PLANE AND SPHERICAL TRIGONOMETRY

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C. I. PALMER

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PLANE AND SPHERICAL TRIGONOMETRY

BY

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PREFACE TO THE FOURTH EDITION

This edition presents a new set of problems in Plane Trigonometry. The type of problem has been preserved, but the details have been changed. The undersigned acknowledges indebtedness to the members of the Department of Mathematics at the Armour Institute of Technology for valuable suggestions and criticisms. He is especially indebted to Profs. S. F. Bibb and W. A. Spencer for their contribution of many new identities and equations and also expresses thanks to Mr. Clark Palmer, son of the late Dean Palmer, for assisting in checking answers to problems and in proofreading and for offering many constructive criticisms.

CHARLES WILBER LEIGH.

CHICAGO,
June, 1934.

PREFACE TO THE FIRST EDITION

This text has been written because the authors felt the need of a treatment of trigonometry that duly emphasized those parts necessary to a proper understanding of the courses taken in schools of technology. Yet it is hoped that teachers of mathematics in classical colleges and universities as well will find it suited to their needs. It is useless to claim any great originality in treatment or in the selection of subject matter. No attempt has been made to be novel only; but the best ideas and treatment have been used, no matter how often they have appeared in other works on trigonometry.

The following points are to be especially noted:

(1) The measurement of angles is considered at the beginning.

(2) The trigonometric functions are defined at once for any angle, then specialized for the acute angle; not first defined for acute angles, then for obtuse angles, and then for general angles. To do this, use is made of Cartesian coordinates, which are now almost universally taught in elementary algebra.

(3) The treatment of triangles comes in its natural and logical order and is not *forced* to the first pages of the book.

(4) Considerable use is made of the line representation of the trigonometric functions. This makes the proof of certain theorems easier of comprehension and lends itself to many useful applications.

(5) Trigonometric equations are introduced early and used often.

(6) Anti-trigonometric functions are used throughout the work, not placed in a short chapter at the close. They are used in the solutions of equations and triangles. Much stress is laid upon the principal values of anti-trigonometric functions as used later in the more advanced subjects of mathematics.

(7) A limited use is made of the so-called "laboratory method" to impress upon the student certain fundamental ideas.

(8) Numerous carefully graded practical problems are given and an abundance of drill exercises.

(9) There is a chapter on complex numbers, series, and hyperbolic functions.

(10) A very complete treatment is given on the use of logarithmic and trigonometric tables. This is printed in connection with the tables, and so does not break up the continuity of the trigonometry proper.

(11) The tables are carefully compiled and are based upon those of Gauss. Particular attention has been given to the determination of angles near 0 and 90°, and to the functions of such angles. The tables are printed in an unshaded type, and the arrangement on the pages has received careful study.

The authors take this opportunity to express their indebtedness to Prof. D. F. Campbell of the Armour Institute of Technology, Prof. N. C. Riggs of the Carnegie Institute of Technology, and Prof. W. B. Carver of Cornell University, who have read the work in manuscript and proof and have made many valuable suggestions and criticisms.

THE AUTHORS.

CHICAGO,
September, 1914.

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GREEK ALPHABET

Α, α	*Alpha*	Ν, ν	*Nu*
Β, β	*Beta*	Ξ, ξ	*Xi*
Γ, γ	*Gamma*	Ο, ο	*Omicron*
Δ, δ	*Delta*	Π, π	*Pi*
Ε, ε	*Epsilon*	Ρ, ρ	*Rho*
Ζ, ζ	*Zeta*	Σ, σ	*Sigma*
Η, η	*Eta*	Τ, τ	*Tau*
Θ, θ	*Theta*	Υ, υ	*Upsilon*
Ι, ι	*Iota*	Φ, φ	*Phi*
Κ, κ	*Kappa*	Χ, χ	*Chi*
Λ, λ	*Lambda*	Ψ, ψ	*Psi*
Μ, μ	*Mu*	Ω, ω	*Omega*

PLANE AND SPHERICAL TRIGONOMETRY

CHAPTER I

INTRODUCTION

GEOMETRY

1. Introductory remarks.—The word trigonometry is derived from two Greek words, *τριγωνον* (trigonon), meaning triangle, and *μετρια* (metria), meaning measurement. While the derivation of the word would seem to confine the subject to triangles, the measurement of triangles is merely a part of the general subject which includes many other investigations involving angles.

