra $\mathfrak{B}_0(\mathcal{H})$ of compact operators is Morita-equivalent to $\mathbb{C}$, with dual pair $\mathfrak{B}_0(\mathcal{H}) \rightleftharpoons \mathcal{H} \rightleftharpoons \mathbb{C}$. In particular, the matrix algebra $\mathfrak{M}^n(\mathbb{C})$ is Morita-equivalent to $\mathbb{C}$.*

This is immediate from (3.29). In the finite-dimensional case one has $\mathfrak{M}^n(\mathbb{C}) \rightleftharpoons \mathbb{C}^n \rightleftharpoons \mathbb{C}$, where $\mathfrak{M}^n(\mathbb{C})$ and $\mathbb{C}$ act on $\mathbb{C}^n$ in the usual way. The double Hilbert C^*-module structure is completed by specifying

$$\begin{aligned} \langle z, w\rangle_{\mathbb{C}} &= \overline{z}^i w^i; \\ (\langle z, w\rangle_{\mathfrak{M}^n(\mathbb{C})})_{ij} &= z^i\overline{w}^j, \end{aligned} \tag{3.43}$$

from which one easily verifies (3.36). ■

Since $\mathfrak{M}^n(\mathbb{C}) \overset{M}{\sim} \mathbb{C}$ and $\mathbb{C} \overset{M}{\sim} \mathfrak{M}^m(\mathbb{C})$, one has $\mathfrak{M}^n(\mathbb{C}) \overset{M}{\sim} \mathfrak{M}^m(\mathbb{C})$. This equivalence is implemented by the dual pair $\mathfrak{M}^n(\mathbb{C}) \rightleftharpoons \mathfrak{M}^{n\times m}(\mathbb{C}) \rightleftharpoons \mathfrak{M}^m(\mathbb{C})$, where $\mathfrak{M}^{n\times m}(\mathbb{C})$ is the space of complex matrices with n rows and m columns. We leave the details as an exercise.

In practice the following way to construct dual pairs, and therefore Morita equivalences, is useful.

Proposition 3.4.4 *Suppose one has*

- *two pre-C^*-algebras $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$;*
- *a full pre-Hilbert $\tilde{\mathfrak{B}}$-module $\tilde{\mathcal{E}}$;*
- *a left-action of $\tilde{\mathfrak{A}}$ on $\tilde{\mathcal{E}}$, such that $\overline{\tilde{\mathcal{E}}}$ can be made into a full pre-Hilbert $\tilde{\mathfrak{A}}$-module with respect to the right-action $\pi_{\mathrm{R}}(A)\Psi := A^*\Psi$;*
- *the identity*
$$\langle\Psi, \Phi\rangle_{\tilde{\mathfrak{A}}} Z = \Psi\langle\Phi, Z\rangle_{\tilde{\mathfrak{B}}} \tag{3.44}$$
(for all $\Psi, \Phi, Z \in \tilde{\mathcal{E}}$) relating the two Hilbert C^*-module structures;
- *the bounds*
$$\langle\Psi B, \Psi B\rangle_{\tilde{\mathfrak{A}}} \leq \| B \|^2 \langle\Psi, \Psi\rangle_{\tilde{\mathfrak{A}}}; \tag{3.45}$$
$$\langle A\Psi, A\Psi\rangle_{\tilde{\mathfrak{B}}} \leq \| A \|^2 \langle\Psi, \Psi\rangle_{\tilde{\mathfrak{B}}} \tag{3.46}$$
for all $A \in \tilde{\mathfrak{A}}$ and $B \in \tilde{\mathfrak{B}}$.

Then $\mathfrak{A} \overset{M}{\sim} \mathfrak{B}$, with dual pair $\mathfrak{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathfrak{B}$, where $\mathcal{E}$ is the completion of $\tilde{\mathcal{E}}$ as a Hilbert $\mathfrak{B}$-module.

Using Corollary 3.2.4 we first complete $\tilde{\mathcal{E}}$ to a Hilbert $\mathfrak{B}$-module $\mathcal{E}$. By (3.46), which implies $\| A\Psi \| \leq \| A \| \| \Psi \|$ for all $A \in \tilde{\mathfrak{A}}$ and $\Psi \in \tilde{\mathcal{E}}$, the action of $\tilde{\mathfrak{A}}$ on $\tilde{\mathcal{E}}$ extends to an action of $\mathfrak{A}$ on $\mathcal{E}$. Similarly, we complete $\overline{\tilde{\mathcal{E}}}$ to a Hilbert $\mathfrak{A}$-module $\overline{\mathcal{E}}_c$; by (3.45) the left-action $\pi_{\mathrm{L}}(B)\Psi := \Psi B^*$ extends to an action of $\mathfrak{B}$ on $\overline{\mathcal{E}}_c$. As in the proof of Theorem 3.3.3, one derives (3.34) and its converse for $\Psi \in \tilde{\mathcal{E}}$, so that the $\mathfrak{B}$-completion $\mathcal{E}$ of $\tilde{\mathcal{E}}$ coincides with the $\mathfrak{A}$-completion $\overline{\mathcal{E}}_c$ of $\overline{\tilde{\mathcal{E}}}$; that is, $\overline{\mathcal{E}}_c = \overline{\mathcal{E}}$.

Since $\overline{\mathcal{E}}$ is a full pre-Hilbert $\tilde{\mathfrak{A}}$-module, the $\mathfrak{A}$-action on $\mathcal{E}$ is injective, hence faithful. It follows from (3.44), Theorem 3.3.3, and (once again) the fullness of $\overline{\mathcal{E}}$, that $\mathfrak{A} \simeq C_0^*(\mathcal{E}, \mathfrak{B})$. In particular, each $A \in \mathfrak{A}$ automatically satisfies (3.19). ■

Clearly, (3.44) is inspired by (3.36), into which it is turned after use of this proposition. We will repeatedly use Proposition 3.4.4 in what follows; see 3.9.3 and 3.10.1.

3.5 Rieffel induction

To formulate and prove the imprimitivity theorem we need a basic technique, which is of interest also in a more general context. Given a Hilbert $\mathfrak{B}$-module $\mathcal{E}$, the goal of the **Rieffel induction** procedure described in this section is to construct a representation π^χ of $C^*(\mathcal{E}, \mathfrak{B})$ from a representation π_χ of $\mathfrak{B}$. In order to explicate that the induction procedure is a generalization of the GNS-construction 2.9.4, we first induce from a state ω_χ on $\mathfrak{B}$, rather than from a representation π_χ.

Construction 3.5.1 *Suppose one has a Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$.*

1. *Given a state ω_χ on $\mathfrak{B}$, define the sesquilinear form $\widetilde{(\,,\,)}_0^\chi$ on $\mathcal{E}$ by*
$$\widetilde{(\Psi, \Phi)}_0^\chi := \omega_\chi(\langle \Psi, \Phi \rangle_{\mathfrak{B}}). \tag{3.47}$$
Since ω_χ and $\langle\,,\,\rangle_{\mathfrak{B}}$ are positive (cf. (3.11)), this form is positive semi-definite. Its null space is
$$\tilde{\mathcal{N}}_\chi = \{\Psi \in \mathcal{E} \mid \widetilde{(\Psi, \Psi)}_0^\chi = 0\}. \tag{3.48}$$
2. *The form $\widetilde{(\,,\,)}_0^\chi$ projects to an inner product $\widetilde{(\,,\,)}^\chi$ on the quotient $\mathcal{E}/\tilde{\mathcal{N}}_\chi$. If $\tilde{V}_\chi : \mathcal{E} \to \mathcal{E}/\tilde{\mathcal{N}}_\chi$ is the canonical projection, then by definition*
$$\widetilde{(\tilde{V}_\chi \Psi, \tilde{V}_\chi \Phi)}^\chi := \widetilde{(\Psi, \Phi)}_0^\chi. \tag{3.49}$$
The Hilbert space $\tilde{\mathcal{H}}^\chi$ is the closure of $\mathcal{E}/\tilde{\mathcal{N}}_\chi$ in this inner product.
3. *The representation $\tilde{\pi}^\chi(C^*(\mathcal{E}, \mathfrak{B}))$ is firstly defined on $\mathcal{E}/\tilde{\mathcal{N}}_\chi \subset \tilde{\mathcal{H}}^\chi$ by*
$$\pi^\chi(A)\tilde{V}_\chi \Psi := \tilde{V}_\chi A\Psi; \tag{3.50}$$
it follows that $\tilde{\pi}^\chi$ is continuous. Since $\mathcal{E}/\tilde{\mathcal{N}}_\chi$ is dense in $\tilde{\mathcal{H}}^\chi$, the operator $\tilde{\pi}^\chi(A)$ may be defined on all of $\tilde{\mathcal{H}}^\chi$ by continuous extension of (3.50), where it satisfies (2.9.1).

The GNS-construction 2.9.4 is a special case of 3.5.1, obtained by choosing $\mathcal{E} = \mathfrak{B} = \mathfrak{A}$, as explained in Example 3.2.2.1.

The analogue of (2.117) of course applies here. The continuity of $\tilde{\pi}^\chi$ follows from (3.50) and (3.49), which imply that $\| \tilde{\pi}^\chi(A)\tilde{V}_\chi \Psi \|^2 = \widetilde{(A\Psi, A\Psi)}_0^\chi$. Using (3.47), (3.20), and (3.18) in succession, one finds that
$$\| \tilde{\pi}^\chi(A) \| \leq \| A \| . \tag{3.51}$$
On the other hand, from the proof of Theorem 2.10.1 one sees that
$$\| A \|^2 = \sup\{|\omega(A^*A)| \mid \omega \in \mathcal{S}(\mathfrak{A})\}. \tag{3.52}$$
Applying (3.52) to $\mathfrak{B}$, used with the definition of $\| A \|$ for $A \in C^*(\mathcal{E}, \mathfrak{B})$, implies that
$$\| A \| = \sup\{\| \tilde{\pi}^\chi(A) \|, \omega_\chi \in \mathcal{S}(\mathfrak{B})\}. \tag{3.53}$$
A similar argument combined with Corollary 2.5.3 shows that $\tilde{\pi}^\chi$ is faithful (hence norm-preserving) when ω_χ is. As a corollary, one infers a useful property, which will be used, e.g., in the proof of Theorem 3.3.3.

Lemma 3.5.2 *Let $A \in C^*(\mathcal{E}, \mathfrak{B})$ satisfy $\langle \Psi, A\Psi \rangle_{\mathfrak{B}} \geq 0$ for all $\Psi \in \mathcal{E}$. Then $A \geq 0$.*

Take a faithful state ω_χ on $\mathfrak{B}$; the condition implies that $\tilde{\pi}^\chi(A) \geq 0$. ■

When one starts from a representation $\pi_\chi(\mathfrak{B})$ rather than from a state, the general construction looks as follows.

Construction 3.5.3 *Start from a Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$.*

1. *Given a representation $\pi_\chi(\mathfrak{B})$ on a Hilbert space $\mathcal{H}_\chi$, with inner product $(\,,\,)_\chi$, the sesquilinear form $(\,,\,)_0^\chi$ is defined on $\mathcal{E} \otimes \mathcal{H}_\chi$ (algebraic tensor product) by sesquilinear extension of*
$$(\Psi \otimes v, \Phi \otimes w)_0^\chi := (v, \pi_\chi(\langle \Psi, \Phi \rangle_{\mathfrak{B}})w)_\chi, \tag{3.54}$$
where $v, w \in \mathcal{H}_\chi$. This form is positive semi-definite, because $(\,,\,)_\chi$ and $\langle\,,\rangle_{\mathfrak{B}}$ are. The null space is
$$\mathcal{N}_\chi = \{\tilde{\Psi} \in \mathcal{E} \otimes \mathcal{H}_\chi | \, (\tilde{\Psi}, \tilde{\Psi})_0^\chi = 0\}. \tag{3.55}$$
As in (2.117), we may equally well write
$$\mathcal{N}_\chi = \{\tilde{\Psi} \in \mathcal{E} \otimes \mathcal{H}_\chi | \, (\tilde{\Psi}, \tilde{\Phi})_0^\chi = 0 \, \forall \, \tilde{\Phi} \in \mathcal{E} \otimes \mathcal{H}_\chi\}. \tag{3.56}$$
2. *The form $(\,,\,)_0^\chi$ projects to an inner product $(\,,\,)^\chi$ on the quotient $\mathcal{E} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$, defined by*
$$(V_\chi \tilde{\Psi}, V_\chi \tilde{\Phi})^\chi := (\tilde{\Psi}, \tilde{\Phi})_0^\chi, \tag{3.57}$$
where $V_\chi : \mathcal{E} \otimes \mathcal{H}_\chi \to \mathcal{E} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$ is the canonical projection. The Hilbert space $\mathcal{H}^\chi$ is the closure of $\mathcal{E} \otimes \mathcal{H}_\chi / \mathcal{N}_\chi$ in this inner product.
3. *The representation $\pi^\chi(C^*(\mathcal{E}, \mathfrak{B}))$ is then defined on $\mathcal{H}^\chi$ by continuous extension of*
$$\pi^\chi(A) V_\chi \tilde{\Psi} := V_\chi(A \otimes \mathbb{I}_\chi \tilde{\Psi}), \tag{3.58}$$
where $\mathbb{I}_\chi$ is the unit operator on $\mathcal{H}_\chi$; the extension in question is possible, since
$$\| \, \pi^\chi(A) \, \| \leq \| \, A \, \| \, . \tag{3.59}$$

To prove that the form defined in (3.54) is positive semi-definite, we assume that $\pi_\chi(\mathfrak{B})$ is cyclic (if not, the argument below is repeated for each cyclic summand; see 2.9.2). With $\tilde{\Psi} = \sum_i \Psi_i v_i$ and $v_i = \pi_\chi(B_i)\Omega$ (where Ω is a cyclic vector for $\pi_\chi(\mathfrak{B})$), one then uses (3.54), (3.14), and (3.10) to find $(\tilde{\Psi}, \tilde{\Psi})_0^\chi = (v, \pi_\chi(\langle \Phi, \Phi \rangle_{\mathfrak{B}})v)_\chi$ with $\Phi := \sum_i \Psi_i B_i$. Hence $(\tilde{\Psi}, \tilde{\Psi})_0^\chi \geq 0$ by (3.10) and the positivity of $\pi_\chi : \mathfrak{B} \to \mathfrak{B}(\mathcal{H}_\chi)$.

Similarly, one computes $\| \, \pi^\chi(A) V_\chi \tilde{\Psi} \, \|^2 = (v, \pi_\chi(\langle A\Phi, A\Phi \rangle_{\mathfrak{B}})v)_\chi$ from (3.57) and (3.58); according to (3.20) and the property $\| \, \pi_\chi(A) \, \| \leq \| \, A \, \|$ (cf. the text after 2.9.1), this is bounded by $\| \, A \, \|^2 \, (v, \pi_\chi(\langle \Phi, \Phi \rangle_{\mathfrak{B}})v)_\chi$. Since the second factor equals $\| \, V_\chi \tilde{\Psi} \, \|^2$, this proves (3.59). ■

Paraphrasing the comment after the first version of the construction, π^χ is faithful when π_χ is. Also, it is not difficult to verify that π^χ is non-degenerate when π_χ is.

