Minkowski Spacetime

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1 Introduction

This document describes the reformulation of the Lorantz Transformation as given by Minkowski. This new formulation gave rise to the concept of *spacetime* as we know today.

2 Rotation

First we consider the situation when a co-ordinate systems becomes rotated with respect to another co-ordinate system.

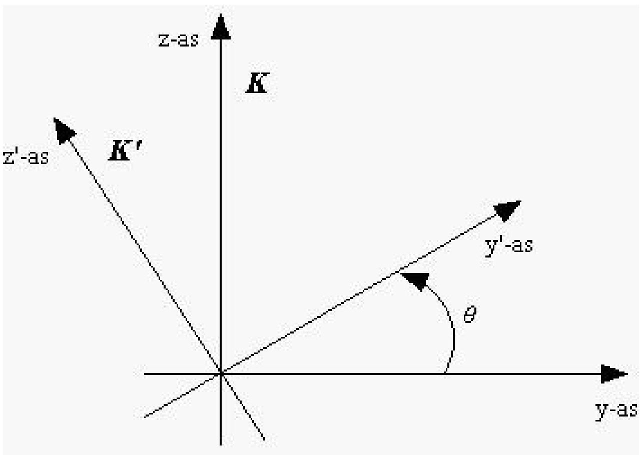

Figure 1: Rotated co-ordinate system

As can be seen in figure (1) the K co-ordinate system has rotated with respect to the K' system along the x-axis. The following relationship can easily be deduced:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

This is called a co-ordinate transformation.

3 Lorentz Transformation

Given two co-ordinate systems as shown in figure (2).

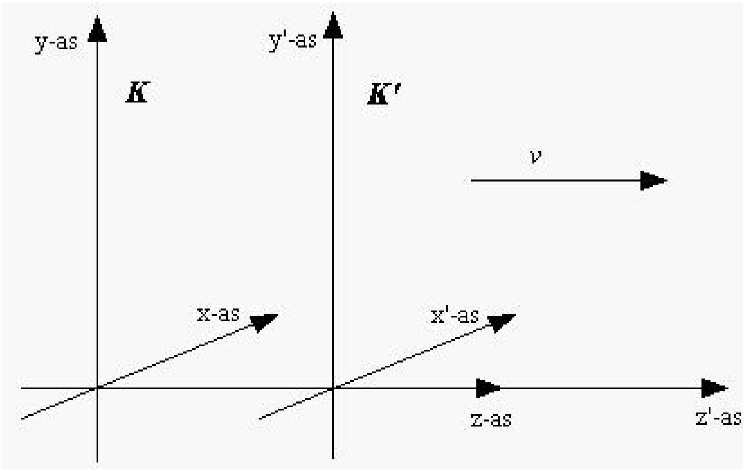

Figure 2: Moving co-ordinate systems

The K' system moves with velocity v along the z-axis. According to Einstein the co-ordinate tranformation between the two systems is given by the Lorentz Transformation (LT):

$$x' = x \tag{2}$$

$$y' = y \tag{3}$$

$$z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4}$$

$$t' = \frac{t - \frac{zt}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

4 Minkowski's formulation

For now lets focus on the z and t co-ordinates. Then the following relationship occurs:

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} 1/\gamma & -\beta c/\gamma \\ -\beta/\gamma c & 1/\gamma \end{pmatrix} \begin{pmatrix} z' \\ t' \end{pmatrix} \tag{6}$$

using

$$\beta = \frac{v}{c} \tag{7}$$

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}} \tag{8}$$

Substituting the relations

$$x_3 = z \tag{9}$$

$$x_4 = ict \tag{10}$$

gives

$$\begin{pmatrix} x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} 1/\gamma & i\beta/\gamma \\ -i\beta/\gamma & 1/\gamma \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{11}$$

Now introduce

$$\cosh\theta = \frac{1}{\gamma} \tag{12}$$

Using the relation $\cosh^2 x - \sinh^2 x = 1$, it can be decduced that

$$\sinh\theta = \frac{\beta}{\gamma} \tag{13}$$

$$\tanh\theta = \beta \tag{14}$$

Finally, substituting this results in

$$\begin{pmatrix} x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} \cos i\theta & \sin i\theta \\ -\sin i\theta & \cos i\theta \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{16}$$

Using the relations $\cos i\theta = \cosh\theta$ and $\sin i\theta = i\sinh\theta$.

The total transformation matrix thus is

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos i\theta & \sin i\theta \\ 0 & 0 & -\sin i\theta & \cos i\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{17}$$

$$= M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{18}$$

Now compare this with the rotation formula (1) and notice the similarity. The Lorentz Transformation appears to be merely a rotation in the complex co-ordinate system (x_1, x_2, x_3, x_4) along the complex angle $i\theta$. A remarkable observation! Formulated in this way the time co-ordinate x_4 acts as just another space co-ordinate. This initiated the idea that space and time are even more intermixed than the Lorentz Transformation in its original formulation already suggested. The complex co-ordinate system (x_1, x_2, x_3, x_4) is what is called *Minkowski spacetime*, or simply *spacetime*.