THE EVAN THOMAS INSTITUTE

MATH FOR MOTHERS

or

YOU Can Help YOUR Child LOVE MATH

BOOK II: Explaining the Cartesian Plane

by

Donald Grey Barnhouse, Jr.

MATH FOR MOTHERS

or
YOU Can Help YOUR Child LOVE MATH

Book 2: Exploring the Cartesian Plane

by
Donald Grey Barnhouse Jr.
in cooperation with
the Institutes for the Achievement of Human Potential

CONTENTS

INTRODUCTION

This book should enable any mother to understand the basics of the Cartesian Plane. When you feel comfortable with this material, you can lead your child to an understanding of this useful device.

The material we cover here introduces a branch of mathematics called Analytic Geometry. It is not usually taught in better schools until students are shown how to graph linear equations, late in the first year of Algebra. If you would like to get a first year algebra textbook as a supplement to this book, the better schools will often help you. They often have books they are no longer using, supplied to them free by state government. They are permitted by law to give them free to other schools, and your home school may be considered eligible by a friendly Math department. If you have a choice of books, as a general rule the oldest ones are better than the newer ones.

Some of the material presented here overlaps topics dealt with in Book 1 of this series. This is because we like a teaching method we call "spiral review." We do not leave a topic behind once it has been taught, never to teach to it again. Vital material is taught several times, each time at a somewhat higher level or with more detail. The treatment of negative numbers and of variables in this book, building on the teaching in Book 1, illustrates these principles.

Students in the Evan Thomas Institute, one of the Institutes for the Achievement of Human Potential, have used the Cartesian Plane for years, and have worked through the material in this book with pleasure and understanding. These are, of course, only the foothills of the great mountain ranges of Mathematics that use the Cartesian Plane, including both differential and integral calculus. As you begin your exploration of this fascinating territory, you will find that each question you put to your child may open up a merry mathematical adventure, for the land you are beginning to walk through has never yet been fully charted.

Have fun!

Chapter 1 -- HOW TO GET STARTED

A. MATERIALS NEEDED

1. This book (you're off to a good start!)

2. A square piece of pegboard 4 feet by 4 feet, available from a lumberyard, building supplies center, or hardware store. (By the way, notice that this board is FOUR FEET SQUARE, NOT FOUR SQUARE FEET! It is sixteen square feet. Four square feet would be two by two.

3. Two lengths of black adhesive plastic tape, 1/8th inch wide and four and a half feet long, to mark the horizontal and vertical axes.

4. Five dozen OR MORE golf tees (available from any sporting goods store) to mark points on the Cartesian Plane. Get AT LEAST eight of each color in as many clearly distinct colors as possible (8 red, 8 white, 8 blue, 8 yellow, etc.). Why skimp here? For a few pennies more you can get ten dozen. They don't take up much room, and they may give your child more fun and flexibility.

5. Ten pieces of rug yarn or knitting yarn, in lengths of four to six feet, of various colors, to connect points on the Cartesian Plane.

6. An Exacto knife, or similar very sharp pointed cutting tool.

7. A roll of masking tape for use in Chapter 4.

8. An algebra text is optional; it MAY be of help along the way.

B. MAKING THE CARTESIAN PLANE

1. Find and mark the hole closest to the center of the piece of pegboard. Stretch a length of adhesive tape across the pegboard **HORIZONTALLY** so that it covers the center hole, and all the other holes on that horizontal line. Make the tape go a bit OVER BOTH EDGES of the board.

2. Using some VERY sharp pointed tool, cut away the parts of the tape that are actually covering the holes, so that golf tees can be inserted in the holes. If the tool is sharp enough, the tape remaining between the holes will form a GOOD STRAIGHT LINE. That line is called "THE X-AXIS." Use the left over scrap of tape to make an "x", and put it on the board at the RIGHT HAND END of the x-axis.

3. Stretch the other length of adhesive tape across the pegboard **VERTICALLY**, covering the center hole and all the holes on the vertical line with it. Again, be sure that the tape goes OVER BOTH EDGES of the pegboard, symbolizing the fact that the axis extends infinitely in both directions, beyond the physical pegboard.

4. Again, cut away the parts of the tape which block the holes, leaving a GOOD STRAIGHT LINE. This line is called "THE Y-AXIS." Label it at its TOP END with a "y" made of the tape scraps. The point where the x-axis and the y-axis intersect is called "THE ORIGIN OF COORDINATES," or simply "THE ORIGIN."

C. MOTHER'S LESSON PREPARATION

The key point about the Cartesian Plane is this: it is **divided** and **marked**. It is divided by the two AXES (plural of AXIS), the lines that intersect at right angles. These lines divide the plane into four areas called QUADRANTS. The upper right area is called the FIRST QUADRANT; the upper left is called the SECOND QUADRANT; the lower left is called the THIRD QUADRANT; and the lower right is called the FOURTH QUADRANT. Theoretically, these all extend outward without limit; they are limited in the Cartesian Plane we construct because we can't get infinite pegboard, or a room to fit it in if we could get it. (Review the concept of the infinite as taught in Book One of this series.)

Notice that the quadrants are numbered in order consecutively if you move around COUNTER-CLOCKWISE from the first quadrant. It may help you and your child to understand this if you visualize a radar weather map, or one of the radar screens used in air traffic control. In both, an area is completely covered by a line which sweeps around a central point, coming back to its starting position and then making another full sweep.

Each axis is essentially a number line, and must be **marked** as such. The x-axis is marked off with the positive numerals running to the RIGHT from the origin, and the negative numerals to the LEFT. The y-axis is marked off with the positive numerals running UP from the origin, and the negative numerals DOWN.

It is a good idea with very young children to print the MATH LANGUAGE TERMS in red on word cards as you come to them, writing the definitions on the backs of the cards in black. Use these as you would any vocabulary words in your teaching program.

(If you are teaching a very young child who learns effectively from "Bits" cards, this is the time to prepare the Bits that show the parts f the Cartesian Plane. Mount the graphics provided with this book on 11" x 11" Bit cards, and paste the corresponding information, found in Appendix I at the back of this book, on to the back of each card.)

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

X-AXIS: the horizontal line on a Cartesian Plane
Y-AXIS: the vertical line on a Cartesian Plane
ORIGIN: the point where the x-axis and the y-axis intersect
FIRST QUADRANT: the upper right quadrant
SECOND QUADRANT: the upper left quadrant
THIRD QUADRANT: the lower left quadrant
FOURTH QUADRANT: the lower right quadrant

All these are very important definitions, used frequently.

Chapter 2 -- TWO KINDS OF NUMBERS

Before your child can learn from the Cartesian PLANE, there are two prerequisites. First, your child must have an understanding of two kinds of numbers which are used constantly in working with the Cartesian Plane: POSITIVE numbers and NEGATIVE numbers. Second, he should be able to handle SIMPLE problems of addition and subtraction with these two kinds of numbers. The development of these skills is covered in Book One in this series.

We also assume that your child has by now some intuitive concept of SPACE, and some idea of what is meant by a PLANE (a flat surface), a LINE (meaning a straight line), and a POINT. In Math language we call curved lines CURVES, and broken lines BROKEN LINES. That should not be too hard to remember. By a LINE we always mean a STRAIGHT LINE unless we specify otherwise. You can explain all this in any way that gets the basic ideas across, and we will refine the definitions later on when it becomes important to do so.

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

POSITIVE NUMBER: a number greater than zero
NEGATIVE NUMBER: a number less than zero
SPACE: something with three dimensions
PLANE: something with two dimensions, a flat surface
LINE: something with one dimension, a straight line
POINT: something with no dimensions, zero dimensions

All these are very important definitions, used frequently.

Chapter 3 -- THE "ONE-DIMENSIONAL CARTESIAN PLANE"

If you have Book One in this series, review the chapter on the NUMBER LINE. If you do not already have one, make one.

Here is an easy way to make one. It won't be permanent, but it will work. Lay out a piece of masking tape at least 10 feet long in a straight line on the floor. A line is NOT a plane, but a number line may be called a ONE-DIMENSIONAL VERSION of a Cartesian Plane." Mark one end with a LARGE RED "+" symbol and the other with a LARGE RED "-" symbol, to show the positive and negative directions.

Put black marks on ONE EDGE of the tape about nine inches apart for very young children, four to six inches apart for older children. On the edge of the tape OPPOSITE the marks, put numerals, but do not mark the negative numbers at first. Let them be a mystery to unfold later, something to intrigue your child. Label a mark near the center "0". Label the marks toward the positive end with the counting numbers: 1, 2, 3, 4, 5, 6, etc.

Instead of masking tape with marks on it, you can use heavy string with knots or tags to mark the numbers, if you wish. If you have a better idea, please tell us! The best will be flexible, portable, durable, and highly visible.