To interrelate the above two formulations, one assumes that π_χ is cyclic, with cyclic vector Ω_χ. Then define a linear map $\tilde{U} : \mathcal{E} \to \mathcal{E} \otimes \mathcal{H}_\chi$ by
$$\tilde{U}\Psi := \Psi \otimes \Omega_\chi. \tag{3.60}$$
According to (3.47), (3.54), and (2.116), this map has the property
$$(\tilde{U}\Psi, \tilde{U}\Phi)_0^\chi = \widetilde{(\tilde{\Psi}, \tilde{\Phi})}_0^\chi. \tag{3.61}$$
By (3.49) and (3.57) the map $\tilde{U}$ therefore quotients to a unitary isomorphism $U : \tilde{\mathcal{H}}^\chi \to \mathcal{H}^\chi$, which by (3.50) and (3.58) duly intertwines $\tilde{\pi}^\chi$ and π^χ.

Of course, any subspace of $C^*(\mathcal{E}, \mathfrak{B})$ may be subjected to the induced representation π^χ. This particularly applies when one has a given (pre-) C^*-algebra $\mathfrak{A}$ and a *-homomorphism $\pi : \mathfrak{A} \to C^*(\mathcal{E}, \mathfrak{B})$, leading to the representation $\pi^\chi(\mathfrak{A})$ on $\mathcal{H}^\chi$. Further to an earlier comment, one verifies that π^χ is non-degenerate when π and π_χ are. With slight abuse of notation we will write $\pi^\chi(A)$ for $\pi^\chi(\pi(A))$. The situation is depicted in Figure 1.

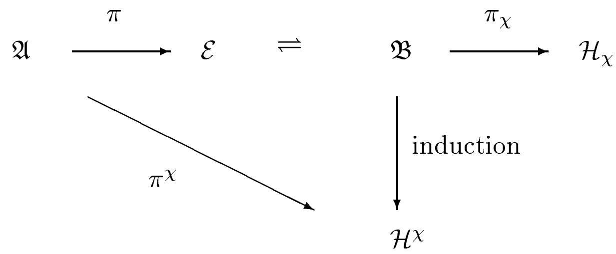

Figure 1: Rieffel induction

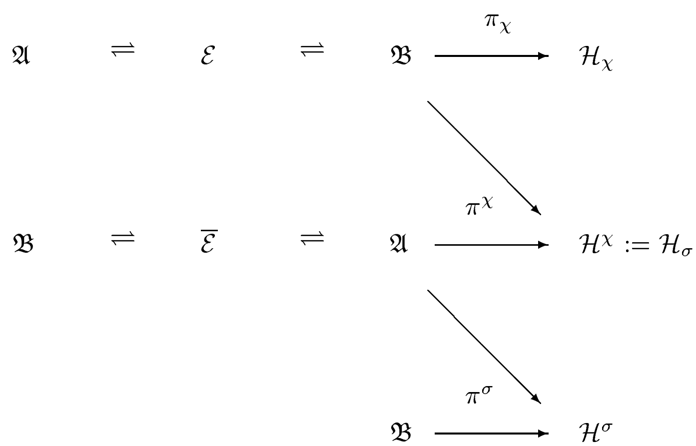

Figure 2: Quantum imprimitivity theorem: $\mathcal{H}^{\sigma} \simeq \mathcal{H}_{\chi}$ and $\pi^{\sigma} \simeq \pi_{\chi}$

3.6 The imprimitivity theorem

After this preparation, we pass to the **imprimitivity theorem**.

Theorem 3.6.1 *There is a bijective correspondence between the non-degenerate representations of Morita-equivalent C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, preserving direct sums and irreducibility. This correspondence is as follows.*

Let the pertinent dual pair be $\mathfrak{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathfrak{B}$. When $\pi_{\sigma}(\mathfrak{A})$ is a representation on a Hilbert space $\mathcal{H}_{\sigma}$ there exists a representation $\pi_{\chi}(\mathfrak{B})$ on a Hilbert space $\mathcal{H}_{\chi}$ such that π_{σ} is equivalent to the Rieffel-induced representation π^{χ} defined by (3.58) and the above dual pair.

In the opposite direction, a given representation $\pi_{\chi}(\mathfrak{B})$ is equivalent to the Rieffel-induced representation π^{σ}, defined with respect to some representation $\pi_{\sigma}(\mathfrak{A})$ and the dual pair $\mathfrak{B} \rightleftharpoons \overline{\mathcal{E}} \rightleftharpoons \mathfrak{A}$.

Taking $\pi_{\sigma}(\mathfrak{A}) = \pi^{\chi}(\mathfrak{A})$ as just defined, one has $\pi^{\sigma}(\mathfrak{B}) \simeq \pi_{\chi}(\mathfrak{B})$. Conversely, taking $\pi_{\chi}(\mathfrak{B}) = \pi^{\sigma}(\mathfrak{B})$, one has $\pi^{\chi}(\mathfrak{A}) \simeq \pi_{\sigma}(\mathfrak{A})$.

See Figure 2. Starting with $\pi_{\chi}(\mathfrak{B})$, we construct $\pi^{\chi}(\mathfrak{A})$ with Rieffel induction from the dual pair $\mathfrak{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathfrak{B}$, relabel this representation as $\pi_{\sigma}(\mathfrak{A})$, and move on to construct $\pi^{\sigma}(\mathfrak{B})$ from Rieffel induction with respect to the dual pair $\mathfrak{B} \rightleftharpoons \overline{\mathcal{E}} \rightleftharpoons \mathfrak{A}$. We then construct a unitary map $U : \mathcal{H}^{\sigma} \rightarrow \mathcal{H}_{\chi}$ which intertwines π^{σ} and π_{χ}.

We first define $\tilde{U} : \overline{\mathcal{E}} \otimes \mathcal{E} \otimes \mathcal{H}_{\chi} \rightarrow \mathcal{H}_{\chi}$ by linear extension of

$$\tilde{U}\Psi \otimes \Phi \otimes v := \pi_{\chi}(\langle \Psi, \Phi \rangle_{\mathfrak{B}})v. \tag{3.62}$$

Note that $\tilde{U}$ is indeed $\mathbb{C}$-linear. Using (3.62), the properties (2.19) and (2.18) with $\varphi \to \pi_\chi$, (3.54), and (3.23), one obtains

$$(\tilde{U}\Psi_1 \otimes \Phi_1 \otimes v_1, \tilde{U}\Psi_2 \otimes \Phi_2 \otimes v_2)_\chi = (\Phi_1 \otimes v_1, T^{\mathfrak{B}}_{\Psi_1,\Psi_2}\Phi_2 \otimes v_2)^\chi_0. \tag{3.63}$$

Now use the assumption $\mathfrak{A} = C^*_0(\mathcal{E}, \mathfrak{B})$ to use (3.36), and subsequently (3.57) and (3.58), all read from right to left. The right-hand side of (3.63) is then seen to be equal to $(V_\chi\Phi_1 \otimes v_1, \pi^\chi(\langle\Psi_1, \Psi_2\rangle_{\mathfrak{A}})V_\chi\Phi_2 \otimes v_2)^\chi$. Now put $\pi^\chi = \pi_\sigma$ and $\mathcal{H}^\chi = \mathcal{H}_\sigma$, and use (3.54) and (3.57) from right to left, with $\chi \to \sigma$. This shows that

$$(\tilde{U}\Psi_1 \otimes \Phi_1 \otimes v_1, \tilde{U}\Psi_2 \otimes \Phi_2 \otimes v_2)_\chi = (V_\sigma(\Psi_1 \otimes V_\chi\Phi_1 \otimes v_1), V_\sigma(\Psi_2 \otimes V_\chi\Phi_2 \otimes v_2))^\sigma. \tag{3.64}$$

In particular, $\tilde{U}$ annihilates $\Psi \otimes \tilde{\Phi}$, where $\tilde{\Phi} \in \mathcal{E} \otimes \mathcal{H}_\chi$, whenever $\tilde{\Phi} \in \mathcal{N}_\chi$ or $\Psi \otimes V_\chi\tilde{\Phi} \in \mathcal{N}_\sigma$. Hence we see from the construction firstly of $\mathcal{H}^\chi = \mathcal{H}_\sigma$ from $\mathcal{E} \otimes \mathcal{H}_\chi$, and secondly of $\mathcal{H}^\sigma$ from $\overline{\mathcal{E}} \otimes \mathcal{H}_\sigma$ (cf. 3.5.3), that $\tilde{U}$ descends to an isometry $U : \mathcal{H}^\sigma \to \mathcal{H}_\chi$, defined by linear extension of

$$UV_\sigma(\Psi \otimes V_\chi\Phi \otimes v) := \tilde{U}\Psi \otimes \Phi \otimes v = \pi_\chi(\langle\Psi, \Phi\rangle_{\mathfrak{B}})v. \tag{3.65}$$

Using the assumptions that the Hilbert C^*-module $\mathcal{E} \rightleftharpoons \mathfrak{B}$ is full and that the representation $\pi_\chi(\mathfrak{B})$ is non-degenerate, we see that the range of $\tilde{U}$ and hence of U is dense in $\mathcal{H}_\chi$, so that U is unitary.

To verify that U intertwines π^σ and π_χ, we use (3.65) and (3.58), with $\chi \to \sigma$, to compute

$$U\pi^\sigma(B)V_\sigma(\Psi \otimes V_\chi\Phi \otimes v) = \pi_\chi(\langle\pi_L(B)\Psi, \Phi\rangle_{\mathfrak{B}})v, \tag{3.66}$$

where the left-action of $B \in \mathfrak{B}$ on $\Psi \in \overline{\mathcal{E}}$ is as defined in 3.3.3. Thus writing $\pi_L(B)\Psi = \Psi B^*$, using (3.14), (2.18) with $\varphi \to \pi_\chi$, and (3.65) from right to left, the right-hand side of (3.66) is seen to be $\pi_\chi(B)UV_\sigma(\Psi \otimes V_\chi\Phi \otimes v)$. Hence $U\pi^\sigma(B) = \pi_\chi(B)U$ for all $B \in \mathfrak{B}$.

Using the proof that the Morita equivalence relation is symmetric (see 3.4.2), one immediately sees that the construction works in the opposite direction as well.

It is easy to verify that $\pi_\chi = \pi_{\chi^1} \oplus \pi_{\chi^2}$ leads to $\pi^\chi = \pi^{\chi^1} \oplus \pi^{\chi^2}$. This also proves that the bijective correspondence $\pi_\chi(\mathfrak{B}) \leftrightarrow \pi^\chi(\mathfrak{A})$ preserves irreducibility: when π_χ is irreducible and π^χ isn't, one puts $\pi^\chi = \pi_\sigma$ as above, decomposes $\pi_\sigma = \pi_{\sigma^1} \oplus \pi_{\sigma^2}$, then decomposes the induced representation $\pi^\sigma(\mathfrak{B})$ as $\pi^\sigma = \pi^{\sigma^1} \oplus \pi^{\sigma^2}$, and thus arrives at a contradiction, since $\pi^\sigma \simeq \pi_\chi$. ∎

Combined with Proposition 3.4.3, this theorem leads to a new proof of Corollary 2.13.10.

3.7 Group C^*-algebras

In many interesting applications, and also in the theory of induced representation as originally formulated for groups by Frobenius and Mackey, the C^*-algebra $\mathfrak{B}$ featuring in the definition of a Hilbert C^*-module and in Rieffel induction is a so-called group C^*-algebra.

We start with the definition of the group algebra $C^*(G)$ of a finite group with $n(G)$ elements; one then usually writes $\mathbb{C}(G)$ instead of $C^*(G)$. As a vector space, $\mathbb{C}(G)$ consist of all complex-valued functions on G, so that $\mathbb{C}(G) = \mathbb{C}^{n(G)}$. This is made into a *-algebra by the **convolution**

$$f * g(x) := \sum_{y,z \in G | yz = x} f(y)g(z) \tag{3.67}$$

and the involution

$$f^*(x) := \overline{f(x^{-1})}. \tag{3.68}$$

It is easy to check that the multiplication $*$ is associative as a consequence of the associativity of the product in G. In similar vein, the operation defined by (3.68) is an involution because of the properties $(x^{-1})^{-1} = x$ and $(xy)^{-1} = y^{-1}x^{-1}$ at the group level.

A representation π of $\mathbb{C}(G)$ on a Hilbert space $\mathcal{H}$ is defined as a morphism $\pi : \mathbb{C}(G) \to \mathfrak{B}(\mathcal{H})$.

Proposition 3.7.1 *There is a bijective correspondence between non-degenerate representations π of the *-algebra $\mathbb{C}(G)$ and unitary representations U of G, which preserves unitary equivalence and direct sums (and therefore preserves irreducibility). This correspondence is given in one direction by*

$$\pi(f) := \sum_{x\in G} f(x)U(x), \tag{3.69}$$

and in the other by

$$U(x) := \pi(\delta_x), \tag{3.70}$$

where $\delta_x(y) := \delta(xy)$.

It is elementary to verify that π is indeed a representation of $\mathbb{C}(G)$ when U is a unitary representation of G, and *vice versa*. Putting $x = e$ in (3.70) yields $\pi(\delta_e) = \mathbb{I}$, so that π cannot be degenerate.

When $U_1(x) = VU_2(x)V^*$ for all $x \in G$ then evidently $\pi_1(f) = V\pi_2(f)V^*$ for all $f \in \mathbb{C}(G)$. The converse follows by choosing $f = \delta_x$. Similarly, $\pi(f) = \pi_1(f) \oplus \pi_2(f)$ for all f iff $U(x) = U_1(x) \oplus U_2(x)$ for all x. ■

We can define a C^*-norm on $\mathbb{C}(G)$ by taking any faithful representation π, and putting $\parallel f \parallel := \parallel \pi(f) \parallel$. Since $\mathbb{C}(G)$ is a finite-dimensional vector space it is complete in this norm, which therefore is independent of the choice of π by Corollary 2.5.3.

Let now G be an arbitrary locally compact group (such as a finite-dimensional Lie group). We also assume that G is **unimodular**; that is, each left Haar measure is also right-invariant. This assumption is not necessary, but simplifies most of the formulae. We denote Haar measure by dx; it is unique up to normalization. Unimodularity implies that the Haar measure is invariant under inversion $x \rightarrow x^{-1}$. When G is compact we choose the normalization so that $\int_G dx = 1$. The Banach space $L^1(G)$ and the Hilbert space $L^2(G)$ are defined with respect to the Haar measure.

The **convolution product** is defined, initially on $C_c(G)$, by

$$f * g(x) := \int_G dy\, f(xy^{-1})g(y); \tag{3.71}$$

it is evident that for a finite group this expression specializes to (3.67). The involution is given by (3.68). As in the finite case, one checks that these operations make $\mathbb{C}_c(G)$ a *-algebra; this time one needs the invariance of the Haar measure at various steps of the proof.