Trigonometry is both geometric and algebraic in nature. Historically, trigonometry developed in connection with astronomy, where distances that could not be measured directly were computed by means of angles and lines that could be measured. The beginning of these methods may be traced to Babylon and Ancient Egypt.

The noted Greek astronomer Hipparchus is often called the founder of trigonometry. He did his chief work between 146 and 126 B. C. and developed trigonometry as an aid in measuring angles and lines in connection with astronomy. The subject of trigonometry was separated from astronomy and established as a distinct branch of mathematics by the great mathematician Leonhard Euler, who lived from 1707 to 1783.

To pursue the subject of trigonometry successfully, the student should know the subjects usually treated in algebra up to and including quadratic equations, and be familiar with plane geometry, especially the theorems on triangles and circles.

Frequent use is made of the protractor, compasses, and the straightedge in constructing figures.

While parts of trigonometry can be applied at once to the solution of various interesting and practical problems, much of it is studied because it is very frequently used in more advanced subjects in mathematics.

ANGLES

2. Definitions.—The definition of an angle as given in geometry admits of a clear conception of small angles only. In trigonometry, we wish to consider *positive* and *negative* angles and these of any size whatever; hence we need a more comprehensive definition of an angle.

If a line, starting from the position OX (Fig. 1), is revolved about the point O and always kept in the same plane, we say the line **generates** an angle. If it revolves from the position OX to the position OA, in the direction indicated by the arrow, the angle XOA is generated.

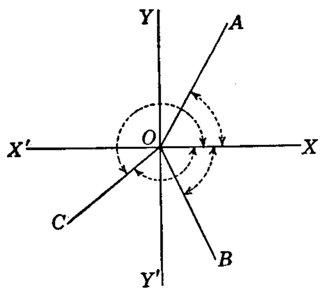

Fig. 1.

The original position OX of the generating line is called the **initial side,** and the final position OA, the **terminal side** of the angle.

If the rotation of the generating line is *counterclockwise*, as already taken, the angle is said to be **positive.** If OX revolves in a *clockwise* direction to a position, as OB, the angle generated is said to be **negative.**

In reading an angle, the letter on the initial side is read first to give the proper sense of direction. If the angle is read in the opposite sense, the negative of the angle is meant.

Thus, $$\angle AOX = -\angle XOA.$$

It is easily seen that this conception of an angle makes it possible to think of an angle as being of any size whatever. Thus, the generating line, when it has reached the position OY, having made a quarter of a revolution in a counterclockwise direction, has generated a right angle; when it has reached the position OX' it has generated two right angles. A complete revolution generates an angle containing four right angles; two revolutions, eight right angles; and so on for any amount of turning.

The right angle is divided into 90 equal parts called degrees (°), each degree is divided into 60 equal parts called minutes (′), and each minute into 60 equal parts called seconds (″).

Starting from any position as initial side, it is evident that for each position of the terminal side, there are two angles less than 360°, one positive and one negative. Thus, in Fig. 1, OC is the terminal side for the positive angle XOC or for the negative angle XOC.

3. Quadrants.—It is convenient to divide the plane formed by a complete revolution of the generating line into four parts by the two perpendicular lines $X'X$ and $Y'Y$. These parts are called **first, second, third,** and **fourth quadrants,** respectively. They are placed as shown by the Roman numerals in Fig. 2.

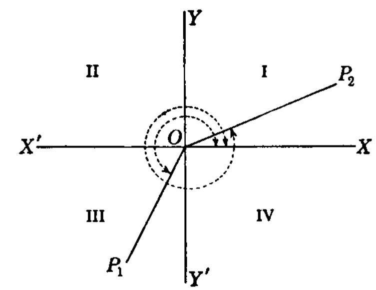

Fig. 2.

If OX is taken as the initial side of an angle, the angle is said to lie in the quadrant in which its terminal side lies. Thus, XOP_1 (Fig. 2) lies in the third quadrant, and XOP_2, formed by more than one revolution, lies in the first quadrant.

An angle lies between two quadrants if its terminal side lies on the line between two quadrants.

4. Graphical addition and subtraction of angles.—Two angles are added by placing them in the same plane with their vertices together and the initial side of the second on the terminal side of the first. The sum is the angle from the initial side of the first to the terminal side of the second.

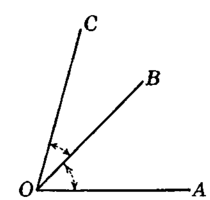

Fig. 3.

Subtraction is performed by adding the negative of the subtrahend to the minuend.