If your child asks about the interval between numerals, explain that it is "arbitrary," meaning it is just chosen at the pleasure of the chooser. This interval is our UNIT. It can be given any name you like, such as "a mamafoot," or "a daddypalm," or "a Joeyspan."

Point out how often people use the length of their walking stride, measuring distance in "paces." You could go on to explain something about the most commonly used units of distance in our world and their arbitrary origins. Your child might enjoy looking into the origins of "foot," "yard," "inch," "mile," "span," "cubit," "fathom," "ell," and "rod," and then into the very different origins of "meter."

You should also explain that everything we learn about numbers is true regardless of the units attached to the numbers. This is one of the reasons numbers are so useful. If 2 + 1 = 3, then two daddyshoes plus one daddyshoe make three daddyshoes, and this goes for parsecs, carlengths, light years, miles, etc. If your child does not ask about the choice of interval, this explanation of units can be safely left to another time and used for its own intrinsic interest.

Show your child a simple addition equation, like 1 + 2 = 3. To work out this equation on the number line, have your child stand on the "1" mark, FACING THE "+" END OF THE LINE. Facing the "+" end of the line means we are about to do an ADDITION. Now have him take 2 steps FORWARD. Repeat this procedure with many equations: 1 + 4 = 5, 5 + 2 = 7, 0 + 6 = 6, 2 + 0 = 2, 3 + 1 = 4, etc. Use every combination you can think of that will give an answer within the limits of the numerals you have marked on your number line.

Now demonstrate a subtraction problem, such as 5 - 3 = 2. Invite your child to stand on the "5" mark, FACING THE NEGATIVE END OF THE LINE. Facing the "-" end of the line means we are about to do a SUBTRACTION. Have him take 3 steps FORWARD. This will place him on the "2" mark. Repeat with other subtraction problems. Multi-step equations may now be presented, such as 5 - 3 + 4 = 6, being careful not to give a problem which would at any stage put your child on the negative side of the zero mark.

IN ALL THESE PROBLEMS YOUR CHILD SHOULD ALWAYS BE WALKING FORWARD, walking forward toward the "+" end for addition and walking forward toward the "-" end for subtraction. That is because we have been adding and subtracting POSITIVE numbers, and positive numbers move us FORWARD in whatever direction we face.

If your child has not already led you to the negative part of the number line by its own questions or suggested examples, ask it "What is 4 - 7?" (Read this as "four MINUS seven.") If your child has raised this issue, move on to this explanation as soon as it does. Where does this bring your child on the number line? Explain that on that side of "0" the marks have different names, representing NEGATIVE NUMBERS. Now label the marks on that end of the line. Practice with other similar problems: 1 - 2 = -1 (Read this as "one MINUS two equals NEGATIVE ONE"), 3 - 5 = -2, 6 - 9 = -3, etc.

Notice that the "-" symbol is read two ways: as "MINUS" when it indicates the operation of subtraction, and as "NEGATIVE" when it is the sign of a number. **Try to read (-2) as "NEGATIVE TWO," not "minus two." This is not worth an ulcer, but it IS helpful in the long run.**

Now introduce ADDING negative numbers, with a problem like 3 + (-2) = ? (Read as "Three plus negative two equals what?") SINCE YOU ARE ADDING, YOU FACE THE POSITIVE END OF THE LINE. The question is: "How do you take (-2) steps?" The answer is that since you walk FORWARD to take 2 steps, two positive steps, you walk BACKWARD to take two negative steps, (-2) steps.

To help establish this VERY IMPORTANT IDEA, practice several problems like 6 + (-6) = ?, 5 + (-5) = ?, 2 + (-2) = ?. This shows that a NEGATIVE NUMBER is the ADDITIVE INVERSE (not a vital term, but nice to know) of the corresponding positive number.

Applying this idea, look at the adding negative two to five, and subtracting positive two from five. 5 + (-2) makes you FACE THE "+" END of the number line and walk BACKWARD two steps, taking you to 3. 5 - 2 makes you FACE THE "-" END of the number line and walk FORWARD two steps, also taking you to 3. Do many of these, allowing your child to intuit the law that adding a negative number has the same effect as subtracting the positive number of the same absolute value. In mathematical notation 7 - 3 is the same as 7 + (-3).

Now introduce SUBTRACTING negative numbers, using a problem like 3 - (-2) = ?. You are SUBTRACTING, so you FACE THE "-" end of the line. A NEGATIVE number is controlling your motion, so you will walk BACKWARD. So your child will take two steps BACKWARD from the "3" mark, while facing the "-", bringing him to the "5" mark. Practice with other similar problems, like 1 - (-1) = 2, 2 - (-4) = 6, -5 - (-8) = 3, etc. Create and solve many such subtractions.

Now look at the relationship between two problems like these: adding positive two to three, and subtracting negative two from three. 3 + 2 makes you face the "+" end of the number line (addition) and walk FORWARD (positive number) two steps, taking you to 5. 3 - (-2) makes you face the "-" end of the number line (subtraction) and walk BACKWARD (negative number) two steps, also taking you to 5. Do several of these, allowing your child to intuit the general law that facing "-" and walking backward gets you to the same place as facing "+" and walking forward. In other words, subtracting a negative number has the same effect as adding the positive number of the same ABSOLUTE VALUE.

Finally, begin to mix up the operations of addition and subtraction using both positive and negative numbers. Remember: Add facing "+"; subtract facing "-"; positive numbers move you FORWARD; negative numbers move you BACKWARD. Your child will find it exciting to work up, gradually, to problems like this:

(-4) + 9 + (-7) - 1 - (-6) + (-8) + 2 - (-3) = ?

4 - 7 - (-8) + (-3) + 2 + (-9) - 1 - (-5) + 6 = ?

8 - 3 + (-7) + 1 - (-4) - 9 + 2 - (-8) + (-6) = ?

SUMMARY:

To ADD a POSITIVE number,	FACE "+" and walk FORWARD.
To ADD a NEGATIVE number,	FACE "+" and walk BACKWARD.
To SUBTRACT a POSITIVE number,	FACE "-" and walk FORWARD.
To SUBTRACT a NEGATIVE number,	FACE "-" and walk BACKWARD

All this **vital review of the basic operations of Math** relates to the Cartesian Plane in two ways. First, the two axes which divide the Cartesian Plane into four quadrants are really two number lines. The x-axis is a number line running from left to right (the "-" end at the left and the "+" end at the right), and the y-axis is a number line running from bottom to top (the "-" end at the bottom and the "+" end at the top). Second, you and your child will be doing a good bit of simple addition, subtraction, and multiplication, using both positive and negative numbers, as you learn the uses of the Cartesian Plane.

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

ABSOLUTE VALUE:

how big the number is, whether it is positive or negative
or, how far the number lies from zero on the number line

ADDITIVE INVERSE:

a number with the same absolute value, but opposite in sign
or, a number the same size, but with a different sign

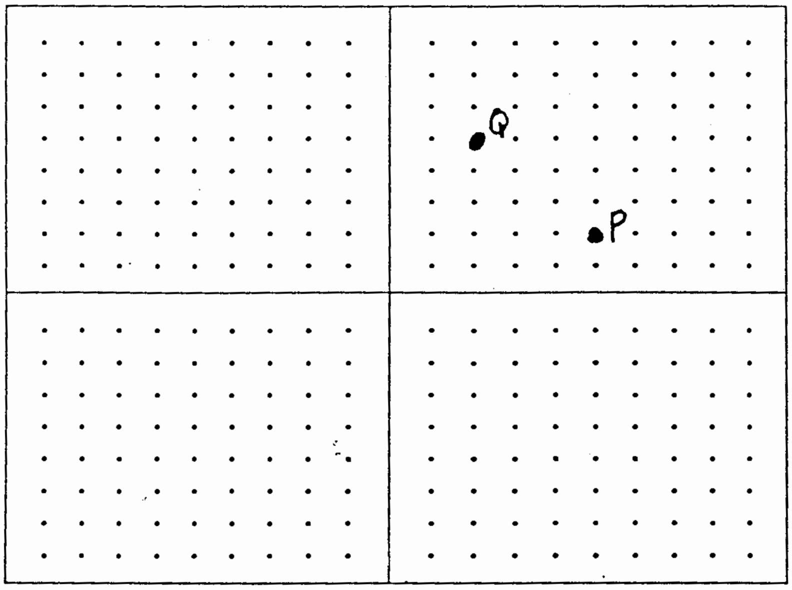
These are NOT very commonly used, but they are useful to know.

Chapter 4 -- THE TWO-DIMENSIONAL CARTESIAN PLANE

Identifying points in the Cartesian Plane is almost as simple as identifying points on a number line. First, count how far you have to move HORIZONTALLY FROM THE ORIGIN to get directly under the point, or over it; then count how far you move VERTICALLY from the x-axis to get to it. These two numbers are called the COORDINATES of the point. This explains why the FULL name of the point we call the "ORIGIN" is the "ORIGIN OF COORDINATES." It is the point from which we start measuring the coordinates. It is the double-zero point.