Proposition 3.7.2 *The operations (3.71) and (3.68) are continuous in the L^1-norm; one has*

$$\parallel f * g \parallel_1 \;\leq\; \parallel f \parallel_1 \parallel g \parallel_1; \tag{3.72}$$
$$\parallel f^* \parallel_1 \;=\; \parallel f \parallel_1 . \tag{3.73}$$

Hence $L^1(G)$ is a Banach *-algebra under the continuous extensions of (3.71) and (3.68) from $C_c(G)$ to $L^1(G)$.

Recall the definition of a Banach *-algebra below 2.1.10.

It is obvious from invariance of the Haar measure under $x \rightarrow x^{-1}$ that $\parallel f^* \parallel_1 = \parallel f \parallel_1$, so that the involution is certainly continuous. The proof of (3.72) is a straightforward generalization of the case $G = \mathbb{R}$; cf. (2.56). This time we have

$$\parallel f * g \parallel_1 = \int_G dx \left| \int_G dy\, f(xy^{-1})g(y)\right| \leq \int_G dy\, |g(y)| \int_G dx\, |f(xy^{-1})|$$
$$= \int_G dy\, |g(y)| \int_G dx\, |f(x)| = \parallel f \parallel_1 \parallel g \parallel_1,$$

which is (3.72). ■

In order to equip $L^1(G)$ with a C^*-norm, we construct a faithful representation on a Hilbert space.

Proposition 3.7.3 *For $f \in L^1(G)$ the operator $\pi_L(f)$ on $L^2(G)$, defined by*

$$\pi_L(f)\Psi := f * \Psi. \tag{3.74}$$

is bounded, satisfying $\| \pi_L(f) \| \leq \| f \|_1$. The linear map $\pi_L : L^1(G) \rightarrow \mathfrak{B}(L^2(G))$ is a faithful representation of $L^1(G)$, seen as a Banach *-algebra as in 3.7.2.

Introducing the **left-regular representation** U_L of G on $L^2(G)$ by

$$U_L(y)\Psi(x) := \Psi(y^{-1}x), \tag{3.75}$$

it follows that

$$\pi_L(f) = \int_G dx\, f(x)U_L(x). \tag{3.76}$$

The boundedness of $\pi_L(f)$ then follows from Lemma 3.7.7 below. One easily verifies that $\pi_L(f*g) = \pi_L(f)\pi_L(g)$ and $\pi_L(f^*) = \pi_L(f)^*$.

To prove that π_L is faithful, we first show that $L^1(G)$ possesses the analogue of an approximate unit (see 2.7.1 for C^*-algebras). When G is finite, the delta-function δ_e is a unit in $\mathbb{C}(G)$. For general locally compact groups one would like to take the Dirac δ-'function' as a unit, but this distribution is not in $L^1(G)$.

Lemma 3.7.4 *The Banach *-algebra $L^1(G)$ has an approximate unit $\mathbb{I}_\lambda$ in the sense that (2.92) - (2.94) hold for all $A \in L^1(G)$, and $\| \cdot \| = \| \cdot \|_1$.*

Pick a basis of neighbourhoods $\mathcal{N}_\lambda$ of e, so that each $\mathcal{N}_\lambda$ is invariant under $x \rightarrow x^{-1}$; this basis is partially ordered by inclusion. Take $\mathbb{I}_\lambda = N_\lambda \chi_{\mathcal{N}_\lambda}$, which is the characteristic function of $\mathcal{N}_\lambda$ times a normalization factor ensuring that $\| \mathbb{I}_\lambda \|_1 = 1$. Eq. (2.92) then holds by virtue of (3.68) and the invariance of $\mathcal{N}_\lambda$ under inversion. By construction, the inequality (2.93) holds as an equality. One has $\mathbb{I}_\lambda * f(x) = N_\lambda \int_{\mathcal{N}_\lambda} dy\, f(y^{-1}x)$ and $f * \mathbb{I}_\lambda(x) = N_\lambda \int_{\mathcal{N}_\lambda} dy\, f(xy^{-1}$. For $f \in C_c(G)$ one therefore has $\lim_\lambda \mathbb{I}_\lambda * f = f$ and $\lim_\lambda f * \mathbb{I}_\lambda = f$ pointwise (i.e., for fixed x). The Lebesgue dominated convergence theorem then leads to (2.94) for all $A \in C_c(G)$, and therefore for all $A \in L^1(G)$, since $C_c(G)$ is dense in $L^1(G)$. ■

To finish the proof of 3.7.3, we now note from (3.74) that $\pi_L(f) = 0$ implies $f * \Psi = 0$ for all $\Psi \in L^2(G)$, and hence certainly for $\Psi = \mathbb{I}_\lambda$. Hence $\| f \|_1 = 0$ by Lemma 3.7.4, so that $f = 0$ and π_L is injective. ■

Definition 3.7.5 *The* **reduced group C^*-algebra** *$C^*_r(G)$ is the smallest C^*-algebra in $\mathfrak{B}(L^2(G))$ containing $\pi_L(C_c(G))$. In other words, $C^*_r(G)$ is the closure of the latter in the norm*

$$\| f \|_r := \| \pi_L(f) \| . \tag{3.77}$$

Perhaps the simplest example of a reduced group algebra is obtained by taking $G = \mathbb{R}$.

Proposition 3.7.6 *One has the isomorphism*

$$C^*_r(\mathbb{R}) \simeq C_0(\mathbb{R}). \tag{3.78}$$

It follows from the discussion preceding 2.3.6 that the Fourier transform (2.59) maps $L^1(G)$ into a subspace of $C_0(\mathbb{R})$ which separates points on $\mathbb{R}$. It is clear that for every $p \in \mathbb{R}$ there is an $f \in L^1(\mathbb{R})$ for which $\hat{f}(p) \neq 0$. In order to apply Lemma 2.4.7, we need to verify that

$$\| f \|_r = \| \hat{f} \|_\infty . \tag{3.79}$$

Since the Fourier transform turns convolution into pointwise multiplication, the left-regular representation π_L on $L^2(\mathbb{R})$ is Fourier-transformed into the action on $L^2(\mathbb{R})$ by multiplication operators. Hence (3.78) follows from Lemma 2.4.7. ■

This example generalizes to arbitrary locally compact abelian groups. Let $\hat{G}$ be the set of all irreducible unitary representations U_γ of G; such representations are necessarily one-dimensional, so that $\hat{G}$ is nothing but the set of characters on G. The generalized Fourier transform $\hat{f}$ of $f \in L^1(G)$ is a function on $\hat{G}$, defined as

$$\hat{f}(\gamma) := \int_G dx\, f(x) U_\gamma(x). \tag{3.80}$$

By the same arguments as for $G = \mathbb{R}$, one obtains

$$C_r^*(G) \simeq C_0(\hat{G}). \tag{3.81}$$

We return to the general case, where G is not necessarily abelian. We have now found a C^*-algebra which may play the role of $\mathbb{C}(G)$ for locally compact groups. Unfortunately, the analogue of Proposition 3.7.1 only holds for a limited class of groups. Hence we need a different construction. Let us agree that here and in what follows, a unitary representation of a topological group is always meant to be continuous.

Lemma 3.7.7 *Let U be an arbitrary unitary representation of G on a Hilbert space $\mathcal{H}$. Then $\pi(f)$, defined by*

$$\pi(f) := \int_G dx\, f(x) U(x) \tag{3.82}$$

is bounded, with

$$\| \pi(f) \| \leq \| f \|_1 . \tag{3.83}$$

The integral (3.82) is most simply defined weakly, that is, by its matrix elements

$$(\Psi, \pi(f)\Phi) := \int_G dx\, f(x) (\Psi, U(x)\Phi).$$

Since U is unitary, we have $|(\Psi, \pi(f)\Psi)| \leq (F, F)_{L^2(G)}$ for all $\Psi \in \mathcal{H}$, where $F(x) := \| \Psi \| \sqrt{|f(x)|}$. The Cauchy-Schwarz inequality then leads to $|(\Psi, \pi(f)\Psi)| \leq \| f \|_1 \| \Psi \|^2$. Lemma 2.12.5 then leads to (3.83). ∎

Alternatively, one may define (3.82) as a **Bochner integral**. We explain this notion in a more general context.

Definition 3.7.8 *Let X be a measure space and let $\mathcal{B}$ be a Banach space. A function $f : X \to \mathcal{B}$ is Bochner-integrable with respect to a measure μ on X iff*

- *f is weakly measurable (that is, for each functional $\omega \in \mathcal{B}^*$ the function $x \to \omega(f(x))$ is measurable);*
- *there is a null set $X_0 \subset X$ such that $\{f(x) | x \in X \backslash X_0\}$ is separable;*
- *the function defined by $x \to \| f(x) \|$ is integrable.*

It will always be directly clear from this whether a given operator- or vector-valued integral may be read as a Bochner integral; if not, it is understood as a weak integral, in a sense always obvious from the context. The Bochner integral $\int_X d\mu(x) f(x)$ can be manipulated as if it were an ordinary (Lebesgue) integral. For example, one has

$$\| \int_X d\mu(x)\, f(x) \| \leq \int_X d\mu(x) \ \| f(x) \| . \tag{3.84}$$

Thus reading (3.82) as a Bochner integral, (3.83) is immediate from (3.84).

The following result generalizes the correspondence between U_L in (3.75) and π_L in (3.76) to arbitrary representations.

Theorem 3.7.9 *There is a bijective correspondence between non-degenerate representations π of the Banach *-algebra $L^1(G)$ which satisfy (3.83), and unitary representations U of G. This correspondence is given in one direction by (3.82), and in the other by*

$$U(x)\pi(f)\Omega := \pi(f^x)\Omega, \tag{3.85}$$

where $f^x(y) := f(x^{-1}y)$. This bijection preserves direct sums, and therefore irreducibility.

Recall from 2.9.2 that any non-degenerate representation of a C^*-algebra is a direct sum of cyclic representations; the proof also applies to $L^1(G)$. Thus Ω in (3.85) stands for a cyclic vector of a certain cyclic summand of $\mathcal{H}$, and (3.82) defines U on a dense subspace of this summand; it will be shown that U is unitary, so that it can be extended to all of $\mathcal{H}$ by continuity.

Given U, it follows from easy calculations that $\pi(f)$ in (3.82) indeed defines a representation. It is bounded by Lemma 3.7.7. The proof of non-degeneracy makes use of Lemma 3.7.4. Since π is continuous, one has $\lim_\lambda \pi(\mathbb{I}_\lambda) = \mathbb{I}$ strongly, proving that π must be non-degenerate.

To go in the opposite direction we use the approximate unit once more; it follows from (3.85) (from which the continuity of U is obvious) that $U(x)\pi(f)\Omega = \lim_\lambda \pi(\mathbb{I}_\lambda^x)\pi(f)\Omega$. Hence $U(x) = \lim_\lambda \pi(\mathbb{I}_\lambda^x)$ strongly on a dense domain. The property $U(x)U(y) = U(xy)$ then follows from (3.85) and (3.71). The unitarity of each $U(x)$ follows by direct calculation, or from the following argument. Since $\| \pi(\mathbb{I}_\lambda^x) \| \leq \| \mathbb{I}_\lambda^x \|_1 = 1$, we infer that $\| U(x) \| \leq 1$ for all x. Hence also $\| U(x^{-1}) \| \leq 1$, which is the same as $\| U(x)^{-1} \| \leq 1$ We see that $U(x)$ and $U(x)^{-1}$ are both contractions; this is only possible when $U(x)$ is unitary.

Finally, if U is reducible there is a projection E such that $[E, U(x)] = 0$ for all $x \in G$. It follows from (3.82) that $[\pi(f), E] = 0$ for all f, hence π is reducible. Conversely, if π is reducible then $[E, \pi(\mathbb{I}_\lambda^x)] = 0$ for all $x \in G$; by the previous paragraph this implies $[E, U(x)] = 0$ for all x. The final claim then follows from Schur's lemma 2.12.3.1. □

This theorem suggests looking at a different object from $C_r^*(G)$. Inspired by 2.10.2 one puts

Definition 3.7.10 *The* **group C^*-algebra** *$C^*(G)$ is the closure of the Banach *-algebra algebra $L^1(G)$ in the norm*

$$\| f \| := \| \pi_u(f) \|, \tag{3.86}$$

where π_u is the direct sum of all non-degenerate representations π of $L^1(G)$ which are bounded as in (3.83).

Equivalently, $C^*(G)$ is the closure of $L^1(G)$ in the norm

$$\| f \| := \sup_\pi \| \pi(f) \|, \tag{3.87}$$

where the sum is over all representations $\pi(L^1(G))$ of the form (3.82), in which U is an irreducible unitary representation of G, and only one representative of each equivalence class of such representations is included.

The equivalence between the two definitions follows from (2.139) and Theorem 3.7.9.

Theorem 3.7.11 *There is a bijective correspondence between non-degenerate representations π of the C^*-algebra $C^*(G)$ and unitary representations U of G, given by (continuous extension of) (3.82) and (3.85). This correspondence preserves irreducibility.*

It is obvious from (2.111) and (3.86) that for any representation $\pi(C^*(G))$ and $f \in L^1(G)$ one has

$$\| \pi(f) \| \leq \| f \| \leq \| f \|_1 . \tag{3.88}$$

Hence the restriction $\pi(L^1(G))$ satisfies (3.83), and therefore corresponds to $U(G)$ by Theorem 3.7.9. Conversely, given $U(G)$ one finds $\pi(L^1(G))$ satisfying (3.83) by 3.7.9; it then follows from (3.88) that one may extend π to a representation of $C^*(G)$ by continuity. ■

In conjunction with (3.79), the second definition of $C^*(G)$ stated in 3.7.11 implies that for abelian groups $C^*(G)$ always coincides with $C_r^*(G)$. The reason is that for $\gamma \in \hat{G}$ one has $\pi_\gamma(f) = \hat{f}(\gamma) \in \mathbb{C}$, so that the norms (3.87) and (3.79) coincide. In particular, one has

$$C^*(\mathbb{R}^n) \simeq C_0(\mathbb{R}^n). \tag{3.89}$$

For general locally compact groups, looking at 3.7.5 we see that

$$C_r^*(G) = \pi_L(C^*(G)) \simeq C^*(G)/\ker(\pi_L). \tag{3.90}$$

A Lie group group is said to be **amenable** when the equality $C_r^*(G) = C^*(G)$ holds; in other words, $\pi_L(C^*(G))$ is faithful iff G is amenable. We have just seen that all locally compact abelian groups are amenable. It follows from the Peter-Weyl theorem that all compact groups are amenable as well. However, non-compact semi-simple Lie groups are not amenable.

3.8 C^*-dynamical systems and crossed products

An **automorphism** of a C^*-algebra $\mathfrak{A}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}$. It follows from Definitions 2.2.2 and 3.8.1 that $\sigma(\alpha(A)) = \sigma(A)$ for any $A \in \mathfrak{A}$ any automorphism α; hence

$$\| \alpha(A) \| = \| A \| \tag{3.91}$$

by (2.80).