Thus, in Fig. 3,

$$\angle AOB + \angle BOC = \angle AOC.$$
$$\angle AOC - \angle BOC = \angle AOC + \angle COB = \angle AOB.$$
$$\angle BOC - \angle AOC = \angle BOC + \angle COA = \angle BOA.$$

EXERCISES

Use the protractor in laying off the angles in the following exercises:

1. Choose an initial side and lay off the following angles. Indicate each angle by a circular arrow. 75°; 145°; 243°; 729°; 456°; 976°. State the quadrant in which each angle lies.

2. Lay off the following angles and state the quadrant that each is in: −40°; −147°; −295°; −456°; −1048°.

3. Lay off the following pairs of angles, using the same initial side for each pair: 170° and −190°; −40° and 320°; 150° and −210°.

4. Give a positive angle that has the same terminal side as each of the following: 30°; 165°; −90°; −210°; −45°; 395°; −390°.

5. Show by a figure the position of the revolving line when it has generated each of the following: 3 right angles; $2\frac{1}{2}$ right angles; $1\frac{1}{3}$ right angles; $4\frac{2}{3}$ right angles.

Unite graphically, using the protractor:

6. 40° + 70°; 25° + 36°; 95° + 125°; 243° + 725°.

7. 75° − 43°; 125° − 59°; 23° − 49°; 743° − 542°; 90° − 270°.

8. 45° + 30° + 25°; 125° + 46° + 95°; 327° + 25° + 400°.

9. 45° − 56° + 85°; 325° − 256° + 400°.

10. Draw two angles lying in the first quadrant but differing by 360°. Two negative angles in the fourth quadrant and differing by 360°.

11. Draw the following angles and their complements: 30°; 210°; 345°; −45°; −300°; −150°.

5. Angle measurement.—Several systems for measuring angles are in use. The system is chosen that is best adapted to the purpose for which it is used.

(1) *The right angle.*—The most familiar unit of measure of an angle is the right angle. It is easy to construct, enters frequently into the practical uses of life, and is almost always used in geometry. It has no subdivisions and does not lend itself readily to computations.

(2) *The sexagesimal system.*—The **sexagesimal system** has for its fundamental unit the degree, which is defined to be the angle formed by $\frac{1}{360}$ part of a revolution of the generating line. This is the system used by engineers and others in making practical numerical computations. The subdivisions of the degree are the minute and the second, as stated in **Art. 2.** The word "sexagesimal" is derived from the Latin word *sexagesimus*, meaning one-sixtieth.

(3) *The centesimal system.*—Another system for measuring angles was proposed in France somewhat over a century ago. This is the **centesimal system.** In it the right angle is divided into 100 equal parts called **grades,** the grade into 100 equal parts called minutes, and the minute into 100 equal parts called seconds. While this system has many admirable features, its use could not become general without recomputing with a great expenditure of labor many of the existing tables.

(4) *The circular or natural system.*—In the **circular** or **natural system** for measuring angles, sometimes called **radian measure** or **π-measure,** the fundamental unit is the radian.

The radian is defined to be the *angle which, when placed with its vertex at the center of a circle, intercepts an arc equal in length to the radius of the circle.* Or it is defined as *the positive angle generated when a point on the generating line has passed through an arc equal in length to the radius of the circle being formed by that point.*

In Fig. 4, the angles AOB, BOC, · · · FOG are each 1 radian, since the sides of each angle intercept an arc equal in length to the radius of the circle.

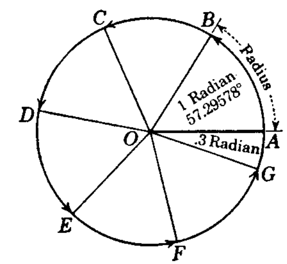

Fig. 4.

The circular system lends itself naturally to the measurement of angles in many theoretical considerations. It is used almost exclusively in the calculus and its applications.

(5) *Other systems.*—Instead of dividing the degree into minutes and seconds, it is sometimes divided into tenths, hundredths, and thousandths. This **decimal scale** has been used more or less ever since decimal fractions were invented in the sixteenth century.

The **mil** is a unit of angle used in artillery practice. The mil is $\frac{1}{6400}$ revolution, or very nearly $\frac{1}{1000}$ radian; hence its name. The scales by means of which the guns in the United States Field Artillery are aimed are graduated in this unit.

6. The radian.—That the circular measure is the natural system to use in measuring an angle is apparent from a consideration of the geometrical basis for the definition of the radian.