You can see how useful it is in working with the Cartesian Plane to have paper printed with a grid pattern on it. We call this kind of paper "graph paper," or "squared paper." It helps us count up or down from a point on the x-axis without drifting to the left or the right. Some packages of business forms give you samples of such paper, with different sized squares, which you can photocopy for your own use. Or you can sometimes find such paper for sale marked off in millimeters, in eighths of an inch, or in tenths of an inch.

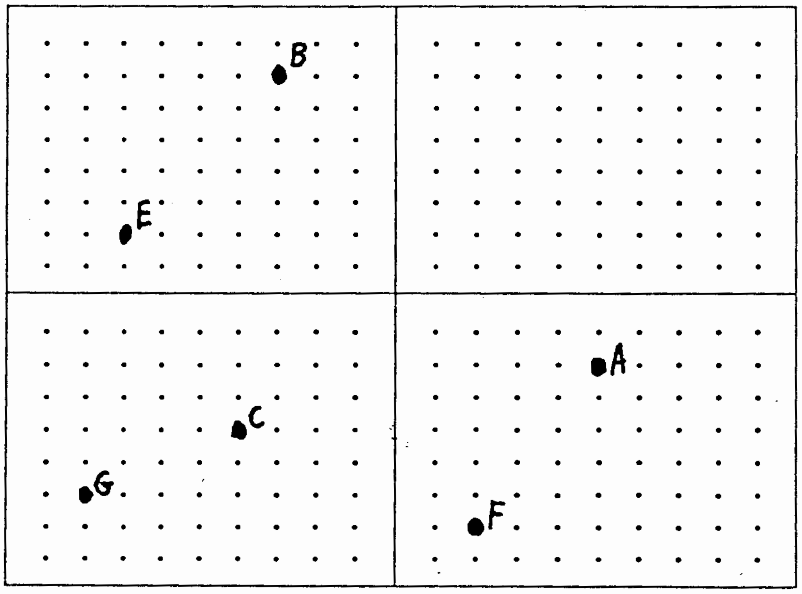
In the diagram below we move from the origin toward the "+" end of the x-axis five units, then up two units parallel to the y-axis, and we get to the point labeled P. We therefore say that the coordinates of the point P are (5,2). This is what is called an ORDERED PAIR of numbers, because the order is significant. The point with coordinates (2,5) would be reached by starting at the origin and moving two units toward the "+" end of the x-axis, and then up five units parallel to the y-axis. That is marked as point Q below, and is obviously a different point.



Five is the x-coordinate of P, and two is its y-coordinate. By convention, the x-coordinate is always listed first, and always read or spoken first. We read the coordinates of the point P as "five, two," and we read the coordinates of the point Q as "two, five." In writing, we then describe the point P simply as "the point (5,2)." When we are reading that aloud, or talking of a point on the plane, we obviously do not put in punctuation, and so you will start speaking of point P as "the point five two." You would speak of point Q as "the point two five." This has nothing to do with decimals, and you have to recognize that from the context of the statement.

What do you suppose the coordinates of the ORIGIN are? See if your child comes up with the right answer, (0,0), on its own.

When we describe points in any quadrant other than the first, we will need to use negative numbers. Again, your child may well lead the way on this if you give it a chance. Consider the point (5,-2). Where would you locate it on the diagram of the Cartesian Plane below? How about the point (-3,7)? How about the point (-4,-4)?



The answers are points A, B, and C. How would you write, and say aloud, the coordinates of point E? Point F? Point G? The answers are, in order, (-7,2), (2,-7), and (-8,-6).

Now you are ready to go over all this material with your child, using golf tees in the holes of your pegboard rather than points on diagrams in this book. Make up your own examples as soon as you feel comfortable doing so, making sure that you illustrate what happens

a) when you interchange the numbers in the ordered pair,
b) when you change the sign of just the x-coordinate,
c) when you change the sign of just the y-coordinate,
d) when you change the signs of both coordinates, and
e) when one of the coordinates is zero.

Let your child pick points and quiz you about the coordinates. Also, let your child suggest coordinates and check on you to see if you can put a golf tee in the right spot.

Spend as much time on this kind of exercise as is needed to make it become easy and quick for your child to do.

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

ORDERED PAIR: A pair of numbers whose order is significant

COORDINATES OF A POINT:
The ordered pair which identifies that point in a plane

POSITIVE DIRECTION ON THE X-AXIS: to the right

POSITIVE DIRECTION ON THE Y-AXIS: up

All these are very important definitions, used frequently.

BEHIND THE SCENES

The name "Cartesian" comes from the name of the great 17th century French mathematician, Rene Descartes. But the basic idea of identifying points by coordinates goes back to the ancient Greek mathematicians. A 14th century French mathematician, Nicole Oresme, developed the idea of coordinates further, calling them longitude and latitude. But the work of Descartes, published in 1637, is what inspired the full use of the concept of coordinates in a plane.

It is said that Descartes got his earliest inspiration for his system of coordinates from staring at the ceiling during a period of prolonged bed rest, thinking up a system for identifying the position of a fly which kept walking around up there over his head.

Certainly his method of identifying points in a plane, by giving each point a unique ordered pair of numbers, opened the door to using all the tools of arithmetic and algebra in dealing with the subject of geometry.

Even so, Descartes never saw anything like what we commonly call graph paper. It was long after his time that the familiar grid system came into use.

Chapter 5 -- VARIABLES

Did you realize that you can talk quite sensibly, and accurately, about unknown quantities? You can. For example, if I tell you that my daughter Joanna was born just three years after my daughter Lucy, you can say something very sensible and accurate about their ages today, even though you don't know how old either one of them is! You can say "Lucy is three years older than Joanna." In the language of Math, we can let L stand for the number of years in Lucy's age and J for the number of years in Joanna's age, and then write:

$$L = J + 3$$

A teenager gets $25 a month allowance, and is trying to save money by spending less than that. We can then say: "The number of dollars he may spend in a month is 25 minus the number of dollars he decides to put in a savings account." Let x stand for the number of $ he saves, and y for the number of $ he spends. We can then write:

$$y = 25 - x$$

The ages of my daughters are changing, and so is the amount which the teenager spends month after month. The symbols we use in the language of math to describe such changing quantities are called "VARIABLES." A variable can represent different numbers as a situation changes. This is among the most important concepts in Math.

We can put it in the form of a rule, and this rule is vital:

VARIABLES MUST REPRESENT NUMBERS

Variables do NOT represent girls, but the number of years in the ages of girls, or the number of inches in their height, or the number of pounds in their weight, and so on. In the examples that follow, please notice that the variables do NOT represent juice or men or fathers or anything like that; they represent the number of some kind of unit which measures something: ounces, millimeters, years, days, degrees, tons, parsecs, miles, fathoms, etc.

This is something for the MOTHER to keep firmly in mind, NOT to lecture the child about. If the teacher is careful to remember this truth about variables, never forgetting, then the student will naturally pick it up and develop the right habit. NEVER SAY "What shall we let the variable stand for?" ALWAYS SAY "What number shall we let the variable stand for?"

How many ounces of pineapple juice do we need to fill a one quart jar of summer punch if we have already put in some guava juice? If we let p represent the number of ounces of pineapple juice we will need, and if we let g represent the number of ounces of guava juice we have already put in, then we can say

$$p = 32 - g$$

Since Gulliver traveled to Lilliput, the people there eat better and exercise more. Now every man there is six millimeters taller than his father. If we let M equal the number of millimeters in a man's height, and let F equal the number of millimeters in his father's height, then we can say

$$M = F + 6$$

This is such a vital concept that we need to present it over and over again. Try to get your child to discuss it with you. Invite him to help you think up other examples, from either fact or fantasy, of situations where we can talk quite accurately about unknown quantities which can change. Notice that we talk about a variable representing a number, standing for a number, or being equal to a number. All these phrases are equivalent, and all are frequently used in Math.

The next step is one your child may have taken already. Even before it puts its mind to work on thinking up other situations where we can write equations using variables, it may have suggested possible values that would fit the sample equations we have presented. For example, it may have said "Lucy could be 7 and Joanna could be 4." Give it lavish praise if it does this! And give it wild hysterical praise if it goes on to suggest more than one set of values, by saying something like "OR, Lucy might be 8 and Joanna might be 5."

Once a child says "OR, ..." in just that tone of voice, it shows that it has grasped the essential idea of VARIABLES: they are symbols which can stand for different numbers within the conditions that we set for them. **Point your conversations toward such a moment!**

The symbol 3, or three, always stands for the same number; but the symbol L, or x, or y, may stand for different numbers under different conditions. Don't be in a hurry to EXPLAIN this! Take time! Fish! It can be a fascinating game for you as the teacher, one through which you may learn a lot about how your child's mind works.