One $\mathfrak{A}$ has a unit, one has

$$\alpha(\mathbb{I}) = \mathbb{I} \tag{3.92}$$

by (2.18) and the uniqueness of the unit. When $\mathfrak{A}$ has no unit, one may extend α to an automorphism $\alpha^{\mathbb{I}}$ of the unitization $\mathfrak{A}_{\mathbb{I}}$ by

$$\alpha^{\mathbb{I}}(A + \lambda\mathbb{I}) := \alpha(A) + \lambda\mathbb{I}. \tag{3.93}$$

Definition 3.8.1 *An* **automorphic action** *α of a group G on a C^*-algebra $\mathfrak{A}$ is a group homomorphism $x \to \alpha_x$ such that each α_x is an automorphism of $\mathfrak{A}$. In other words, one has*

$$\alpha_x \circ \alpha_y(A) = \alpha_{xy}(A); \tag{3.94}$$
$$\alpha_x(AB) = \alpha_x(A)\alpha_x(A); \tag{3.95}$$
$$\alpha(A^*) = \alpha(A)^* \tag{3.96}$$

for all $x, y \in G$ and $A, B \in \mathfrak{A}$.

A C^***-dynamical system** *$(G, \mathfrak{A}, \alpha)$ consists of a locally compact group G, a C^*-algebra $\mathfrak{A}$, and an automorphic action of G on $\mathfrak{A}$ such that for each $A \in \mathfrak{A}$ the function from G to $\mathfrak{A}$, defined by $x \to \| \alpha_x(A) \|$, is continuous.*

The term 'dynamical system' comes from the example $G = \mathbb{R}$ and $\mathfrak{A} = C_0(S)$, where $\mathbb{R}$ acts on S by $t : \sigma \to \sigma(t)$, and $\alpha_t(f) : \sigma \to \sigma(t)$. Hence a general C^*-dynamical system is a non-commutative analogue of a dynamical system.

Proposition 3.8.2 *Let $(G, \mathfrak{A}, \alpha)$ be a C^*-dynamical system, and define $L^1(G, \mathfrak{A}, \alpha)$ as the space of all measurable functions $f : G \to \mathfrak{A}$ for which*

$$\| f \|_1 := \int_G dx \ \| f(x) \| \tag{3.97}$$

is finite. The operations

$$f * g(x) := \int_G dy \, f(y)\alpha_y(g(y^{-1}x)); \tag{3.98}$$
$$f^*(x) := \alpha_x(f(x^{-1})^*) \tag{3.99}$$

turn $L^1(G, \mathfrak{A}, \alpha)$ into a Banach *-algebra.

As usual, we have assumed that G is unimodular; with a slight modification one may extend these formulae to the non-unimodular case. The integral (3.98) is defined as a Bochner integral; the assumptions in Definition 3.7.8 are satisfied as a consequence of the continuity assumption in the definition of a C^*-dynamical system. To verify the properties (3.72) and (3.73) one follows the same derivation as for $L^1(G)$, using (3.84) and (3.91). The completeness of $L^1(G, \mathfrak{A}, \alpha)$ is proved as in the case $\mathfrak{A} = \mathbb{C}$, for which $L^1(G, \mathfrak{A}, \alpha) = L^1(G)$. ■

In order to generalize Theorem 3.7.9, we need

Definition 3.8.3 *A* **covariant representation** *of a C^*-dynamical system $(G, \mathfrak{A}, \alpha)$ consists of a pair $(U, \tilde{\pi})$, where U is a unitary representation of G, and $\tilde{\pi}$ is a non-degenerate representation of $\mathfrak{A}$ which for all $x \in G$ and $A \in \mathfrak{A}$ satisfies*

$$U(x)\tilde{\pi}(A)U(x)^* = \tilde{\pi}(\alpha_x(A)). \tag{3.100}$$

Here is an elegant and useful method to construct covariant representations.

Proposition 3.8.4 *Let $(G, \mathfrak{A}, \alpha)$ be a C^*-dynamical system, and suppose one has a state ω on $\mathfrak{A}$ which is G-invariant in the sense that*

$$\omega(\alpha_x(A)) = \omega(A) \tag{3.101}$$

for all $x \in G$ and $A \in \mathfrak{A}$. Consider the GNS-representation $\pi_\omega(\mathfrak{A})$ on a Hilbert space $\mathcal{H}_\omega$ with cyclic vector Ω_ω. For $x \in G$, define an operator $U(x)$ on the dense subspace $\pi_\omega(\mathfrak{A})\Omega_\omega$ of $\mathcal{H}_\omega$ by

$$U(x)\pi_\omega(A)\Omega_\omega := \pi_\omega(\alpha_x(A))\Omega_\omega. \tag{3.102}$$

This operator is well defined, and defines a unitary representation of G on $\mathcal{H}_\omega$.

If $\pi_\omega(A)\Omega_\omega = \pi_\omega(B)\Omega_\omega$ then $\omega((A-B)^*(A-B)) = 0$ by (2.120). Hence $\omega(\alpha_x(A-B)^*\alpha_x(A-B)) = 0$ by (3.101), so that $\| \pi_\omega(\alpha_x(A-B))\Omega_\omega \|^2 = 0$ by (2.120). Hence $\pi_\omega(\alpha_x(A))\Omega_\omega = \pi_\omega(\alpha_x(B))$, so that $U(x)\pi_\omega(A)\Omega_\omega = U(x)\pi_\omega(B)\Omega_\omega$.

Furthermore, (3.94) implies that $U(x)U(y) = U(xy)$, whereas (3.102) and (3.101) imply that

$$(U(x)\pi_\omega(A)\Omega_\omega, U(x)\pi_\omega(B)\Omega_\omega) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega).$$

This shows firstly that $U(x)$ is bounded on $\pi_\omega(\mathfrak{A})\Omega_\omega$, so that it may be extended to $\mathcal{H}_\omega$ by continuity. Secondly, $U(x)$ is a partial isometry, which is unitary from $\mathcal{H}_\omega$ to the closure of $U(x)\mathcal{H}_\omega$. Taking $A = \alpha_{x^{-1}}(B)$ in (3.102), one sees that $U(x)\mathcal{H}_\omega = \pi_\omega(\mathfrak{A})\Omega_\omega$, whose closure is $\mathcal{H}_\omega$ because π_ω is cyclic. Hence $U(x)$ is unitary. ■

Note that (3.102) with (3.92) or (3.93) implies that

$$U(x)\Omega_\omega = \Omega_\omega. \tag{3.103}$$

Proposition 3.8.4 describes the way unitary representations of the Poincaré group are constructed in algebraic quantum field theory, in which ω is then taken to be the vacuum state on the algebra of local observables of the system in question. Note, however, that not all covariant representations of a C^*-dynamical system arise in this way; a given unitary representation $U(G)$ may may not contain the trivial representation as a subrepresentation; cf. (3.103).

In any case, the generalization of Theorem 3.7.9 is as follows. Recall (3.97).

Theorem 3.8.5 *Let $(G, \mathfrak{A}, \alpha)$ be a C^*-dynamical system. There is a bijective correspondence between non-degenerate representations π of the Banach *-algebra $L^1(G, \mathfrak{A}, \alpha)$ which satisfy (3.83), and covariant representations $(U(G), \tilde{\pi}(\mathfrak{A}))$. This correspondence is given in one direction by*

$$\pi(f) = \int_G dx\, \tilde{\pi}(f(x))U(x); \tag{3.104}$$

in the other direction one defines $Af : x \to Af(x)$ *and* $\tilde{\alpha}_x(f) : y \to \alpha_x(f(x^{-1}y))$, *and puts*

$$U(x)\pi(f)\Omega = \pi(\tilde{\alpha}_x(f))\Omega; \tag{3.105}$$
$$\tilde{\pi}(A)\pi(f)\Omega = \pi(Af)\Omega, \tag{3.106}$$

where Ω *is a cyclic vector for a cyclic summand of* $\pi(C^*(G,\tilde{\mathfrak{A}}))$.

This bijection preserves direct sums, and therefore irreducibility.

The proof of this theorem is analogous to that of 3.7.9. The approximate unit in $L^1(G,\mathfrak{A},\alpha)$ is constructed by taking the tensor product of an approximate unit in $L^1(G)$ and an approximate unit in $\mathfrak{A}$. The rest of the proof may then essentially be read off from 3.7.9. ■

Generalizing 3.7.10, we put

Definition 3.8.6 *Let* $(G,\mathfrak{A},\alpha)$ *be a* C^**-dynamical system. The* **crossed product** $C^*(G,\mathfrak{A},\alpha)$ *of* G *and* $\mathfrak{A}$ *is the closure of the Banach* **-algebra algebra* $L^1(G,\mathfrak{A},\alpha)$ *in the norm*

$$\| f \| := \| \pi_u(f) \|, \tag{3.107}$$

where π_u *is the direct sum of all non-degenerate representations* π *of* $L^1(G,\mathfrak{A},\alpha)$ *which are bounded as in (3.83).*

Equivalently, $C^*(G,\mathfrak{A},\alpha)$ *is the closure of* $L^1(G,\mathfrak{A},\alpha)$ *in the norm*

$$\| f \| := \sup_{\pi} \| \pi(f) \|, \tag{3.108}$$

where the sum is over all representations $\pi(L^1(G,\mathfrak{A},\alpha))$ *of the form (3.104), in which* $(U,\tilde{\pi})$ *is an irreducible covariant representation of* $(G,\mathfrak{A},\alpha)$, *and only one representative of each equivalence class of such representations is included.*

Here we simply say that a covariant representation $(U,\tilde{\pi})$ is irreducible when the only bounded operator commuting with all $U(x)$ and $\tilde{\pi}(A)$ is a multiple of the unit. The equivalence between the two definitions follows from (2.139) and Theorem 3.8.5.

Theorem 3.8.7 *Let* $(G,\mathfrak{A},\alpha)$ *be a* C^**-dynamical system. There is a bijective correspondence between non-degenerate representations* π *of the crossed product* $C^*(G,\mathfrak{A},\alpha)$ *and covariant representations* $(U(G),\tilde{\pi}(\mathfrak{A}))$. *This correspondence is given by (continuous extension of) (3.104) and (3.105), (3.106). This correspondence preserves direct sums, and therefore irreducibility.*

The proof is identical to that of 3.7.11. ■

3.9 Transformation group C^*-algebras

We now come to an important class of crossed products, in which $\mathfrak{A} = C_0(Q)$, where Q is a locally compact Hausdorff space, and α_x is defined as follows.

Definition 3.9.1 *A (left-)* **action** L *of a group* G *on a space* Q *is a map* $L : G \times Q \to Q$, *satisfying* $L(e,q) = q$ *and* $L(x, L(y,q)) = L(xy,q)$ *for all* $q \in Q$ *and* $x, y \in G$. *If* G *and* Q *are locally compact we assume that* L *is continuous. If* G *is a Lie group and* Q *is a manifold we assume that* L *is smooth. We write* $L_x(q) = xq := L(x,q)$.

We assume the reader is familiar with this concept, at least at a heuristic level. The main example we shall consider is the canonical action of G on the coset space G/H (where H is a closed subgroup of G). This action is given by

$$x[y]_H := [xy]_H, \tag{3.109}$$

where $[x]_H := xH$; cf. 3.1.6 etc. For example, when $G = SO(3)$ and $H = SO(2)$ is the subgroup of rotations around the z-axis, one may identify G/H with the unit two-sphere S^2 in $\mathbb{R}^3$. The $SO(3)$-action (3.109) is then simply the usual action on $\mathbb{R}^3$, restricted to S^2.

Assume that Q is a locally compact Hausdorff space, so that one may form the commutative C^*-algebra $C_0(Q)$; cf. 2.4. A G-action on Q leads to an automorphic action of G on $C_0(Q)$, given by

$$\alpha_x(\tilde{f}) : q \to \tilde{f}(x^{-1}q). \tag{3.110}$$

Using the fact that G is locally compact, so that e has a basis of compact neighbourhoods, it is easy to prove that the continuity of the G-action on Q implies that

$$\lim_{x\to e} \parallel \alpha_x(\tilde{f}) - \tilde{f} \parallel = 0 \tag{3.111}$$

for all $\tilde{f} \in C_c(Q)$. Since $C_c(Q)$ is dense in $C_0(Q)$ in the sup-norm, the same is true for $\tilde{f} \in C_0(Q)$. Hence the function $x \to \alpha_x(\tilde{f})$ from G to $C_0(Q)$ is continuous at e (as $\alpha_e(\tilde{f}) = \tilde{f}$). Using (3.94) and (3.91), one sees that this function is continuous on all of G. Hence $(G, C_0(Q), \alpha)$ is a C^*-dynamical system.

It is quite instructive to look at covariant representations $(U, \tilde{\pi})$ of $(G, C_0(Q), \alpha)$ in the special case that G is a Lie group and Q is a manifold. Firstly, given a unitary representation U of a Lie group G on a Hilbert space $\mathcal{H}$ one can construct a representation of the Lie algebra $\mathfrak{g}$ by

$$dU(X)\Psi := \frac{d}{dt}U(\text{Exp}(tX))\Psi_{|t=0}. \tag{3.112}$$

When $\mathcal{H}$ is infinite-dimensional this defines an unbounded operator, which is not defined on all of $\mathcal{H}$. Eq. (3.112) makes sense when Ψ is a **smooth vector** for a U; this is an element $\Psi \in \mathcal{H}$ for which the map $x \to U(x)\Psi$ from G to $\mathcal{H}$ is smooth. It can be shown that the set $\mathcal{H}_U^\infty$ of smooth vectors for U is a dense linear subspace of $\mathcal{H}$, and that the operator $idU(X)$ is essentially self-adjoint on $\mathcal{H}_U^\omega$. Moreover, on $\mathcal{H}_U^\infty$ one has

$$[dU(X), dU(Y)] = dU([X, Y]). \tag{3.113}$$

Secondly, given a Lie group action one defines a linear map $X \to \xi_X$ from $\mathfrak{g}$ to the space of all vector fields on Q by

$$\xi_X \tilde{f}(q) := \frac{d}{dt}\tilde{f}(\text{Exp}(tX)q)_{|t=0}, \tag{3.114}$$

where $\text{Exp} : \mathfrak{g} \to G$ is the usual exponential map.