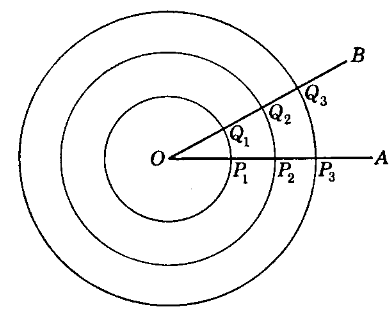

Fig. 5.

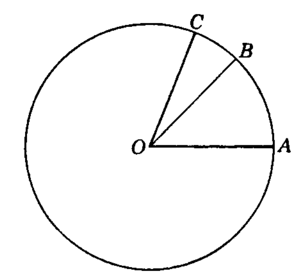

Fig. 6.

(1) Given several concentric circles and an angle AOB at the center as in Fig. 5, then

$$\frac{\text{arc } P_1Q_1}{OP_1} = \frac{\text{arc } P_2Q_2}{OP_2} = \frac{\text{arc } P_3Q_3}{OP_3}, \text{ etc.}$$

That is, the ratio of the intercepted arc to the radius of that arc is a constant for all circles when the angle is the same. *The angle at the center which makes this ratio unity is then a convenient unit for measuring angles. This is* 1 *radian.*

(2) In the same or equal circles, two angles at the center are in the same ratio as their intercepted arcs. That is, in Fig. 6,

$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC}.$$

Here, if $\angle AOC$ is unity when arc $AC = r$, $\angle AOB = \frac{\text{arc } AB}{r}$, or, in general, $\theta = \frac{s}{r}$, where θ is the angle at the center measured in radians, s the arc length, and r the radius of the circle.

7. Relations between radian and degree.—The relations between a degree and a radian can be readily determined from their definitions. Since the circumference of a circle is 2π times the radius,

$$2\pi \text{ radians} = 1 \text{ revolution.}$$

Also $$360° = 1 \text{ revolution.}$$

Then $$2\pi \text{ radians} = 360°.$$

$$\therefore 1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = 57.29578° -$$

$$= 206264.8'' + = 57°\ 17'\ 44.8'' +.$$

For less accurate work 1 radian is taken as 57.3°.

Conversely, **180° = π radians.**

$$\therefore 1° = \frac{\pi}{180} = 0.0174533 - \text{ radian.}$$

To convert radians to degrees, multiply the number of radians by $\frac{180}{\pi}$, *or* 57.29578 −.

To convert degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$, *or* 0.017453 +.

In writing an angle in degrees, minutes, and seconds, the signs °, ′, ″ are always expressed. In writing an angle in circular measure, usually no abbreviation is used. Thus, the angle 2 means an angle of 2 radians, the angle $\frac{1}{2}\pi$ means an angle of $\frac{1}{2}\pi$ radians. One should be careful to note that $\frac{1}{2}\pi$ does not denote

an angle, it simply tells how many radians the angle contains. Sometimes radian is abbreviated as follows: 3^r, $3^{(r)}$, 3ρ, or 3 rad. When the word "radians" is omitted, the student should be careful to supply it mentally.

Many of the most frequently used angles are conveniently expressed in radian measure by using π. In this manner the values are expressed accurately and long decimals are avoided. Thus, $180° = \pi$ radians, $90° = \frac{1}{2}\pi$ radians, $60° = \frac{1}{3}\pi$ radians, $135° = \frac{3}{4}\pi$ radians, $30° = \frac{1}{6}\pi$ radians. These forms are more convenient than the decimal form. For instance, $\frac{1}{3}\pi$ radians = 1.0472 radians.

Example 1.—Reduce 2.5 radians to degrees, minutes, and seconds.

Solution.—1 radian = 57.29578°.

Then 2.5 radians = 2.5 × 57.29578° = 143.2394°.

To find the number of minutes, multiply the decimal part of the number of degrees by 60.

$$0.2394° = 60 \times 0.2394 = 14.364'.$$

Likewise, $0.364' = 60 \times 0.364 = 21.8''$.

$$\therefore 2.5 \text{ radians} = 143°\ 14'\ 22''.$$

Example 2.—Reduce 22° 36′ 30″ to radians.

Solution.—First, change to degrees and decimal of degree.

This gives $22°\ 36'\ 30'' = 22.6083°$.

$$1° = 0.017453 \text{ radian.}$$

$$22.6083° = 22.6083 \times 0.017453 = 0.3946 \text{ radian.}$$

$$\therefore 22°\ 36'\ 30'' = 0.3946 \text{ radian.}$$

EXERCISES

The first eight exercises are to be done orally.