If your child has not spontaneously come to this insight, don't be concerned! Mine didn't. After all, this took the brilliant Greek mathematicians centuries to figure out. Just keep working and try to fish for that moment of comprehension with suggestive questions.

The relationship of two children's ages is ideal for this, and will probably work better than the examples about money or mixing punch. "Could we write a math sentence about your age and Betsy's?" "What about Ryan and Timmy?" "How old was Mark when Stephanie had her fifth birthday party? How old will Mark be when she has her sixth birthday party?" "Can we write some math sentences about the ages of these friends?"

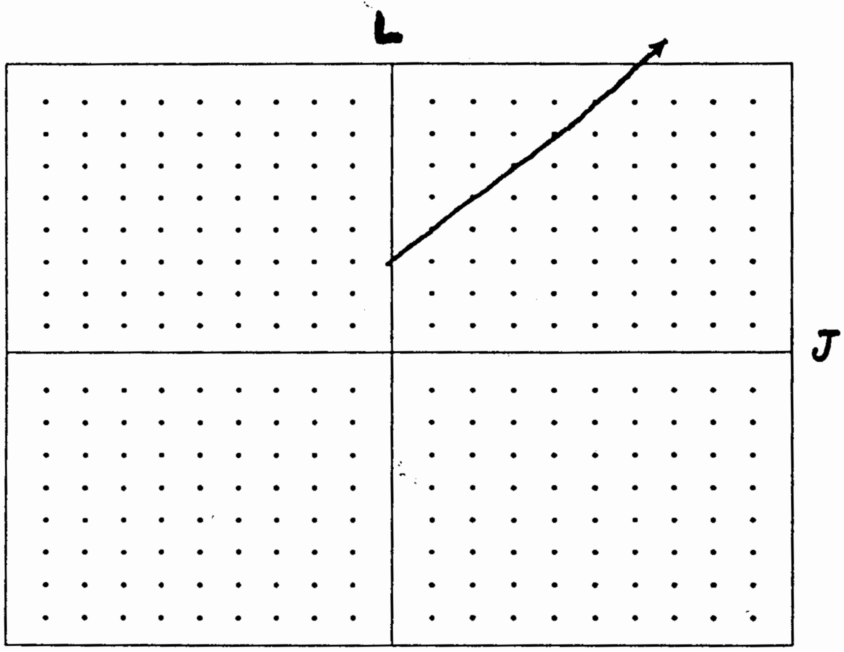
When the insight has come, here is what we do next. Pick one of these equations, such as L = J + 3, and work out some of the values which will fit. The variable for which we CHOOSE values is called the "INDEPENDENT VARIABLE." It will be plotted on the x-axis. The one for which we then SOLVE for values is called the "DEPENDENT VARIABLE." It will be plotted on the y-axis.

Let's decide to choose possible ages for Joanna and solve for the age Lucy must be at that time. In words, we will say "If Joanna is ___ years old, then Lucy must be ___ years old." So we will use the x-axis as the J-axis and the y-axis as the L-axis. (We could just as easily have done it the other way around in this case, and in most of the cases we will be dealing with for quite a while.)

What we will get is a series of pairs of numbers, and this is the form in which we usually list them:

J	L
1	4
2	5
3	6
4	7
5	8
6	9
7	10
8	11
9	12
10	13
15	18
20	23

Each ordered pair in the list is called a "TRUTH SET" for that equation, a set of values for the variables in an equation which make the equation true. The usefulness of the Cartesian plane lies in the fact that each ordered pair can then be considered as the coordinates of a point, a point which can then be plotted on the plane. So get out the golf tees (all one color) and go to work.



Now you can use some of your yarn, looping it around both the first and the last golf tee. You should end up with a straight line.

Here are the steps in what we did.

1) We CHOSE VARIABLES, deciding on letters we wanted to use to represent numbers, numbers which measured SOMETHING GIVEN or SOMETHING ASKED FOR in the situation.

2) We translated the information given us from English into Math language, FORMING A RELATIONSHIP, AN EQUATION.

3) From the equation we got A LIST OF ORDERED PAIRS, by picking values for one variable (the independent variable) and calculating the corresponding values for the other variable (the dependent variable) from the equation.

4) Then, on our Cartesian Plane, we PLOTTED THE POINTS determined by the ordered pairs.

This process is called GRAPHING. An equation which gives us a straight line is called a LINEAR EQUATION. That's a piece of Math language that should be easy to remember.

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

VARIABLES: Symbols to represent numbers which measure changing quantities

TRUTH SET: A set of values for the variables in an equation which make the equation true

INDEPENDENT VARIABLE: The variable for which we choose values when we are listing the truth sets

DEPENDENT VARIABLE: The variable for which we calculate values when we are listing the truth sets

GRAPHING: Finding truth sets for an equation and then plotting the ordered pairs on a Cartesian Plane

LINEAR EQUATION: An equation whose graph is a straight line

Chapter 6 -- GRAPHING LINEAR EQUATIONS

What you learned to do together in the last chapter is called "GRAPHING" an equation. An equation from which we get a straight line, by finding truth sets and then plotting the ordered pairs, is called a LINEAR EQUATION.

Now invite your child to see some of the uses of the line you have made together. You can read off sets of values which you had not figured out, such as Lucy is 9 when Joanna is 6. You can also see that Lucy is eight and a half when Joanna is five and a half. Can you also see that the relation between their ages goes on past the points you can fit on your pegboard? Can you see that Lucy will be 23 when Joanna is 20? Don't press these points, but just touching on them now will lay a foundation for when we deal in greater depth with the solution of linear equations.

You are now ready to make lists of truth sets for some other equations, those in the last chapter and some more of your own. Use the ordered pairs to plot points on your Cartesian Plane, using golf tees of a different color for each equation and yarn of the same color as the tees (or close) to form the straight lines you have graphed.

It is just as true in math as in learning to play the piano that practice makes perfect.

Chapter 7 -- GETTING OUT OF THE FIRST QUADRANT

The equations we have graphed so far have been far from typical. The situations described by the equations have all been such that all the truth sets involved only positive numbers and zero. This has meant that all our ordered pairs led us to points which were either in the first quadrant or on one of the axes.

There are some real situations, however, which lead us readily to the use of negative numbers. Let's start by thinking about the weather. A cold front is moving steadily down from Canada in such a way that the temperature, which stands right now at 0 degrees, has been dropping steadily by one degree every hour. What was the temperature 24 hours ago? What will it be 24 hours from now?

First we must choose variables and set up the equation. Let T stand for the number of degrees in the temperature, and let h stand for the number of hours measured from now. In English, the facts we are given could be stated this way: "The temperature is zero degrees, minus the number of hours that have passed." In Math language:

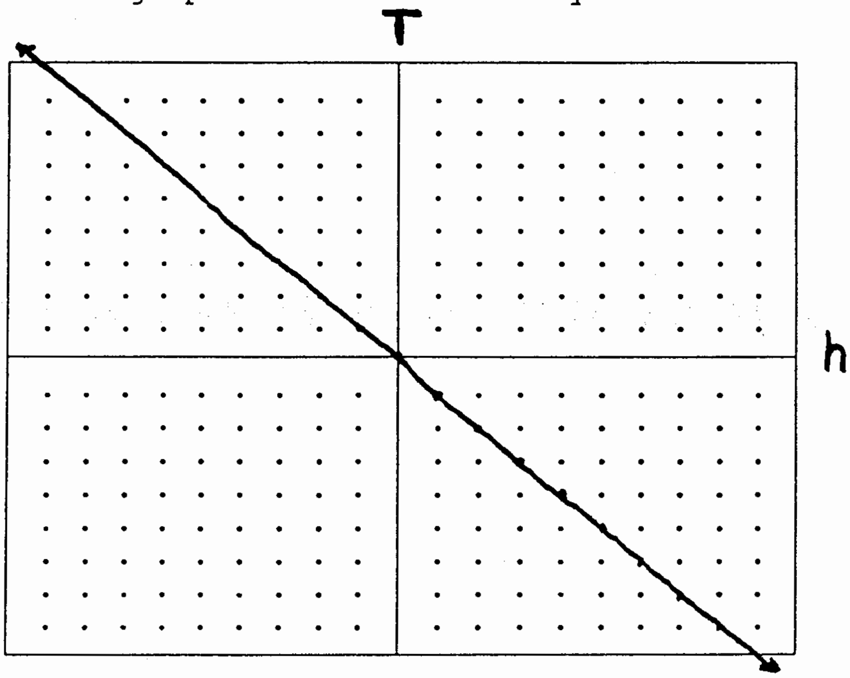
$$T = 0 - h$$

Again let's list the values which will make the equation true, a list of truth sets, a list of ordered pairs which tell us what the temperature was or will be within the range which our graph can reach on our Cartesian Plane. But this time, negative values of both time and temperature are realistic, and of interest. Negative values of time cover the past.

h	T
-24	24
-18	18
-12	12
-6	6
0	0
6	-6
12	-12
18	-18
24	-24

To get the truth sets, we picked an arbitrary value for h, either a value of special interest to us or a value chosen to make our work easier, and put it in the equation to see what T would come out to be for that value of h. For example, if h = -24, T = 0 - (-24). That gives us T = 24. So our first ordered pair is (-24,24).