The meaning of the covariance condition (3.100) on the pair $(U, \tilde{\pi})$ may now be clarified by re-expressing it in infinitesimal form. For $X \in \mathfrak{g}$, $\tilde{f} \in C_c^\infty(Q)$, and $\hbar \in \mathbb{R}\backslash\{0\}$ we put

$$\mathcal{Q}_\hbar^\pi(\tilde{X}) := i\hbar dU(X); \tag{3.115}$$

$$\mathcal{Q}_\hbar^\pi(\tilde{f}) := \tilde{\pi}(\tilde{f}). \tag{3.116}$$

From the commutativity of $C_0(Q)$, (3.113), and (3.100), respectively, we then obtain

$$\frac{i}{\hbar}[\mathcal{Q}_\hbar^\pi(\tilde{f}), \mathcal{Q}_\hbar^\pi(\tilde{g})] = 0; \tag{3.117}$$

$$\frac{i}{\hbar}[\mathcal{Q}_\hbar^\pi(\tilde{X}), \mathcal{Q}_\hbar^\pi(\tilde{Y})] = \mathcal{Q}_\hbar^\pi(-\widetilde{[X, Y]}); \tag{3.118}$$

$$\frac{i}{\hbar}[\mathcal{Q}_\hbar^\pi(\tilde{X}), \mathcal{Q}_\hbar^\pi(\tilde{f})] = \mathcal{Q}_\hbar^\pi(\xi_X \tilde{f}). \tag{3.119}$$

These equations hold on the domain $\mathcal{H}_U^\infty$, and may be seen as a generalization of the canonical commutation relations of quantum mechanics. To see this, consider the case $G = Q = \mathbb{R}^n$, where the G-action is given by $L(x, q) := q + x$. If $X = T_k$ is the k'th generator of $\mathbb{R}^n$ one has $\xi_k := \xi_{T_k} = \partial/\partial q^k$. Taking $f = q^l$, the l'th co-ordinate function on $\mathbb{R}^n$, one therefore obtains $\xi_k q^l = \delta_k^l$. The relations (3.117) - (3.119) then become

$$\frac{i}{\hbar}[\mathcal{Q}_\hbar^\pi(q^k), \mathcal{Q}_\hbar^\pi(q^l)] = 0; \tag{3.120}$$

$$\frac{i}{\hbar}[\mathcal{Q}_\hbar^\pi(\tilde{T}_k), \mathcal{Q}_\hbar^\pi(\tilde{T}_l)] = 0; \tag{3.121}$$

$$\frac{i}{\hbar}[\mathcal{Q}_\hbar^\pi(\tilde{T}_k), \mathcal{Q}_\hbar^\pi(q^l)] = \delta_k^l. \tag{3.122}$$

Hence one may identify $\mathcal{Q}_\hbar^\pi(q^k)$ and $\mathcal{Q}_\hbar^\pi(\tilde{T}_k)$ with the quantum position and momentum observables, respectively. (It should be remarked that $\mathcal{Q}_\hbar^\pi(q^k)$ is an unbounded operator, but one may show from the representation theory of the Heisenberg group that $\mathcal{Q}_\hbar^\pi(q^k)$ and $\mathcal{Q}_\hbar^\pi(\tilde{T}_k)$ always possess a common dense domain on which (3.120) - (3.122) are valid.)

Definition 3.9.2 *Let L be a continuous action of a locally compact group on a locally compact space Q. The* **transformation group C^*-algebra** *$C^*(G,Q)$ is the crossed product $C^*(G, C_0(Q), \alpha)$ defined by the automorphic action (3.110).*

Conventionally, the G-action L on Q is not indicated in the notation $C^*(G,Q)$, although the construction clearly depends on it.

One may identify $L^1(G, C_0(Q))$ with a subspace of the space of all (measurable) functions from $G \times Q$ to $\mathbb{C}$; an element f of the latter defines $F \in L^1(G, C_0(Q))$ by $F(x) = f(x, \cdot)$. Clearly, $L^1(G, C_0(Q))$ is then identified with the space of all such functions f for which

$$\parallel f \parallel_1 = \int_G dx \sup_{q \in Q} |f(x,q)| \tag{3.123}$$

is finite; cf. (3.97). In this realization, the operations (3.98) and (3.99) read

$$f * g(x,q) = \int_G dy\, f(y,q) g(y^{-1}x, y^{-1}q); \tag{3.124}$$

$$f^*(x,q) = \overline{f(x^{-1}, x^{-1}q)}. \tag{3.125}$$

As always, G is here assumed to be unimodular. Here is a simple example.

Proposition 3.9.3 *Let a locally compact group G act on $Q = G$ by $L(x,y) := xy$. Then $C^*(G,G) \simeq \mathfrak{B}_0(L^2(G))$ as C^*-algebras.*

We start from $C_c(G \times G)$, regarded as a dense subalgebra of $C^*(G,G)$. We define a linear map $\pi : C_c(G \times G) \to \mathfrak{B}(L^2(G))$ by

$$\pi(f)\Psi(x) := \int_G dy\, f(xy^{-1}, x)\Psi(y). \tag{3.126}$$

One verifies from (3.124) and (3.125) that $\pi(f)\pi(g) = \pi(f * g)$ and $\pi(f^*) = \pi(f)^*$, so that π is a representation of the *-algebra $C_c(G \times G)$. It is easily verified that the Hilbert-Schmidt-norm (2.141) of $\pi(f)$ is

$$\parallel \pi(f) \parallel_2^2 = \int_G \int_G dx\, dy\, |f(xy^{-1}, x)|^2. \tag{3.127}$$

Since this is clearly finite for $f \in C_c(G \times G)$, we conclude from (2.154) that $\pi(C_c(G \times G)) \subseteq \mathfrak{B}_0(L^2(G))$. Since $\pi(C_c(G \times G))$ is dense in $\mathfrak{B}_2(L^2(G))$ in the Hilbert-Schmidt-norm (which is a standard fact of Hilbert space theory), and $\mathfrak{B}_2(L^2(G))$ is dense in $\mathfrak{B}_0(L^2(G))$ in the usual operator norm (since by Definition 2.13.1 even $\mathfrak{B}_f(L^2(G))$ is dense in $\mathfrak{B}_0(L^2(G))$), we conclude that the closure of $\pi(C_c(G \times G))$ in the operator norm coincides with $\mathfrak{B}_0(L^2(G))$.

Since π is evidently faithful, the equality $\pi(C^*(G,G)) = \mathfrak{B}_0(L^2(G))$, and therefore the isomorphism $C^*(G,G) \simeq \mathfrak{B}_0(L^2(G))$, follows from the previous paragraph if we can show that the norm defined by (3.108) coincides with the operator norm of $\pi(\cdot)$. This, in turn, is the case if all irreducible representations of the *-algebra $C_c(G \times G)$ are unitarily equivalent to π.

To prove this, we proceed as in Proposition 3.4.4, in which we take $\tilde{\mathfrak{A}} = C_c(G \times G)$, $\tilde{\mathfrak{B}} = \mathbb{C}$, and $\tilde{\mathcal{E}} = C_c(G)$. The pre-Hilbert C^*-module $C_c(G) \rightleftharpoons \mathbb{C}$ is defined by the obvious $\mathbb{C}$-action on $C_c(G)$, and the inner product

$$\langle \Psi, \Phi \rangle_{\mathbb{C}} := (\Psi, \Phi)_{L^2(G)}. \tag{3.128}$$

The left-action of $\tilde{\mathfrak{A}}$ on $\tilde{\mathcal{E}}$ is π as defined in (3.126), whereas the $C_c(G \times G)$-valued inner product on $C_c(G)$ is given by

$$\langle \Psi, \Phi \rangle_{C_c(G \times G)} := \Psi(y)\overline{\Phi}(x^{-1}y). \tag{3.129}$$

It is not necessary to consider the bounds (3.45) and (3.46). Following the proof of Theorem 3.6.1, one shows directly that there is a bijective correspondence between the representations of $C_c(G \times G)$ and of $\mathbb{C}$. ■

3.10 The abstract transitive imprimitivity theorem

We specialize to the case where $Q = G/H$, where H is a closed subgroup of G, and the G-action on G/H is given by (3.109). This leads to the transformation group C^*-algebra $C^*(G, G/H)$.

Theorem 3.10.1 *The transformation group C^*-algebra $C^*(G, G/H)$ is Morita-equivalent to $C^*(H)$.*

We need to construct a full Hilbert C^*-module $\mathcal{E} \rightleftharpoons C^*(H)$ for which $C_0^*(\mathcal{E}, C^*(H))$ is isomorphic to $C^*(G, G/H)$. This will be done on the basis of Proposition 3.4.4. For simplicity we assume that both G and H are unimodular. In 3.4.4 we take

- $\tilde{\mathfrak{A}} = C_c(G, G/H)$, seen as a dense subalgebra of $\mathfrak{A} = C^*(G, G/H)$ as explained prior to (3.123);
- $\tilde{\mathfrak{B}} = C_c(H)$, seen as a dense subalgebra of $\mathfrak{B} = C^*(H)$;
- $\tilde{\mathcal{E}} = C_c(G)$.

We make a pre-Hilbert $C_c(H)$-module $C_c(G) \rightleftharpoons C_c(H)$ by means of the right-action

$$\pi_{\mathrm{R}}(f)\Psi = \Psi f : x \to \int_H dh\, \Psi(xh^{-1}) f(h). \tag{3.130}$$

Here $f \in C_c(H)$ and $\Psi \in C_c(G)$. The $C_c(H)$-valued inner product on $C_c(G)$ is defined by

$$\langle \Psi, \Phi \rangle_{C_c(H)} : h \to \int_G dx\, \overline{\Psi(x)} \Phi(xh). \tag{3.131}$$

Interestingly, both formulae may be written in terms of the right-regular representation U_R of H on $L^2(G)$, given by

$$U_R(h)\Psi(x) := \Psi(xh). \tag{3.132}$$

Namely, one has

$$\pi_{\mathrm{R}}(f) = \int_H dh\, f(h) U(h^{-1}), \tag{3.133}$$

which should be compared with (3.82), and

$$\langle \Psi, \Phi \rangle_{C_c(H)} : h \to (\Psi, U(h)\Phi)_{L^2(G)}. \tag{3.134}$$

The properties (3.9) and (3.10) are easily verified from (3.68) and (3.71), respectively. To prove (3.11), we take a vector state ω_χ on $C^*(H)$, with corresponding unit vector $\Omega_\chi \in \mathcal{H}_\chi$. Hence for $f \in C_c(H) \subset L^1(H)$ one has

$$\omega_\chi(f) = (\Omega_\chi, \pi_\chi(f)\Omega_\chi) = \int_H dh\, f(h)(\Omega_\chi, U_\chi(h)\Omega_\chi), \tag{3.135}$$

where U_χ is the unitary representation of H corresponding to $\pi_\chi(C^*(H))$; see Theorem 3.7.11 (with $G \to H$). We note that the Haar measure on G and the one on H define a unique measure ν on G/H, satisfying

$$\int_G dx\, f(x) = \int_{G/H} d\nu(q) \int_H dh\, f(s(q)h) \tag{3.136}$$

for any $f \in C_c(G)$, and any measurable map $s : G/H \to G$ for which $\tau \circ s = \mathrm{id}$ (where $\tau : G \to G/H$ is the canonical projection $\tau(x) := [x]_H = xH$). Combining (3.135), (3.131), and (3.136), we find

$$\omega_\chi(\langle \Psi, \Psi \rangle_{C_c(H)}) = \int_{G/H} d\nu(q) \parallel \int_H dh\, \Psi(s(q)h) U_\chi(h)\Omega_\chi \parallel^2 . \tag{3.137}$$

Since this is positive, this proves that $\pi_\chi(\langle \Psi, \Psi \rangle_{C_c(H)})$ is positive for all representations π_χ of $C^*(H)$, so that $\langle \Psi, \Psi \rangle_{C_c(H)}$ is positive in $C^*(H)$ by Corollary 2.10.3. This proves (3.11). Condition

(3.12) easily follows from (3.137) as well, since $\langle\Psi,\Psi\rangle_{C_c(H)}=0$ implies that the right-hand side of (3.137) vanishes for all χ. This implies that the function $(q,h)\rightarrow\Psi(s(q)h)$ vanishes almost everywhere for arbitrary sections s. Since one may choose s so as to be piecewise continuous, and $\Psi\in C_c(G)$, this implies that $\Psi=0$.

We now come to the left-action $\pi_{\rm L}$ of $\tilde{\mathfrak{A}}=C_c(G,G/H)$ on $C_c(G)$ and the $C_c(G,G/H)$-valued inner product $\langle\, ,\,\rangle_{C_c(G,G/H)}$ on $C_c(G)$. These are given by

$$\pi_{\rm L}(f)\Psi(x) = \int_G dy\, f(xy^{-1},[x]_H)\Psi(y); \tag{3.138}$$

$$\langle\Psi,\Phi\rangle_{C_c(G,G/H)} : (x,[y]_H)\rightarrow\int_H dh\,\Psi(yh)\overline{\Phi(x^{-1}yh)}. \tag{3.139}$$

Using (3.124) and (3.125), one may check that $\pi_{\rm L}$ is indeed a left-action, and that $\overline{C_c(G)}\rightleftharpoons C_c(G,G/H)$ is a pre-Hilbert C^*-module with respect to the right-action of $C_c(G,G/H)$ given by $\pi_{\rm R}(f)\Psi:=\pi_{\rm L}(f^*)\Psi$; cf. 3.4.4. Also, using (3.139), (3.138), (3.130), and (3.131), it is easy to verify the crucial condition (3.44).

To complete the proof, one needs to show that the Hilbert C^*-modules $C_c(G)\rightleftharpoons C_c(H)$ and $\overline{C_c(G)}\rightleftharpoons C_c(G,G/H)$ are full, and that the bounds (3.45) and (3.46) are satisfied. This is indeed the case, but an argument that is sufficiently elementary for inclusion in these notes does not seem to exist. Enthusiastic readers may find the proof in M.A. Rieffel, Induced representations of C^*-algebras, *Adv. Math.* **13** (1974) 176-257. ∎

3.11 Induced group representations

The theory of induced group representations provides a mechanism for constructing a unitary representation of a locally compact group G from a unitary representation of some closed subgroup H. Theorem 3.10.1 then turns out to be equivalent to a complete characterization of induced group representations, in the sense that it gives a necessary and sufficient criterion for a unitary representation to be induced.

In order to explain the idea of an induced group representation from a geometric point of view, we return to Proposition 3.1.6. The group G acts on the Hilbert bundle H^χ defined by (3.5) by means of

$$\mathsf{U}^\chi(x):[y,v]_H\rightarrow[xy,v]_H. \tag{3.140}$$

Since the left-action $x:y\rightarrow xy$ of G on itself commutes with the right-action $h:y\rightarrow yh$ of H on G, the action (3.140) is clearly well defined.

The G-action U^χ on the vector bundle H^χ induces a natural G-action $U^{(\chi)}$ on the space of continuous sections $\Gamma(\mathsf{H}^\chi)$ of H^χ, defined on $\Psi^{(\chi)}\in\Gamma(\mathsf{H}^\chi)$ by

$$U^{(\chi)}(x)\Psi^{(\chi)}(q):=\mathsf{U}^\chi(x)\Psi^{(\chi)}(x^{-1}q)). \tag{3.141}$$

One should check that $U^{(\chi)}(x)\Psi^{(\chi)}$ is again a section, in that $\tau_\chi(U^{(\chi)}(x)\Psi^{(\chi)}(q))=q$; see (3.6). This section is evidently continuous, since the G-action on G/H is continuous.

There is a natural inner product on the space of sections $\Gamma(\mathsf{H}^\chi)$, given by

$$(\Psi^{(\chi)},\Phi^{(\chi)}):=\int_{G/H} d\nu(q)\,(\Psi^{(\chi)}(q),\Phi^{(\chi)}(q))_\chi, \tag{3.142}$$

where ν is the measure on G/H defined by (3.136), and $(\, ,\,)_\chi$ is the inner product in the fiber $\tau_\chi^{-1}(q)\simeq\mathcal{H}_\chi$. Note that different identifications of the fiber with $\mathcal{H}_\chi$ lead to the same inner product. The Hilbert space $L^2(\mathsf{H}^\chi)$ is the completion of the space $\Gamma_c(\mathsf{H}^\chi)$ of continuous sections of H^χ with compact support (in the norm derived from this inner product).

When the measure ν is G-invariant (which is the case, for example, when G and H are unimodular), the operator $U^{(\chi)}(x)$ defined by (3.141) satisfies

$$(U^{(\chi)}(x)\Psi^{(\chi)},U^{(\chi)}(x)\Phi^{(\chi)})=(\Psi^{(\chi)},\Phi^{(\chi)}). \tag{3.143}$$

When ν fails to be G-invariant, it can be shown that it is still quasi-invariant in the sense that $\nu(\cdot)$ and $\nu(x^{-1}\cdot)$ have the same null sets for all $x \in G$. Consequently, the Radon-Nikodym derivative $q \to d\nu(x^{-1}(q))/d\nu(q)$ exists as a measurable function on G/H. One then modifies (3.141) to

$$U^{(\chi)}(x)\Psi^{(\chi)}(q) := \sqrt{\frac{d\nu(x^{-1}(q))}{d\nu(q)}}\,\mathsf{U}^{\chi}(x)\Psi^{(\chi)}(x^{-1}q)). \tag{3.144}$$

Proposition 3.11.1 *Let G be a locally compact group with closed subgroup H, and let U_χ be a unitary representation of H on a Hilbert space $\mathcal{H}_\chi$. Define the Hilbert space $L^2(\mathsf{H}^\chi)$ of L^2-sections of the Hilbert bundle H^χ as the completion of $\Gamma_c(\mathsf{H}^\chi)$ in the inner product (3.142), where the measure ν on G/H is defined by (3.136).*

The map $x \to U^{(\chi)}(x)$ given by (3.144) with (3.140) defines a unitary representation of G on $L^2(\mathsf{H}^\chi)$. When ν is G-invariant, the expression (3.144) simplifies to (3.141).

One easily verifies that the square-root precisely compensates for the lack of G-invariance of ν, guaranteeing the property (3.143). Hence $U^{(\chi)}(x)$ is isometric on $\Gamma_c(\mathsf{H}^\chi)$, so that it is bounded, and can be extended to $L^2(\mathsf{H}^\chi)$ by continuity. Since $U^{(\chi)}(x)$ is invertible, with inverse $U^{(\chi)}(x^{-1})$, it is therefore a unitary operator. The property $U^{(\chi)}(x)U^{(\chi)}(y) = U^{(\chi)}(xy)$ is easily checked. ■

The representation $U^{(\chi)}(G)$ is said to be **induced** by $U_\chi(H)$.

Proposition 3.11.2 *In the context of 3.11.1, define a representation $\tilde{\pi}^{(\chi)}(C_0(G/H))$ on $L^2(\mathsf{H}^\chi)$ by*

$$\tilde{\pi}^{(\chi)}(\tilde{f})\Psi^{(\chi)}(q) := \tilde{f}(q)\Psi^{(\chi)}(q). \tag{3.145}$$

The pair $(U^{(\chi)}(G), \tilde{\pi}^{(\chi)}(C_0(G/H)))$ is a covariant representation of the C^*-dynamical system $(G, C_0(G/H), \alpha)$, where α is given by (3.110).

Given 3.11.1, this follows from a simple computation. ■

Note that the representation (3.145) is nothing but the right-action (3.1) of $(C_0(G/H))$ on $L^2(\mathsf{H}^\chi)$; this right-action is at the same time a left-action, because $(C_0(G/H))$ is commutative.

We now give a more convenient unitarily equivalent realization of this covariant representation. For this purpose we note that a section $\Psi^{(\chi)} : Q \to \mathsf{H}^\chi$ of the bundle H^χ may alternatively be represented as a map $\Psi^\chi : G \to \mathcal{H}_\chi$ which is H-equivariant in that

$$\Psi^\chi(xh^{-1}) = U_\chi(h)\Psi^\chi(x). \tag{3.146}$$

Such a map defines a section $\Psi^{(\chi)}$ by

$$\Psi^{(\chi)}(\tau(x)) = [x, \Psi^\chi(x)]_H, \tag{3.147}$$

where $\tau : G \to G/H$ is given by (3.7). The section $\Psi^{(\chi)}$ thus defined is independent of the choice of $x \in \tau^{-1}(\tau(x))$ because of (3.146).

For $\Psi^{(\chi)}$ to lie in $\Gamma_c(\mathsf{H}^\chi)$, the projection of the support of Ψ^χ from G to G/H must be compact. In this realization the inner product on $\Gamma_c(\mathsf{H}^\chi)$ is given by

$$(\Psi^\chi, \Phi^\chi) := \int_{G/H} d\nu(\tau(x))\,(\Psi^\chi(x), \Phi^\chi(x))_\chi; \tag{3.148}$$

the integrand indeed only depends on x through $\tau(x)$ because of (3.146).

Definition 3.11.3 *The Hilbert space $\mathcal{H}^\chi$ is the completion in the inner product (3.148) of the set of continuous functions $\Psi^\chi : G \to \mathcal{H}_\chi$ which satisfy the equivariance condition (3.146), and the projection of whose support to G/H is compact.*

Given (3.147), we define the induced G-action U^χ on Ψ^χ by

$$[y, U^\chi(x)\Psi^\chi(y)]_H := U^{(\chi)}(x)\Psi^{(\chi)}(\tau(y)). \tag{3.149}$$

Using (3.141), (3.147), and (3.140), as well as the definition $x\tau(y) = x[y]_H = [xy]_H = \tau(xy)$ of the G-action on G/H (cf. (3.7)), we obtain

$$U^{(\chi)}(x)\Psi^{(\chi)}(\tau(y)) = \mathsf{U}^\chi(x)\Psi^{(\chi)}(x^{-1}\tau(y))) = \mathsf{U}^\chi(x)[x^{-1}y, \Psi^\chi(x^{-1}y)]_H = [y, \Psi^\chi(x^{-1}y)]_H.$$

Hence we infer from (3.149) that

$$U^\chi(y)\Psi^\chi(x) = \Psi^\chi(y^{-1}x). \tag{3.150}$$

Replacing (3.141) by (3.144) in the above derivation yields

$$U^\chi(y)\Psi^\chi(x) = \sqrt{\frac{d\nu(\tau(y^{-1}x))}{d\nu(\tau(x))}}\,\Psi^\chi(y^{-1}x). \tag{3.151}$$

Similarly, in the realization $\mathcal{H}^\chi$ the representation (3.145) reads

$$\tilde{\pi}^\chi(\tilde{f})\Psi^\chi(x) := \tilde{f}([x]_H)\Psi^\chi(x). \tag{3.152}$$

Analogous to 3.11.2, we then have

Proposition 3.11.4 *In the context of 3.11.1, define a representation $\tilde{\pi}^\chi(C_0(G/H))$ on $\mathcal{H}^\chi$ (cf. 3.11.3) by (3.152). The pair $(U^\chi(G), \tilde{\pi}^\chi(C_0(G/H)))$, where U^χ is given by (3.151), is a covariant representation of the C^*-dynamical system $(G, C_0(G/H), \alpha)$, where α is given by (3.110).*

This pair is unitarily equivalent to the pair $(U^{(\chi)}(G), \tilde{\pi}^{(\chi)}(C_0(G/H)))$ by the unitary map $V : \mathcal{H}^\chi \to \mathcal{H}^{(\chi)}$ given by

$$V\Psi^\chi(\tau(x)) := [x, \Psi^\chi(x)]_H, \tag{3.153}$$

in the sense that

$$VU^\chi(y)V^{-1} = U^{(\chi)}(y) \tag{3.154}$$

for all $y \in G$, and

$$V\tilde{\pi}^\chi(\tilde{f})V^{-1} = \tilde{\pi}^{(\chi)}(\tilde{f}) \tag{3.155}$$

for all $\tilde{f} \in C_0(G/H)$.

Comparing (3.153) with (3.147), it should be obvious from the argument leading from (3.149) to (3.151) that (3.154) holds. An analogous but simpler calculation shows (3.155). ■

3.12 Mackey's transitive imprimitivity theorem

In the preceding section we have seen that the unitary representation $U^\chi(G)$ induced by a unitary representation U_χ of a closed subgroup $H \subset G$ can be extended to a covariant representation $(U^\chi(G), \tilde{\pi}^\chi(C_0(G/H))$. The original imprimitivity theorem of Mackey, which historically preceded Theorems 3.6.1 and 3.10.1, states that all covariant pairs $(U(G), \tilde{\pi}(C_0(G/H))$ arise in this way.

Theorem 3.12.1 *Let G be a locally compact group with closed subgroup H, and consider the C^*-dynamical system $(G, C_0(G/H), \alpha)$, where α is given by (3.110). Recall (cf. 3.8.3) that a covariant representation of this system consists of a unitary representation $U(G)$ and a representation $\tilde{\pi}(C_0(G/H))$, satisfying the covariance condition*

$$U(x)\tilde{\pi}(\tilde{f})U(x)^{-1} = \tilde{\pi}(\tilde{f}^x) \tag{3.156}$$

for all $x \in G$ and $\tilde{f} \in C_0(G/H$; here $\tilde{f}^x(q) := \tilde{f}(x^{-1}q)$.

Any unitary representation $U_\chi(H)$ leads to a covariant representation $(U^\chi(G), \tilde{\pi}^\chi(C_0(G/H))$ of $(G, C_0(G/H), \alpha)$, given by 3.11.3, (3.151) and (3.152). Conversely, any covariant representation $(U, \tilde{\pi})$ of $(G, C_0(G/H), \alpha)$ is unitarily equivalent to a pair of this form.

This leads to a bijective correspondence between the space of equivalence classes of unitary representations of H and the space of equivalence classes of covariant representations $(U, \tilde{\pi})$ of the C^*-dynamical system $(G, C_0(G/H), \alpha)$, which preserves direct sums and therefore irreducibility (here the equivalence relation is unitary equivalence).

The existence of the bijective correspondence with the stated properties follows by combining Theorems 3.10.1 and 3.6.1, which relate the representations of $C^*(H)$ and $C^*(G,G/H)$, with Theorems 3.7.11 and 3.8.7, which allow one to pass from $\pi(C^*(H))$ to $U(H)$ and from $\pi(C^*(G,G/H))$ to $(U(G),\tilde{\pi}(C_0(G/H)))$, respectively.

The explicit form of the correspondence remains to be established. Let us start with a technical point concerning Rieffel induction in general. Using (3.57), (3.47), and (3.13), one shows that $\| \tilde{V}\Psi \| \leq \| \Psi \|$, where the norm on the left-hand side is in $\tilde{\mathcal{H}}^\chi$, and the norm on the right-hand side is the one defined in (3.13). It follows that the induced space $\tilde{\mathcal{H}}^\chi$ obtained by Rieffel-inducing from a pre-Hilbert C^*-module is the same as the induced space constructed from its completion. The same comment, of course, applies to $\mathcal{H}^\chi$.

We will use a gerenal technique that is often useful in problems involving Rieffel induction.

Lemma 3.12.2 *Suppose one has a Hilbert space $\mathcal{H}^\chi_*$ (with inner product denoted by $(\,,\,)^\chi_*$) and a linear map $\tilde{U}:\mathcal{E}\otimes\mathcal{H}_\chi\rightarrow\mathcal{H}^\chi_*$ satisfying*

$$(\tilde{U}\tilde{\Psi},\tilde{U}\tilde{\Phi})^\chi_* = (\tilde{\Psi},\tilde{\Phi})^\chi_0 \tag{3.157}$$

for all $\tilde{\Psi},\tilde{\Phi}\in\mathcal{E}\otimes\mathcal{H}_\chi$.

Then $\tilde{U}$ quotients to an isometric map between $\mathcal{E}\otimes\mathcal{H}_\chi/\mathcal{N}^\chi$ and the image of $\tilde{U}$ in $\mathcal{H}^\chi_*$. When the image is dense this map extends to a unitary isomorphism $U:\mathcal{H}^\chi\rightarrow\mathcal{H}^\chi_*$. Otherwise, U is unitary between $\mathcal{H}^\chi$ and the closure of the image of $\tilde{U}$.

In any case, the representation $\pi^\chi(C^*(\mathcal{E},\mathfrak{B}))$ is equivalent to the representation $\pi^\chi_*(C^*(\mathcal{E},\mathfrak{B}))$, defined by continuous extension of

$$\pi^\chi_*(A)\tilde{U}\tilde{\Psi} := \tilde{U}(A\otimes I_\chi\tilde{\Psi}). \tag{3.158}$$

It is obvious that $\mathcal{N}^\chi=\ker(\tilde{U})$, so that, comparing with (3.58), one indeed has $U\circ\pi^\chi = \pi^\chi_*\circ U$. ∎

We use this lemma in the following way. To avoid notational confusion, we continue to denote the Hilbert space $\mathcal{H}^\chi$ defined in Construction 3.5.3, starting from the pre-Hilbert C^*-module $C_c(G)\rightleftharpoons C_c(H)$ defined in the proof of 3.10.1, by $\mathcal{H}^\chi$. The Hilbert space $\mathcal{H}^\chi$ defined below (3.148), however, will play the role $\mathcal{H}^\chi_*$ in 3.12.2, and will therefore be denoted by this symbol.

Consider the map $\tilde{U}:C_c(G)\otimes\mathcal{H}_\chi\rightarrow\mathcal{H}^\chi_*$ defined by linear extension of

$$\tilde{U}\Psi\otimes v(x) := \int_H dh\,\Psi(xh)U_\chi(h)v. \tag{3.159}$$

Note that the equivariance condition (3.146) is indeed satisfied by the left-hand side, as follows from the invariance of the Haar measure.

Using (3.54), (3.131), and (3.82), with $G\rightarrow H$, one obtains

$$(\Psi\otimes v,\Phi\otimes w)^\chi_0 = \int_H dh\,(\Psi,U_R(h)\Phi)_{L^2(G)}(v,U_\chi(h)w)_\chi = \int_H dh\int_G dx\,\overline{\Psi(x)}\Phi(xh)(v,U_\chi(h)w)_\chi; \tag{3.160}$$

cf. (3.132). On the other hand, from (3.159) and (3.148) one has

$$(\tilde{U}\Psi\otimes v,\tilde{U}\Phi\otimes w)_{\mathcal{H}^\chi_*} = \int_H dh\int_{G/H} d\nu(\tau(x))\int_H dk\,\overline{\Psi(xk)}\Phi(xh)(U_\chi(k)v,U_\chi(h)w)_\chi. \tag{3.161}$$

Shifting $h\rightarrow kh$, using the invariance of the Haar measure on H, and using (3.136), one verifies (3.157). It is clear that $\tilde{U}(C_c(G)\otimes\mathcal{H}_\chi)$ is dense in $\mathcal{H}^\chi$, so by Proposition 3.12.2 one obtains the desired unitary map $U:\mathcal{H}^\chi\rightarrow\mathcal{H}^\chi_*$.