We calculate the other truth sets in the same way. Each gives us an ordered pair to enter in our list. Each ordered pair represents a point to plot on our graph of this linear equation.



Did you plot the graph using the x-axis as the h-axis, and the y-axis as the T-axis? That is the proper way of doing it, since we chose the values for h and solved for values of T. In this equation, h was the independent variable, and T was the dependent variable. The temperature depended on the time.

Let's graph another equation which involves negative values of the variables. A young doctor who took out student loans to pay for her education is still in debt, owing $4,000. From her earnings, she puts $3,000 a year into a savings account from which she has been paying off the debt. When will her savings account show a positive balance of $8,000 with which to buy a good used car? What if she wants to wait until she has $23,000 for a new luxury car? How many years has she been paying on the debt if it was $22,000 to start with?

Again we must choose some variables and set up the equation. Let W stand for the number of thousands dollars in her net worth. If we let it stand for the number of dollars, we would have a hard time graphing our truth sets! This is a good time to review the nature of variables, and the arbitrary nature of units. Now let n stand for the number of years measured from right now. In English, the facts we are given could be simply stated this way: "Her net worth is negative $4,000 plus $3,000 times the number of years that pass." In math we put it this way:

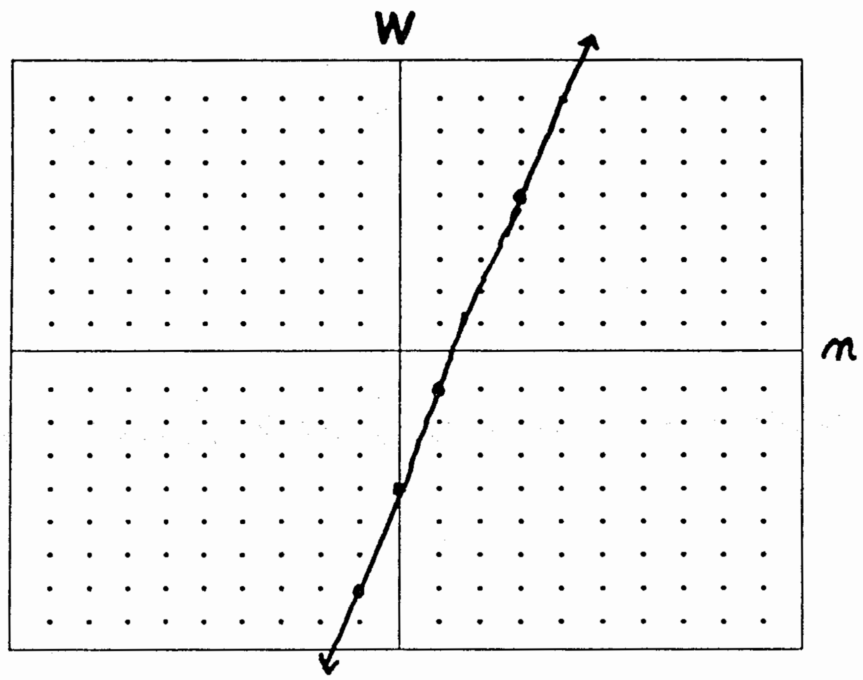
$$W = -4 + 3n$$

Again let's list the values which will make the equation true, a list of truth sets, a list of ordered pairs which tell us what the young doctor's net worth will be within the range which our graph can reach on our Cartesian Plane. Again this time negative values of both variables are useful. Try to let your child discover that negative net worth is debt, that positive net worth is savings, that negative time is the past, and that positive time is the future. This time let's demonstrate that you don't need so many points to get a graph.

n	W
0	-4
1	-1
3	5
5	11
-1	-7
-5	-19

This equation is a good example of one where it doesn't matter which variable we call independent. The calculations are easier if we choose values for n and solve for W than if we do it the other way around, and so we do it the way that makes it easier for us.

Also, have you noticed that we don't need all these points to determine the graph? You're right. But plotting them is good practice.



Now we can use the graph to read off the answers we were asked for in the question (word problem). When will she have $8,000 to buy that good used car? We can see from the line we have graphed that W will be 8 when n = 4. When was the debt $22,000? That is another way of asking when her net worth was NEGATIVE 22,000 dollars. We can see from the graph that W = -22 when n = -6. So she has been paying off this debt for six years already. When will she have $23,000 for that new luxury car she wants? We have to look for the value of n when W has grown to 23, and we find that n = 9. Nine years from now she will be able to buy that car for cash if she keeps saving at this rate. My guess is she'll borrow the money and buy it sooner; what do you think?

Now you see that we have found how useful the Cartesian Plane can be in solving word problems. We find an equation, graph it, and read off the answers we need.

BY THE WAY: If this problem, or any other problem or example you find here, seems inappropriate for your child -- too complex, too easy, too sophisticated, or just out of tune with your framework of orientation and devotion -- feel free to make up something similar that suits you and your child better. Or just skip it and go on. It will be all right!

Chapter 8 - FUNCTIONS

We are now ready to introduce one of the key concepts in Math: the concept of a "FUNCTION."

We have already been working with functions for several chapters, and from a list of the examples of FUNCTIONS you have seen you can get a pretty good idea of what a function IS. Look back at the equations we have been working with. We represented Lucy's age as a FUNCTION of Joanna's age. Temperature was represented as a FUNCTION of the hours passing as the cold front went through. The number of dollars the teenager had left over to spend in any month was a FUNCTION of what it had put in a savings account. The number of ounces of pineapple juice we had to add was a FUNCTION of the number of ounces of guava juice we had already put in; the state of the young doctor's finances was a FUNCTION of the number of years since she finished school and started setting aside a fixed amount each year to pay off her debts.

Encourage your child to try, from this information, to formulate a definition of "FUNCTION." There will almost certainly be value in whatever you or your child may come up with. It is almost certain that nothing you or your child suggest will be "wrong." Look for the value in your own ideas.

Math teachers have many different approaches to teaching this concept. One of the simplest is to say: "A FUNCTION IS A RULE." This is good because it is easy to remember, but it is really clear only after some illustrations like those above.

Another useful teaching approach is to speak of the FUNCTION as a MACHINE, a machine into which you stuff a value of the INDEPENDENT VARIABLE, and from which you then crank out a value of the DEPENDENT VARIABLE.

A combination of the two approaches is this definition: "A FUNCTION is a RULE for turning any value of the independent variable into a corresponding value of the dependent variable." That's pretty complete, and pretty clear if enough illustrations are given along with it.

Let's look at some function rules, expressed in both English and math language, to make sure that we have an understanding of what a function is.

1. Multiply the number of dollars Joe has in his piggy bank by three and you get seven dollars more than Mary has: $3j = m + 7$

2. Double the number of dots on Willy's card, and then add five, and you get the number of dots on Yolanda's card: $y = 2w + 5$

3. Professor Wild says that in an old oak forest you will always have six times as many squirrels as there are trees: $s = 6t$

Because of the actual situations described, these functions can only be graphed in the first quadrant, where values of both variables are positive. It will be good practice to graph them, and we will take another look at them in the next chapter as we look at a new use for the Cartesian Plane.

Now let's look at functions where the values of both variables may be either positive or negative, and the English translation of the rule still sounds like Math language. Here are four such functions, and here is the English tranlation of just the first: "One number is equal to three less than four times another number."

4. $y = 4x - 3$

5. $y = 2x + 5$

6. $2y = x - 8$

7. $3y = x$

Take this chapter slowly. Have your child try to state the FUNCTION RULE described by each equation. Find several TRUTH SETS for each equation. GRAPH each equation. If the graphs are not straight lines, check your work. If you feel the least bit shaky about anything, go back and review earlier chapters. When you are confident and comfortable, make up equations of your own with y as a function of x. Then for each of them state the FUNCTION RULE, find several TRUTH SETS, and GRAPH them. Let your child show off to friends.

And now, just because it is occasionally healthy to throw SOME children a curve, there are a couple of mild curves in problems 6. and 7. above. And if you have a hotshot that needs to be cooled down a little, ask him to graph the area of a square as a function of the length of the side of the square. Aha! He'll have to be patient and take in some more Math before we can explain that one fully.