Using (3.158) and (3.138), one finds that the induced representation of $C^*(G,G/H)$ on $\mathcal{H}^\chi_*$ is given by

$$\pi^\chi(f)\Psi^\chi(x) = \int_G dy\,f(xy^{-1},[x]_H)\Psi^\chi(y); \tag{3.162}$$

this looks just like (3.138), with the difference that Ψ in (3.138) lies in $C_c(G)$, whereas Ψ^χ in (3.162) lies in $\mathcal{H}^\chi_*$. Indeed, one should check that the function $\pi^\chi(f)\Psi^\chi$ defined by (3.162) satisfies the equivariance condition (3.146).

Finally, it is a simple exercise the verify that the representation $\pi^\chi(C^*(G,G/H))$ defined by (3.162) corresponds to the covariant representation $(U^\chi(G),\tilde{\pi}^\chi(C_0(G/H))$ by the correspondence (3.104) - (3.106) of Theorem 3.8.5. ■

4 Applications to quantum mechanics

4.1 The mathematical structure of classical and quantum mechanics

In classical mechanics one starts from a phase space S, whose points are interpreted as the pure states of the system. More generally, mixed states are identified with probability measures on S. The observables of the theory are functions on S; one could consider smooth, continuous, bounded, measurable, or some other other class of real-vaued functions. Hence the space $\mathfrak{A}_\mathbb{R}$ of observables may be taken to be $C^\infty(S,\mathbb{R})$, $C_0(S,\mathbb{R})$, $C_b(S,\mathbb{R})$, or $L^\infty(S,\mathbb{R})$, etc.

There is a pairing $\langle\, ,\, \rangle : \mathcal{S}\times\mathfrak{A}_\mathbb{R}\rightarrow\mathbb{R}\cup\infty$ between the state space $\mathcal{S}$ of probability measures μ on S and the space $\mathfrak{A}_\mathbb{R}$ of observables f. This pairing is given by

$$\langle\mu,f\rangle := \mu(f) = \int_S d\mu(\sigma)\, f(\sigma). \tag{4.1}$$

The physical interpretation of this pairing is that in a state μ the observable f has expectation value $\langle\mu,f\rangle$. In general, this expectation value will be unsharp, in that $\langle\mu,f\rangle^2\neq\langle\mu,f^2\rangle$. However, in a pure state σ (seen as the Dirac measure δ_σ on S) the observable f has sharp expectation value

$$\delta_\sigma(f) = f(\sigma). \tag{4.2}$$

In elementary quantum mechanics the state space consists of all density matrices ρ on some Hilbert space $\mathcal{H}$; the pure states are identified with unit vectors Ψ. The observables are taken to be either all unbounded self-adjoint operators A on $\mathcal{H}$, or all bounded self-adjoint operators, or all compact self-adjoint operators, etc. This time the pairing between states and observables is given by

$$\langle\rho,A\rangle = \mathrm{Tr}\,\rho A. \tag{4.3}$$

In a pure state Ψ one has

$$\langle\Psi,A\rangle = (\Psi,A\Psi). \tag{4.4}$$

A key difference between classical and quantum mechanics is that even in pure states expectation values are generally unsharp. The only exception is when an observable A has discrete spectrum, and Ψ is an eigenvector of A.

In these examples, the state space has a convex structure, whereas the set of observables is a real vector space (barring problems with the addition of unbounded operators on a Hilbert space). We may, therefore, say that a physical theory consists of

- a convex set $\mathcal{S}$, interpreted as the state space;
- a real vector space $\mathfrak{A}_\mathbb{R}$, consisting of the observables;
- a pairing $\langle\, ,\, \rangle : \mathcal{S}\times\mathfrak{A}_\mathbb{R}\rightarrow\mathbb{R}\cup\infty$, which assigns the expectation value $\langle\omega,f\rangle$ to a state ω and an observable f.

In addition, one should specify the dynamics of the theory, but this is not our concern here.

The situation is quite neat if $\mathcal{S}$ and $\mathfrak{A}_\mathbb{R}$ stand in some duality relation. For example, in the classical case, if S is a locally compact Hausdorff space, and we take $\mathfrak{A}_\mathbb{R} = C_0(S,\mathbb{R})$, then the space of all probability measures on S is precisely the state space of $\mathfrak{A} = C_0(S)$ in the sense of Definition 2.8.1; see Theorem 2.8.2. In the same sense, in quantum mechanics the space of all density matrices on $\mathcal{H}$ is the state space of the C^*-algebra $\mathfrak{B}_0(\mathcal{H})$ of all compact operators on $\mathcal{H}$;

see Corollary 2.13.10.1. On the other hand, with the same choice of the state space, if we take $\mathfrak{A}_{\mathbb{R}}$ to be the space $\mathfrak{B}(\mathcal{H})_{\mathbb{R}}$ of all bounded self-adjoint operators on $\mathcal{H}$, then the space of observables is the dual of the (linear space spanned by the) state space, rather then *vice versa*; see Theorem 2.13.8.

In the C^*-algebraic approach to quantum mechanics, a general quantum system is specified by some C^*-algebra $\mathfrak{A}$, whose self-adjoint elements in $\mathfrak{A}_{\mathbb{R}}$ correspond to the observables of the theory. The state space of $\mathfrak{A}_{\mathbb{R}}$ is then given by Definition 2.8.1. This general setting allows for the existence of **superselection rules**. We will not go into this generalization of elementary quantum mechanics here, and concentrate on the choice $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$.

4.2 Quantization

The physical interpretation of quantum mechanics is a delicate matter. Ideally, one needs to specify the physical meaning of any observable $A \in \mathfrak{A}_{\mathbb{R}}$. In practice, a given quantum system arises from a classical system by 'quantization'. This means that one has a classical phase space S and a linear map $\mathcal{Q} : \mathfrak{A}^0_{\mathbb{R}} \rightarrow \mathcal{L}(\mathcal{H})$, where $\mathfrak{A}^0_{\mathbb{R}}$ stands for $C^\infty(S,\mathbb{R})$, or $C_0(S,\mathbb{R})$, etc, and $\mathcal{L}(\mathcal{H})$ denotes some space of self-adjoint operators on $\mathcal{H}$, such as $\mathfrak{B}_0(\mathcal{H})_{\mathbb{R}}$ or $\mathfrak{B}(\mathcal{H})_{\mathbb{R}}$. Given the physical meaning of a classical observable f, one then ascribes the same physical interpretation to the corresponding quantum observable $\mathcal{Q}(f)$. This provides the physical meaning of al least all operators in the image of $\mathcal{Q}$. It is desirable (though not strictly necessary) that $\mathcal{Q}$ preserves positivity, as well as the (approximate) unit.

It is quite convenient to assume that $\mathfrak{A}^0_{\mathbb{R}} = C_0(S,\mathbb{R})$, which choice discards what happens at infinity on S. We are thus led to the following

Definition 4.2.1 *Let X be a locally compact Hausdorff space. A* **quantization** *of X consists of a Hilbert space $\mathcal{H}$ and a positive map $\mathcal{Q} : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$. When X is compact it is required that $\mathcal{Q}(1_X) = \mathbb{I}$, and when X is non-compact one demands that $\mathcal{Q}$ can be extended to the unitization $C_0(X)_{\mathbb{I}}$ by a unit-preserving positive map.*

Here $C_0(X)$ and $\mathfrak{B}(\mathcal{H})$ are, of course, regarded as C^*-algebras, with the intrinsic notion of positivity given by 2.6.1. Also recall Definition 2.8.4 of a positive map. It follows from 2.6.5 that a positive map automatically preserves self-adjointness, in that

$$\mathcal{Q}(\overline{f}) = \mathcal{Q}(f)^* \tag{4.5}$$

for all $f \in C_0(X)$; this implies that $f \in C_0(X,\mathbb{R})$ is mapped into a self-adjoint operator.

There is an interesting reformulation of the notion of a quantization in the above sense.

Definition 4.2.2 *Let X be a set with a σ-algebra Σ of subsets of X. A* **positive-operator-valued measure** *or* **POVM** *on X in a Hilbert space $\mathcal{H}$ is a map $\Delta \rightarrow A(\Delta)$ from Σ to $\mathfrak{B}(\mathcal{H})^+$ (the set of positive operators on $\mathcal{H}$), satisfying $A(\emptyset) = 0$, $A(X) = \mathbb{I}$, and $A(\cup_i \Delta_i) = \sum_i A(\Delta_i)$ for any countable collection of disjoint $\Delta_i \in \Sigma$ (where the infinite sum is taken in the weak operator topology).*

A **projection-valued measure** *or* **PVM** *is a POVM which in addition satisfies $A(\Delta_1 \cap \Delta_2) = A(\Delta_1)A(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Sigma$.*

Note that the above conditions force $0 \leq A(\Delta) \leq \mathbb{I}$. A PVM is usually written as $\Delta \rightarrow E(\Delta)$; it follows that each $E(\Delta)$ is a projection (take $\Delta_1 = \Delta_2$ in the definition). This notion is familiar from the spectral theorem.

Proposition 4.2.3 *Let X be a locally compact Hausdorff space, with Borel structure Σ. There is a bijective correspondence between quantizations $\mathcal{Q} : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$, and POVM's $\Delta \rightarrow A(\Delta)$ on S in $\mathcal{H}$, given by*

$$\mathcal{Q}(f) = \int_S dA(x)\, f(x). \tag{4.6}$$

The map $\mathcal{Q}$ is a representation of $C_0(X)$ iff $\Delta \rightarrow A(\Delta)$ is a PVM.

The precise meaning of (4.6) will emerge shortly. Given the assumptions, in view of 2.3.7 and 2.4.6 we may as well assume that X is compact.

Given $\mathcal{Q}$, for arbitrary $\Psi \in \mathcal{H}$ one constructs a functional $\hat{\mu}_{\Psi,\Psi}$ on $C(X)$ by $\hat{\mu}_{\Psi,\Psi}(f) := (\Psi, \mathcal{Q}(f)\Psi)$ Since $\mathcal{Q}$ is linear and positive, this functional has the same properties. Hence the Riesz representation theorem yields a probability measure $\mu_{\Psi,\Psi}$ on X. For $\Delta \in \Sigma$ one then puts $(\Psi, A(\Delta)\Psi) := \mu_{\Psi,\Psi}(\Delta)$, defining an operator $A(\Delta)$ by polarization. The ensuing map $\Delta \to A(\Delta)$ is easily checked to have the properties required of a POVM.

Conversely, for each pair $\Psi, \Phi \in \mathcal{H}$ a POVM $\Delta \to A(\Delta)$ in $\mathcal{H}$ defines a signed measure $\mu_{\Psi,\Phi}$ on X by means of $\mu_{\Psi,\Phi}(\Delta) := (\Psi, A(\Delta)\Phi)$. This yields a positive map $\mathcal{Q} : C(X) \to \mathfrak{B}(\mathcal{H})$ by $(\Psi, \mathcal{Q}(f)\Phi) := \int_X d\mu_{\Psi,\Phi}(x)\, f(x)$; the meaning of (4.6) is expressed by this equation.

Approximating $f, g \in C(X)$ by step functions, one verifies that the property $E(\Delta)^2 = E(\Delta)$ is equivalent to $\mathcal{Q}(fg) = \mathcal{Q}(f)\mathcal{Q}(g)$. ∎

Corollary 4.2.4 *Let $\Delta \to A(\Delta)$ be a POVM on a locally compact Hausdorff space X in a Hilbert space $\mathcal{H}_\chi$. There exist a Hilbert space $\mathcal{H}^\chi$, a projection p on $\mathcal{H}^\chi$, a unitary map $U : \mathcal{H}_\chi \to p\mathcal{H}^\chi$, and a PVM $\Delta \to E(\Delta)$ on $\mathcal{H}^\chi$ such that $UA(\Delta)U^{-1} = pE(\Delta)p$ for all $\Delta \in \Sigma$.*

Combine Theorem 2.11.2 with Proposition 4.2.3. ∎

When X is the phase space S of a physical system, the physical interpretation of the map $\Delta \to A(\Delta)$ is contained in the statement that the number

$$p_\rho(\Delta) := \mathrm{Tr}\, \rho A(\Delta) \tag{4.7}$$

is the probability that, in a state ρ, the system in question is localized in $\Delta \subset S$.

When X is a configuration space Q, it is usually sufficient to take the positive map $\mathcal{Q}$ to be a representation π of $C_0(Q)$ on $\mathcal{H}$. By Proposition 4.2.3, the situation is therefore described by a PVM $\Delta \to E(\Delta)$ on Q in $\mathcal{H}$. The probability that, in a state ρ, the system is localized in $\Delta \subset Q$ is

$$p_\rho(\Delta) := \mathrm{Tr}\, \rho E(\Delta). \tag{4.8}$$

4.3 Stinespring's theorem and coherent states

By Proposition 2.11.4, a quantization $\mathcal{Q} : C_0(X) \to \mathfrak{B}(\mathcal{H})$ is a completely positive map, and Definition 4.2.1 implies that the conditions for Stinespring's Theorem 2.11.2 are satisfied. We will now construct a class of examples of quantization in which one can construct an illuminating explicit realization of the Hilbert space $\mathcal{H}^\chi$ and the partial isometry W.

Let S be a locally compact Hausdorff space (interpreted as a classical phase space), and consider an embedding $\sigma \to \Psi^\sigma$ of S into some Hilbert space $\mathcal{H}$, such that each Ψ^σ has unit norm (so that a pure classical state is mapped into a pure quantum state). Moreover, there should be a measure μ on S such that

$$\int_S d\mu(\sigma)(\Psi_1, \Psi^\sigma)(\Psi^\sigma, \Psi_2) = (\Psi_1, \Psi_2). \tag{4.9}$$

for all $\Psi_1, \Psi_2 \in \mathcal{H}$. The Ψ^σ are called **coherent states** for S.

Condition (4.9) guarantees that we may define a POVM on S in $\mathcal{H}$ by

$$A(\Delta) = \int_\Delta d\mu(\sigma)\, [\Psi^\sigma], \tag{4.10}$$

where $[\Psi]$ is the projection onto the one-dimensional subspace spanned by Ψ (in Dirac's notation one would have $[\Psi] = |\Psi><\Psi|$).

The positive map $\mathcal{Q}$ corresponding to the POVM $\Delta \to A(\Delta)$ by Proposition 4.2.3 is given by

$$\mathcal{Q}(f) = \int_S d\mu(\sigma)\, f(\sigma)[\Psi^\sigma]. \tag{4.11}$$

In particular, one has $\mathcal{Q}(1_S) = \mathbb{I}$.

For example, when $S = T^*\mathbb{R}^3 = \mathbb{R}^6$, so that $\sigma = (p,q)$, one may take

$$\Psi^{(p,q)}(x) = (\pi)^{-n/4} e^{-\frac{1}{2}ipq+ipx} e^{-(x-q)^2/2} \tag{4.12}$$

in $\mathcal{H} = L^2(\mathbb{R}^3)$. Eq. (4.9) then holds with $d\mu(p,q) = d^3pd^3q/(2\pi)^3$. Extending the map $\mathcal{Q}$ from $C_0(S)$ to $C^\infty(S)$ in a heuristic way, one finds that $\mathcal{Q}(q_i)$ and $\mathcal{Q}(p_i)$ are just the usual position- and momentum operators in the Schrödinger representation.