Other children need the sense of total mastery at all stages of the learning process, and it might be discouraging to them to have this thrown at them. Use your judgment as a parent, remembering that a parent always has the potential of being the best teacher. **Modify or skip any example or problem that doesn't fit you or your child.**

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

FUNCTION: A RULE for turning any value of an independent variable into a corresponding value of a dependent variable.

Chapter 9 -- DIVISION REVIEWED

Before learning more about linear equations, we need to review briefly division and fractions. Our review of division must start with a review of the definition of multiplication:

MULTIPLICATION IS REPEATED ADDITION.

You can hear it when we talk about multiplication in English. We say "three times six," and that means simply "take six three times." It means three sixes added together. Students often learn to write this as 3 X 6 before they start using letters to represent variables. After that we use two other ways of indicating multiplication. One is a dot between the two quantities. Another is just to put them next to each other UNLESS they are both numerals. For example, xy means x times y, and 3t means 3 times t.

One special case of this juxtaposition is when we put quantities in parentheses, and then put the parentheses next to each other. That means that the quantities inside the parentheses are to be multiplied together. So (3)(6) is another way of writing three times six in Math language.

$$(3)(6) = 6 + 6 + 6 = 18$$

(Why not use 36 to mean three times six? Because that already has a meaning: three tens added to six ones, or thirty six.)

Similarly, (6 - 2)(8 - 5) means 4 times 3.

Now we are ready to review the operation called DIVISION. Ask yourself, and your child, how many TIMES can you fit 6 into 18? This is called DIVIDING 18 by 6, and, as you can easily see, the answer is three. This explains why we often call multiplication and division INVERSE OPERATIONS.

We write divisions in Math language in three different ways. Two are used when we need to put the whole operation on one line. 18 divided by 6 equals 3 is written as

$18/6 = 3$ or $18 \div 6 = 3$

Ask your child to make up many examples like the following, showing the inverse relationship of multiplication and division:

Since (2)(5) = 10, then 10/2 = 5, and 10/5 = 2
Since (3)(7) = 21, then 21/3 = 7, and 21/7 = 3
Since (4)(6) = 24, then 24/4 = 6, and 24/6 = 4
Since (5)(3) = 15, then 15/5 = 3, and 15/3 = 5

A question which will soon come up, if your child has not raised it already, is this: What happens when you try to fit a larger number into a smaller one, such as 7 into 3, or 6 into 1?

The first thing we must remember is how to write these quantities in Math language. We write them as 3/7 and 1/6, and we read them as "three divided by seven" or "three sevenths," and "one divided by six" or "one sixth." When we learned division, we called these expressions "fractions."

Review now with your child the other way of writing fractions, with the fraction line horizontal instead of slanted, and the first number on top with the second number on the bottom. Point out that the fraction line has the very same function, whether it is written horizontally or slanted: it separates the DIVISOR from the DIVIDEND.

When we write fractions using letters, which are representing variables, y divided by x is written y/x, and is usually read as "y over x." We do not read it as "y xths" because that is awkward to pronounce. The use of the word "over" comes very naturally from the fact that when you write the fraction the first number is, literally, over the second, above the fraction line while the second is under it.

NOW, FINALLY, you and your child are ready to get a vivid picture of the reality of what fractions really mean. We'll begin in the next chapter, by asking you to think about either pizza or sticks. Just be sure not to eat the sticks, or build your house out of pizza!

Before we close this chapter, notice that we may use fractions to write the equations in problems 6. and 7. from the last chapter in somewhat different form, a form that might make it a little clearer, or a little easier, for us to figure out the truth sets we need to graph the equations. Instead of writing them as we did there,

6. $2y = x - 8$

7. $3y = x$

we could write them this way:

6. $y = x/2 - 4$

7. $y = x/3$

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

DIVISOR That part of a fraction doing the dividing

DIVIDEND That part of a fraction being divided

Chapter 10 -- SOLVING SIMPLE LINEAR EQUATIONS

It may not have been quite clear to you how we transformed those two equations at the end of the last chapter. We were building on the definition of an equation, as found in Book One of this series:

AN EQUATION IS A BALANCE

Because an equation is a balance, we may do anything we want to either side of the equation as long as we do exactly the same thing to the other side, since the two sides will still be in balance. For example, we may ADD the same thing to both sides. We may subtract the same thing from both sides. We may multiply both sides by the same quantity. We may divide both sides by the same quantity. That's not all, but it is enough for now.

What we did at the end of the last chapter when we had these two equations:

$2y = x - 8$ and $3y = x$

was to divide both sides of the first by two, and to divide both sides of the second by three. That gave us:

$y = x/2 - 4$ and $y = x/3$

(Notice that in dividing the right hand side of the first equation by two, we had to divide both terms by two. Just as if you had ten toys and four dresses to divide between two girls, you would first divide the ten toys in half, and then divide the four dresses by two also.)

Now that we know a bit more about division, this enables us to solve simple linear equations. Practice on these, trying to find the value of the unknown (the quantity represented by a variable).

1. $x - 8 = 0$
2. $y - 7 = 3$
3. $x + 5 = 9$
4. $y + 1 = 16$

In the first, you add eight to both sides, and get $x = 8$. In the second, you add seven to both sides and get $y = 10$. In the third you add negative five to both sides (or subtract five from both sides) and you get $x = 4$. In the fourth you add negative one to both sides (or subtract one from both sides) and you get $y = 15$. Here are more:

5. $3x = 15$
6. $2y = 6$
7. $5x - 3 = 17$
8. $3y - 5 = y + 9$

In the first, you divide both sides of the equation by three (fit three into both sides) and you get $x = 5$. In the second, you divide both sides by two (fit two into both sides) and get $y = 3$. The third is a tiny bit more complex. You first add three to both sides and get $5x = 20$. Then you divide both sides by 5 and get $x = 4$. The fourth is again a hair more advanced. First you add five to both sides and get $3y = y + 14$. Then you add negative y to both sides (subtract y from both sides) and get $2y = 14$. Finally you divide both sides by seven (fit seven into both sides) and get $y = 2$.

Invite your child to help you make up more examples, not making them much harder than these unless he is sailing along through this material. AGAIN, WE REPEAT: IF YOU CHILD IS NOT SAILING ALONG, BACK UP (WITHOUT ANY BLAME ON EITHER OF YOU) AND TALK IT THROUGH UNTIL YOU ARE VERY COMFORTABLE AND CONFIDENT WITH ALL WE ARE DOING. THAT IS A VITAL GENERAL RULE THROUGHOUT THESE BOOKS.

Chapter 11 -- MORE ABOUT FRACTIONS

When a family orders a big pizza, the person who serves it often cuts it into pieces. The first two cuts are usually like the x-axis and the y-axis of a Cartesian Plane, dividing the pizza into four pieces. We call each of those pieces a fourth of the pizza, or a quarter, or a quadrant, and the words "quarter" and "quadrant" both come from the Latin word for four. Notice also that 1/4 of a dollar is called a quarter. Ask your child why we do that. If it thinks the answer is obvious, great. It is. But it is also vital; and if this step is skipped a child's understanding of fractions may come hard.

Similarly a long stick may be broken into equal pieces in many ways: two equal pieces, or three, or four or five or six or twenty. The word "fraction," like the word "fracture," comes from the Latin word for broken.

The key fact here is that we name fractions by the number beneath the fraction line. That number is called the DENOMINATOR of the fraction, and the word denominator comes from the Latin word for "name." Remember that when we speak of someone being "nominated" to a position, or "named" to a position, we usually mean almost exactly the same thing.

Let's look at some fractions and their names. Cover up the column showing the names, if you like, and ask your child to name these fractions:

1/4	one fourth
1/6	one sixth

1/7	one seventh
1/8	one eighth
1/10	one tenth
1/12	one twelfth
1/20	one twentieth
1/100	one hundredth
1/1000	one thousandth

The point of this exercise is to emphasize the fact that the number UNDER the fraction line, the DENOMINATOR, NAMES the KIND of fraction it is.

If the person serving your pizza runs the pizza cutter through it only twice, with cuts like the x-axis and the y-axis of the Cartesian Plane, you have four pieces, and each piece is one fourth of the pizza.

If the waiter then makes two more long cuts, cutting once through each quadrant, you have eight pieces, and each piece is called one eighth.

If instead of that he cuts TWICE through each quadrant (four extra cuts after the first two), carefully, then you have twelve equal pieces, each one called one twelfth.

Can your child sketch each of these situations?

Now we are ready to look at the number ABOVE the fraction line. It is called the NUMERATOR, and it COUNTS the pieces. Your child should remember how to read the following fractions, and again you can start by covering the right hand column, then write and read many more examples.