In Theorem 2.11.2 we now put $\mathfrak{A} = C_0(S)$, $\mathfrak{B} = \mathfrak{B}(\mathcal{H})$, $\pi_\chi(A) = A$ for all A. We may then verify the statement of the theorem by taking

$$\mathcal{H}^\chi = L^2(S, d\mu). \tag{4.13}$$

The map $W : \mathcal{H} \to \mathcal{H}^\chi$ is then given by

$$W\Psi(\sigma) := (\Psi^\sigma, \Psi). \tag{4.14}$$

It follows from (4.9) that W is a partial isometry. The representation $\pi(C_0(S))$ is given by

$$\pi(f)\Phi(\sigma) = f(\sigma)\Phi(\sigma). \tag{4.15}$$

Finally, for (2.122) one has the simple expression

$$\tilde{\mathcal{Q}}(f) = U\mathcal{Q}(f)U^{-1} = pfp. \tag{4.16}$$

Eqs. (4.13) and (4.16) form the core of the realization of quantum mechanics on phase space. One realizes the state space as a closed subspace of $L^2(S)$ (defined with respect to a suitable measure), and defines the quantization of a classical observable $f \in C_0(S)$ as multiplication by f, sandwiched between the projection onto the subspace in question. This should be contrasted with the usual way of doing quantum mechanics on $L^2(Q)$, where Q is the configuration space of the system.

In specific cases the projection $p = WW^*$ can be explicitly given as well. For example, in the case $S = T^*\mathbb{R}^3$ considered above one may pass to complex variables by putting $z = (q - ip)/\sqrt{2}$. We then map $L^2(T^*\mathbb{R}^3, d^3pd^3q/(2\pi)^3)$ into $\mathcal{K} := L^2(\mathbb{C}^3, d^3zd^3\overline{z}\exp(-z\overline{z})/(2\pi i)^3)$ by the unitary operator V, given by

$$V\Phi(z,\overline{z}) := e^{\frac{1}{2}z\overline{z}}\Phi(p = (\overline{z} - z)/\sqrt{2}, q = (\overline{z} + z)/\sqrt{2}). \tag{4.17}$$

One may then verify from (4.14) and (4.12) that VpV^{-1} is the projection onto the space of entire functions in $\mathcal{K}$.

4.4 Covariant localization in configuration space

In elementary quantum mechanics a particle moving on $\mathbb{R}^3$ with spin $j \in \mathbb{N}$ is described by the Hilbert space

$$\mathcal{H}^j_{\text{QM}} = L^2(\mathbb{R}^3) \otimes \mathcal{H}_j, \tag{4.18}$$

where $\mathcal{H}_j = \mathbb{C}^{2j+1}$ carries the irreducible representation $U_j(SO(3))$ (usually called $\mathcal{D}_j$). The basic physical observables are represented by unbounded operators Q^S_k (position), P^S_k (momentum), and J^S_k (angular momentum), where $k = 1,2,3$. These operators satisfy the commutation relations (say, on the domain $\mathcal{S}(\mathbb{R}^3) \otimes \mathcal{H}_j$)

$$[Q^S_k, Q^S_l] = 0; \tag{4.19}$$
$$[P^S_k, Q^S_l] = -i\hbar\delta_{kl}; \tag{4.20}$$
$$[J^S_k, Q^S_l] = i\hbar\epsilon_{klm}Q^S_m; \tag{4.21}$$
$$[P^S_k, P^S_l] = 0; \tag{4.22}$$
$$[J^S_k, J^S_l] = i\hbar\epsilon_{klm}J^S_m; \tag{4.23}$$
$$[J^S_k, P^S_l] = i\hbar\epsilon_{klm}P^S_m, \tag{4.24}$$

justifying their physical interpretation.

The momentum and angular momentum operators are most conveniently defined in terms of a unitary representation $U^j_{\rm QM}$ of the Euclidean group $E(3) = SO(3) \ltimes \mathbb{R}^3$ on $\mathcal{H}^j_{\rm QM}$, given by

$$U^j_{\rm QM}(R,a)\Psi(q) = U_j(R)\Psi(R^{-1}(q-a)). \tag{4.25}$$

In terms of the standard generators P_k and T_k of $\mathbb{R}^3$ and $SO(3)$, respectively, one then has $P^S_k = i\hbar dU^j_{\rm QM}(P_k)$ and $J^S_k = i\hbar dU^j_{\rm QM}(T_k)$; see (3.112). The commutation relations (4.22) - (4.24) follow from (3.113) and the commutation relations in the Lie algebra of $E(3)$.

Moreover, we define a representation $\tilde{\pi}^j_{\rm QM}$ of $C_0(\mathbb{R}^3)$ on $\mathcal{H}^j_{\rm QM}$ by

$$\tilde{\pi}^j_{\rm QM}(\tilde{f}) = \tilde{f} \otimes \mathbb{I}_j, \tag{4.26}$$

where $\tilde{f}$ is seen as a multiplication operator on $L^2(\mathbb{R}^3)$. The associated PVM $\Delta \rightarrow E(\Delta)$ on $\mathbb{R}^3$ in $\mathcal{H}^j_{\rm QM}$ (see 4.2.3) is $E(\Delta) = \chi_\Delta \otimes \mathbb{I}_j$, in terms of which the position operators are given by $Q^S_k = \int_{\mathbb{R}^3} dE(x)x_k$; cf. the spectral theorem for unbounded operators. Eq. (4.19) then reflects the commutativity of $C_0(\mathbb{R}^3)$, as well as the fact that $\tilde{\pi}^j_{\rm QM}$ is a representation.

Identifying $Q = \mathbb{R}^3$ with $G/H = E(3)/SO(3)$ in the obvious way, one checks that the canonical left-action of $E(3)$ on $E(3)/SO(3)$ is identified with its defining action on $\mathbb{R}^3$. It is then not hard to verify from (4.25) that the pair $(U^j_{\rm QM}(E(3)), \tilde{\pi}^j_{\rm QM}(C_0(\mathbb{R}^3)))$ is a covariant representation of the C^*-dynamical system $(E(3), C_0(\mathbb{R}^3), \alpha)$, with α given by (3.110). The commutation relations (4.20), (4.21) are a consequence of the covariance relation (3.156).

Rather than using the unbounded operators Q^S_k, P^S_k, and J^S_k, and their commutation relations, we therefore state the situation in terms of the pair $(U^j_{\rm QM}(E(3)), \tilde{\pi}^j_{\rm QM}(C_0(\mathbb{R}^3)))$. Such a pair, or, equivalently, a non-degenerate representation $\pi^j_{\rm QM}$ of the transformation group C^*-algebra $C^*(E(3), \mathbb{R}^3)$ (cf. 3.8.7, then by definition describes a quantum system which is localizable in $\mathbb{R}^3$. and covariant under the defining action of $E(3)$. It is natural to require that $\pi^j_{\rm QM}$ be irreducible, in which case the quantum system itself is said to be irreducible.

Proposition 4.4.1 *An irreducible quantum system which is localizable in $\mathbb{R}^3$ and covariant under $E(3)$ is completely characterized by its spin $j \in \mathbb{N}$. The corresponding covariant representation $(U^j(E(3)), \tilde{\pi}^j(C_0(\mathbb{R}^3)))$, given by 3.11.3, (3.151), and (3.152), is equivalent to the one described by (4.18), (4.25), and (4.26).*

This follows from Theorem 3.12.1. The representation $U^j_{\rm QM}(E(3))$ defined in (4.25) is unitarily equivalent to the induced representation U^j. To see this, check that the unitary map $V : \mathcal{H}^j \rightarrow \mathcal{H}^j_{\rm QM}$ defined by $V\Psi^j(q) := \Psi^j(e,q)$ intertwines U^j and $U^j_{\rm QM}$. In addition, it intertwines the representation (3.152) with $\tilde{\pi}^j_{\rm QM}$ as defined in (4.26). ■

This is a neat explanation of spin in quantum mechanics.

Generalizing this approach to an arbitrary homogeneous configuration space $Q = G/H$, a non-degenerate representation π of $C^*(G, G/H)$ on a Hilbert space $\mathcal{H}$ describes a quantum system which is localizable in G/H and covariant under the canonical action of G on G/H. By 3.8.7 this is equivalent to a covariant representation $(U(G), \tilde{\pi}(C_0(G/H)))$ on $\mathcal{H}$, and by Proposition 4.2.3 one may instead assume one has a PVM $\Delta \rightarrow E(\Delta)$ on G/H in $\mathcal{H}$ and a unitary representation $U(G)$, which satisfy

$$U(x)E(\Delta)U(x)^{-1} = E(x\Delta) \tag{4.27}$$

for all $x \in G$ and $\Delta \in \Sigma$; cf. (4.29). The physical interpretation of the PVM is given by (4.8); the operators defined in (3.115) play the role of quantized momentum observables. Generalizing Proposition 4.4.1, we have

Theorem 4.4.2 *An irreducible quantum system which is localizable in $Q = G/H$ and covariant under the canonical action of G is characterized by an irreducible unitary representation of H. The system of imprimitivity $(U^\chi(G), \tilde{\pi}^j(C_0(G/H)))$ is equivalent to the one described by (3.151) and (3.152).*

This is immediate from Theorem 3.12.1. ■

For example, writing the two-sphere S^2 as $SO(3)/SO(2)$, one infers that $SO(3)$-covariant quantum particles on S^2 are characterized by an integer $n \in \mathbb{Z}$. For each unitary irreducible representation U of $SO(2)$ is labeled by such an n, and given by $U_n(\theta) = \exp(in\theta)$.

4.5 Covariant quantization on phase space

Let us return to quantization theory, and ask what happens in the presence of a symmetry group. The following notion, which generalizes Definition 3.8.3, is natural in this context.

Definition 4.5.1 *A* **generalized covariant representation** *of a C^*-dynamical system $(G, C_0(X), \alpha)$, where α arises from a continuous G-action on X by means of (3.110), consists of a pair $(U, \mathcal{Q})$, where U is a unitary representation of G on a Hilbert space $\mathcal{H}$, and $\mathcal{Q} : C_0(X) \to \mathfrak{B}(\mathcal{H})$ is a quantization of $C_0(X)$ (in the sense of Definition 4.2.1), which for all $x \in G$ and $\tilde{f} \in C_0(X)$ satisfies the covariance condition*

$$U(x)\mathcal{Q}(\tilde{f})U(x)^* = \mathcal{Q}(\alpha_x(\tilde{f})). \tag{4.28}$$

This condition may be equivalently stated in terms of the POVM $\Delta \to A(\Delta)$ associated to $\mathcal{Q}$ (cf. 4.2.3) by

$$U(x)A(\Delta)U(x)^{-1} = A(x\Delta). \tag{4.29}$$

Every (ordinary) covariant representation is evidently a generalized one as well, since a representation is a particular example of a quantization. A class of examples of truly generalized covariant representations arises as follows. Let $(U(G), \tilde{\pi}(C_0(G/H))$ be a covariant representation on a Hilbert space $\mathcal{K}$, and suppose that $U(G)$ is reducible. Pick a projection p in the commutant of $U(G)$; then $(pU(G), p\tilde{\pi}p)$ is a generalized covariant representation on $\mathcal{H} = p\mathcal{K}$. Of course, $(U, \tilde{\pi})$ is described by Theorem 3.12.1, and must be of the form $(U^\chi, \tilde{\pi}^\chi)$. This class actually turns out to exhaust all possibilities. What follows generalizes Theorem 3.12.1 to the case where the representation $\tilde{\pi}$ is replaced by a quantization $\mathcal{Q}$.

Theorem 4.5.2 *Let $(U(G), \mathcal{Q}(C_0(G/H)))$ be a generalized covariant representation of the C^*-dynamical system $(G, C_0(G/H), \alpha)$, defined with respect to the canonical G-action on G/H.*

There exists a unitary representation $U_\chi(H)$, with corresponding covariant representation $(U^\chi, \tilde{\pi}^\chi)$ of $(G, C_0(G/H), \alpha)$ on the Hilbert space $\mathcal{H}^\chi$, as described by 3.11.3, (3.151), and (3.152), and a projection p on $\mathcal{H}^\chi$ in the commutant of $U^\chi(G)$, such that $(pU^\chi(G), p\tilde{\pi}^\chi p)$ and $(U(G), \mathcal{Q}(C_0(G/H)))$ are equivalent.

We apply Theorem 2.11.2. To avoid confusion, we denote the Hilbert space $\mathcal{H}^\chi$ and the representation π^χ in Construction 2.11.3 by $\tilde{\mathcal{H}}^\chi$ and $\tilde{\pi}^\chi$, respectively; the space defined in 3.11.3 and the induced representation (3.151) will still be called $\mathcal{H}^\chi$ and π^χ, as in the formulation of the theorem above. Indeed, our goal is to show that $(\tilde{\pi}^\chi, \tilde{\mathcal{H}}^\chi)$ may be identified with $(\pi^\chi, \mathcal{H}^\chi)$. We identify $\mathfrak{B}$ in 2.11.2 and 2.11.3 with $\mathfrak{B}(\mathcal{H})$, where $\mathcal{H}$ is specified in 4.5.2; we therefore omit the representation π_χ occurring in 2.11.2 etc., putting $\mathcal{H}_\chi = \mathcal{H}$.

For $x \in G$ we define a linear map $\tilde{U}(x)$ on $C_0(G/H) \otimes \mathcal{H}$ by linear extension of

$$\tilde{U}(x) f \otimes \Psi := \alpha_x(f) \otimes U(x)\Psi. \tag{4.30}$$

Since $\alpha_x \circ \alpha_y = \alpha_{xy}$, and U is a representation, $\tilde{U}$ is clearly a G-action. Using the covariance condition (4.28) and the unitarity of $U(x)$, one verifies that

$$(\tilde{U}(x) f \otimes \Psi, \tilde{U}(x) g \otimes \Phi)_0^\chi = (f \otimes \Psi, g \otimes \Phi)_0^\chi, \tag{4.31}$$

where $(\, , \,)_0^\chi$ is defined in (2.123). Hence $\tilde{U}(G)$ quotients to a representation $\tilde{U}^\chi(G)$ on $\tilde{\mathcal{H}}^\chi$. Computing on $C_0(G/H) \otimes \mathcal{H}$ and then passing to the quotient, one checks that $(\tilde{U}^\chi, \tilde{\pi}^\chi)$ is a covariant representation on $\tilde{\mathcal{H}}^\chi$. By Theorem 3.12.1, this system must be of the form $(U^\chi, \tilde{\pi}^\chi)$ (up to unitary equivalence).

Finally, the projection p defined in 2.11.2 commutes with all $\tilde{U}^\chi(x)$. This is verified from (2.127), (2.128), and (4.28). The claim follows. ■

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