2/7	two sevenths
2/8	two eighths
3/8	three eighths

4/8	four eighths
5/8	five eighths
6/8	six eighths
3/10	three tenths
5/10	five tenths
5/12	five twelfths

There is one more thing we want you to notice at this level of learning about fractions. You and your child may have already noticed it as you have thought together about sticks and pizza, and cherry pie: two (or more) different fractions can represent the same real quantity.

2/4	is the same as	1/2
4/8	is the same as	1/2
5/10	is the same as	1/2
2/6	is the same as	1/3
3/12	is the same as	1/4
2/8	is the same as	1/4

Spend a LOT of time on this. Think of more examples, and let your child offer ideas to explain them. Work with actual sticks, or pies, or pizzas, if possible. Otherwise be sure to use drawings for visualizing. We'll come back to fractions as our studies go on.

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

DENOMINATOR: the number below the fraction line, the name of the fraction, naming the kind of piece the fraction is

NUMERATOR: the number above the fraction line, counting how many of this kind of piece are in this particular fraction

Chapter 12 -- SLOPE

As in all languages, the terms in the language of Math are often quite arbitrary. You have seen that none of the equations we have used as examples have turned into perfectly horizontal lines, or perfectly vertical lines, when you graphed them. They have all had something which we might call a slant, or a slope, or a tilt, or an angle. Some have been very steep, and some have not. A Math term was needed to describe this quality in the graphs of linear equations, and the term chosen was "SLOPE." Slant or tilt would have done just as well, logically, but SLOPE is what was chosen and what is used.

It may not be very logical that the English words for a thing we can fly to Europe in (plane) and a tool for getting rid of bumps on boards (plane) are the same. But that's the way it is, and that's the way Russians and French and Chinese have to learn it if they want to understand us and have us understand them. Similarly, there is no way of avoiding the necessity of learning and remembering the terms in the language of Mathematics.

SLOPE is a particularly important Math language term, since it is used all the way up through Calculus and Solid Analytical Geometry. Its meaning is quite easy to grasp, however, since it means simply what it means in English: it is a measure of the direction in which the line is going.

The Cartesian Plane makes it very easy to define a way of measuring this direction, or steepness, accurately. This is a good point at which to stop and ask your child how it thinks this measurement of SLOPE might be done. Your child will probably have some ideas about this, and most of them will probably be very sound. Explore them, and refine them as much as possible before going on to teach the "standard" way of measuring slope.

When you come to teaching the "standard" way, try to relate it as much as possible to the method or ideas you and your child thought of. This will almost certainly be feasible. The standard method of measuring slope is to look at the graph of a linear equation and think of it as the path of a point, moving left to right across the Cartesian Plane. We consider any two points on its path and look at how far the point moved from left to right, and how far it moved up or down. The distance moved from left to right we call the "RUN," and the movement up or down we call the "RISE." What if it was a fall instead of a rise? We still speak of the "rise," but it is then a negative number.

The next step is just as reasonable. We define the SLOPE as the ratio of the RISE to the RUN. What is a ratio? A RATIO is a fraction. The RATIO of two quantities is what we get when we divide the first by the second. We might also call it what we get when we try to fit the second quantity into the first. This is a good time to review the simplest basics of division, and what a fraction is.

If the point moving along a straight line runs five units for every unit it rises, then the slope is one fifth (1/5). That is steep for a railroad, but not as steep as most of the lines we get from graphing linear equations. Your child might enjoy researching what is the largest slope allowed in normal railroading, the slope of some of the funicular railroads in Switzerland, the largest slope allowed in road construction in the interstate highway system, the slope of some steep hills near you, the slope of some of the steeper San Francisco streets, the slope of a roof (often called the "pitch" of the roof), etc. They may be defined as a rise of three feet in a thousand, or seven feet in a hundred, or one foot in thirty, or one foot in twelve, or one foot in six. These are slopes of 3/1000, 7/100, 1/30, 1/12, and 1/6. A roof line might have a slope of 2/3, or even 1.

What if the moving point drops one unit in a run of five? Then the slope is negative one fifth (-1/5). You and your child might enjoy asking a truck driver how steep a down slope has to be before he must use a lower gear instead of just his brakes. Ask a skier about ski slopes. Investigations like this forge mental links between the real world and the Math we use to describe it. These links between the symbols and the reality are vitally important, both as keys to the application of Math and as ways of keeping a child's interest.

When the moving point rises one unit for every unit it runs, the slope of the line is one. Lines which rise less sharply have FRACTIONAL slopes, like those we have been considering. Lines which rise more sharply have larger slopes. For example, if the moving point which we think of as generating the line rises three units for every unit of run, then we have a slope of three.

You are ready now to go back and look at all the graphs you have made, and at each new one you make, and calculate their slopes.

An important point to note is that the slope does not change from one part of the line to another. This is why it doesn't matter which PART of the line you look at when you calculate the slope. (You and your child can guess that simply by looking at the lines.) If in one part of the line it rises four units for every unit of run, it will also rise four units for any other unit of run you choose to look at.

GRAPH ANY LINE AND CHECK THIS OUT. Here's how, using either your pegboard or paper. Select any two of the ordered pairs in your list of truth sets. Look at the rise between the two points, and the run, and look at their ratio. Do the same thing again using two different truth sets. As often as you do this, you will find that the value of the slope comes out the same. Then do the same thing again, using points which you see are on the line even though they are not in your table of ordered pairs. In every case the value of the slope you get by taking the rise (the difference of the y-coordinates) divided by the run (the difference of the x-coordinates) should be the same.

The ability to graph, and the ability to calculate slope by taking the rise divided by the run, are both so vital in many branches of Math that they should be repeated frequently.

Here is one way to come at them from slightly different directions. 1) Ask your child to draw a line with a slope of 1/4. Ask it to draw another with a slope of 3. Ask it to draw one with a slope of 2/3. Let it have turns giving you a slope and asking you to draw a line with that slope. Let this be a game with the pegboard Cartesian Plane, or on graph paper if he prefers.

To make that game a little more interesting, ask your child to draw a line with a slope of 1/4 and passing through the ORIGIN. Then ask it to draw another line with the same slope, passing through the point (0,-3). Then ask it to draw another with the same slope, passing through the point (0,5). What do you learn from this? Lines with the same slope are PARALLEL. We can also say that PARALLEL LINES have the same slope.

MATH LANGUAGE TERMS INTRODUCED IN THIS CHAPTER:

SLOPE: the steepness of a line, the ratio of RISE to RUN

RATIO: what you get when you divide one quantity by another

RISE: the difference between the y coordinates of two points

RUN: the difference between the x coordinates of two points

PARALLEL: having the same slope

Chapter 13 -- A MATHEMATICAL PROOF

One thing we learned in the last chapter was that the slope of a line is the same in all parts of the line. You and your child can PROVE this conclusion, and doing the proof can be both fun and highly instructive. When you have only one line up on your Cartesian Plane, ask your child to take a golf tee and put it in some new hole which it believes SHOULD BELONG on the line. Now, and this is VERY IMPORTANT, ask your child how it decided on that location for the golf tee! You will find, almost certainly, that it picked out a point by visually following the DIRECTION of the line, the SLOPE of the line.

Next, ask it to read off the coordinates of that new point, the ordered pair which defines that point. When you have done that, put the first number, the x-coordinate, in for x in the equation which led you to the graph of the line, and put the y-coordinate, the second number, in for y. You will find that those values make the equation true; they are a TRUTH SET for the equation.

Mathematicians often say those values "satisfy" the equation. That proves that the point your child selected by visually following the slope of the line actually does belong on the line.

Repeat this whole process often, and discuss it thoroughly. Use as many different ways of talking about it as possible. All those ways of describing what you are doing will help your child understand the reality. Then when you go over it all again, this time using of the Math language for these actions, it will connect the reality with the language, and so take another step toward being able to understand and use the language of Math.

In the end, the standard Math language terms SHOULD emerge as the most precise and economical of the various words we can use in talking about these things. But Math, like all languages, has been gradually evolving for centuries, and continues to evolve. If good new ways of talking about these things are found, there is a good chance that they will eventually be accepted into the language.

You may have noticed, by the way, that all this activity we have been doing with finding TRUTH SETS and SLOPES has been reviewing the basic skills of handling positive and negative numbers, adding and subracting, multiplying and dividing. Review through actual use like this is the best kind of review, because it teaches at the same time WHY the skill was learned in the first place.

This may be a good time to show you (and you may want to show your child) the GENERAL expression for the slope of any line. Name one point on the line (x_1, y_1), and another (x_2, y_2). Let's decide to represent the slope by the letter "m". Then the run between the two points is $(x_2 - x_1)$, and the rise is $(y_2 - y_1)$. The slope, the rise over the run, then must be:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This is a very, very useful formula, and it will acquire a firm place in your child's mind as you do more and more graphing of linear equations and calculation of their slopes, and also as you work backwards from slopes to graphs, and shortly going backward also from graphs to linear equations.

Chapter 14 -- DERIVING A LINEAR EQUATION FROM A GRAPH

THIS CHAPTER IS ADVANCED WORK, but try it. Choose an equation, find and list a few truth sets and draw the graph. Now when you work back from the graph and get an equation you will know you have done the work right if you come out with the equation you started with.

Working backward from a line to an equation is possible because of the fact we proved in the last chapter: **the slope of a straight line is the same in all parts of the line.**

Here's how we do it. First take any two points on the line and write an equation for the slope, using the formula at the end of the last chapter.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Now do the same thing again, but this time using the general point (x,y) as one of the points. We have chosen PARTICULAR points and named them (x_1, y_1) and (x_2, y_2). But (x,y) is the name that describes any general point at all, a point about which we know nothing yet. What we are doing here is finding the slope of a line connecting ANY general point on the whole plane with ONE of the two points you just chose from the straight line we are working with.

$$m = \frac{y - y_1}{x - x_1}$$

That describes the slope of a straight line drawn from ANY point on the plane, what mathematicians call "a GENERAL point," a point you might think of as a "wild card" point, to ONE of our chosen points, the point (x_1, y_1), on the line we are working with. We'll call this the slope of a "wild card" line, or just the slope of a "wild" line through the point (x_1, y_1).

Slow down here to make sure this is getting through. Draw this new line, or peg it (complete with yarn), so that you can see it is NOT the same line as we are working with. It is a line coming out of anywhere and cutting our line at the point (x_1, y_1). Make it visible by winding some yarn around the peg at the point (x_1, y_1) and around a loose peg you hold in your hand and move to any spot on the pegboard.

That general point, which can be ANYWHERE on our Cartesian Plane, will be on our original line, the line whose equation we are trying to find, if and only if the slope of the wild line coming out of anywhere happens to be equal to the slope of our line. Take that movable peg and swing it around, showing that when you have in just the right position, actually on our line, the wild line coincides with our original line. That will happen if and only if the two slopes we have calculated are equal. Here, then, is the condition for that general point being on our line:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

What this equation says is that the slope of the wild line equals the slope of our line. Obviously this will not be true for every value of x and y; it will only be true for those "wild card" points which are actually on our line. This equation, then, states the conditions under which a GENERAL point is on the line we started with. This means that it is the equation of the line. Its truth sets will be the truth sets which will give us that line when they are graphed.

Chapter 15 -- TWO SPECIAL SLOPES

There are two interesting questions about slope which you can now ask your child. Go back to the discussion of slopes of highways and rail lines, slopes of 1/12, 1/40, 1/100, and 1/1000, and then ask what it thinks a line with a slope of 1/1,000,000,000 might look like. Then ask what a ZERO SLOPE might indicate.

After some thought, or perhaps right away, your child will discover that a zero slope line is, of course, a line that does not rise at all, no matter how far it runs. In other words, it is a horizontal line, a line parallel to the x-axis.

Looking at the fractions which would ordinarily define the slope of a line, we find that they all have a zero numerator. Depending on which points you might choose in order to calculate the slope, you would get rise over run fractions like 0/3, or 0/7, or 0/12. Dividing something into zero is like fitting something into zero: it won't fit at all. The answer is always zero.

What would the equation of such a line look like? Put one up on your Cartesian Plane, with pegs and yarn, and think about it together. The answer: $y = 5$, or $y = -12$, or $y = 0$ (the equation of the x-axis).

All the horizontal lines on a piece of graph paper have equations like this. You can practice pointing to some and asking your child to tell you its equation. Then let your child choose one to point to and let you know if you are correct or not when you give its equation.

The other very interesting question about slope to ask your child is how large a number can you get for a slope. Gently steer your child away from the answer "infinity," as though that were some number; it is not. There is no such thing as the biggest number. One can always think of a number one bigger.

The slope of a line, therefore, can get as large as you can imagine. In other words, the line can get as steep as you like. Be sure to discuss with your child the practical difficulties of graphing lines with very large slopes, and of seeing any difference between them.

It may help your child to think about this in mathematical language if you remind it that the definition of our Math-word SLOPE is the ratio of rise to run. When we are making steeper and steeper lines, we can finally make a line that does not run at all, no matter how far it rises.

We say that such a line has "INFINITE" slope. It is vertical, parallel to the y-axis. The run is zero, and this relates to the important mathematical law that we cannot divide by zero. All the rise over run fractions we might try to use to calculate the slope have zero as the denominator. The answer to a division problem with a zero denominator is UNDEFINED. There is no largest number.

What would the equation of a line like this look like? Again, put some up on your Cartesian Plane, with pegs and yarn, and think about it together. The answer: $x = 18$, or $x = -7$, or $x = 0$ (the equation of the y-axis).

All the vertical lines on a piece of graph paper have equations like this, and you can play the same game with your child as you did with the horizontal lines.

Chapter 16 -- A PROVOCATIVE CLOSING QUESTION

Every advance of understanding in Math, as in all fields of knowledge, leads to new questions. As the territory of the known expands, the frontier with the unknown expands also. So it is appropriate that we should bring this book to a close with a question. We hope that it may stimulate your child to some enjoyable flights of fancy. Most reason, after all, begins as fancy, and Math has a closer relation to fancy than any other field of human knowledge.

Through all your consideration of large slopes and small slopes, you have probably been talking only about large and small POSITIVE slopes. It is interesting to go through the whole line of reasoning again using NEGATIVE slopes, looking first at slopes of -1/10, -1/100, -1/1,000,000, etc. This progression will again, just as before with your consideration of small positive slopes, lead you to zero. But then look at slopes of steeply falling lines, lines with slopes of -4, -100, -1,000,000, and so on.

Now ask your child to muse with you about the apparently strange fact that a line with a huge positive slope and a line with a huge negative slope can be going so nearly in the same direction that the naked eye cannot tell the difference. But be careful; a question like that MIGHT get a child so hooked on Math that you will never get it unhooked!

CONCLUSION

This is an unusual Math book. Remember what the title tells you: it is "Math For Mothers." The books are not as fat as the Math texts used in schools because the books are written for you; they are not written for your child. Your student may enjoy parts of them with you, but they are written for you.

We talked at the very beginning of the first book in the series about how to use these books, and that needs to be repeated. They are to give you enough understanding and mastery and confidence that you can teach your child OUT OF YOUR HEAD -- NOT OUT OF THE BOOK. Ideally you should be several books ahead of your child, not just a few pages ahead. We expect the books to be useful to you as you teach but they will not accomplish our aims if you just follow them slavishly.

We like our order of topics. We have given it a great deal of thought. But you need to be free to skip around, to move from topic to topic as your child's interest and questions lead.

You may need to go three books deep into division and fractions, for example, before you draw a single graph. You may need to pursue multiplication up into the land of exponents before doing very much with fractions. You may need to teach all about numeration systems before trying to deal thoroughly with multiplication. At all times you must play it by ear, listening to the mind of your child as it works on the material you present.

Is it possible to turn you into such a capable and confident teacher in only three or four thin volumes? Absolutely. By the end of these few introductory books you will have the heart of Math in your heart, and its bones in your bones. Remember that some of the greatest mathematicians in the history of the world had no books at all; all they had was an attitude. They were the ancient Greeks, and

the attitude they had was the combimation of curiosity with the love of precision. They knew that Math is just careful thought, and all of you are surely capable of careful thought. In fact, I'm sure you are LOVERS of careful thought. You would not be taking such an interest in helping your child develop its full intellectual potential if that were not true.

In the centuries since the Greeks, the most important thing that has changed is the emphasis on STANDARD MATH LANGUAGE. This has come about because we live in a world of many cultures and many languages, and they all need a precise understanding of each other's Math. **So we need to learn the language as we go along, and keep using it properly.**

This may be tiresome for those who would like to enjoy the pure rocket ride of unfettered logical thought, but it also makes possible higher flights. We can take off from higher and higher bases as we master the work of history's logical pioneers.

Finally, if you THINK THROUGH these books, you will be able to read any Math book ever written. You will know when they are spending too much time on something trivial, and when they are skimming too fast over something important. Trust your judgment as you read them. They are not written by giants, but by teachers subject to changing fads and commercial pressures. So the older Math texts are the best.

Go to yard sales and look for books written BEFORE 1960. Before 1940 is better yet. They all teach the same stuff. Old books of chemistry and physics have been made obsolete by new discoveries, but there have been no new discoveries in Math below the Ph.D. level for centuries. I know all about what Einstein did to Euclid's axioms, but as Einstein himself said, Euclid's geometry is still valid on its own terms.

So launch out! Enjoy yourself! Sail with confidence into this sea of logic and power and delight, and you WILL FIND that your joy and confidence will be absorbed by your child.