1. Express the angles of the following numbers of radians in degrees: $\frac{1}{2}\pi$; $\frac{2}{3}\pi$; $\frac{3}{2}\pi$; $\frac{5}{6}\pi$; $\frac{4}{3}\pi$; $\frac{3}{4}\pi$; $\frac{11}{6}\pi$; $\frac{7}{6}\pi$.

2. Express the following angles as some number of π radians: 30°; 90°; 180°; 135°; 120°; 240°; 270°; 330°; 225°; 315°; 81°; 360°; 720°.

3. Express the angles of the following numbers of right angles in radians, using π; 2; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{4}{3}$; $3\frac{1}{2}$; $2\frac{1}{3}$; $1\frac{2}{3}$; $3\frac{1}{3}$.

4. Express in radians each angle of an equilateral triangle. Of a regular hexagon. Of an isosceles triangle if the vertex angle is a right angle.

5. How many degrees does the minute hand of a watch turn through in 15 min.? In 20 min.? How many radians in each of these angles?

6. What is the measure of 90° when the right angle is taken as the unit of measure? Of 135°? Of 60°? Of 240°? Of 540°? Of −270°? Of −360°? Of −630°?

7. What is the measure of each of the angles of the previous exercise when the radian is taken as the unit of measure?

8. What is the angular velocity of the second hand of a watch in radians per minute? What is the angular velocity of the minute hand?

Reduce the following angles to degrees, minutes and integral seconds:

9. 2.3 radians. *Ans.* 131° 46′ 49″.
10. 1.42 radians. *Ans.* 81° 21′ 36″.
11. 3.75 radians. *Ans.* 214° 51′ 33″.
12. 0.25 radian. *Ans.* 14° 19′ 26″.
13. $\frac{3}{16}\pi$ radian. *Ans.* 33° 45′.
14. $\frac{11}{4}\pi$ radians. *Ans.* 495°.
15. 0.0074 radian. *Ans.* 25′ 16″.
16. 6.28 radians. *Ans.* 359° 49′ 3″.

Reduce the following angles to radians correct to four decimals, using **Art. 7**:

17. 55°. **18.** 103°. **19.** 265°. **20.** 17°.
21. 24° 37′ 27″. *Ans.* 0.4298.
22. 285° 28′ 56″. *Ans.* 4.9825.
23. 416° 48′ 45″. *Ans.* 7.2746.

Reduce the following angles to radians, using **Table V**, of **Tables.**

24. 25° 14′ 23″. *Ans.* 0.4405162.
25. 175° 42′ 15″. *Ans.* 3.0666162.
26. 78° 15′ 30″. *Ans.* 1.3658655.
27. 243° 35′ 42″. *Ans.* 4.2515348.
28. 69° 25′ 8″. *Ans.* 1.2115882.
29. 9° 9′ 9″. *Ans.* 0.1597412.

30. Compute the equivalents given in **Art. 7.**

31. Show that 1 mil is very nearly 0.001 radian, and find the per cent of error in using 1 mil = 0.001 radian. *Ans.* 1.86 per cent.

32. What is the measure of each of the following angles when the right angle is taken as the unit of measure: 1 radian, 2π radians, 650°, 2.157 radians? *Ans.* 0.6366; 4; 7.222; 1.373.

33. An angular velocity of 10 revolutions per second is how many radians per minute? *Ans.* 3769.91.

34. An angular velocity of 30 revolutions per minute is how many π radians per second? *Ans.* One-π radians.

35. An angular velocity of 80 radians per minute is how many degrees per second? *Ans.* 76.394°.

36. Show that nine-tenths the number of grades in an angle is the number of degrees in that angle.

37. The angles of a triangle are in the ratio of 2:3:7. Express the angles in radians. *Ans.* $\frac{1}{6}\pi$; $\frac{1}{4}\pi$; $\frac{7}{12}\pi$.

38. Express an interior angle of each of the following regular polygons in radians: octagon, pentagon, 16-gon, 59-gon.

39. Express 48° 22′ 25″ in the centesimal system in grades, minutes, and seconds. *Ans.* 53 grades 74 min. 84 sec.

ANGLE AT CENTER OF CIRCLE

8. Relations between angle, arc, and radius.—In **Art. 6,** it is shown that, if the central angle is measured in radians and the arc

length and the radius are measured in the same linear unit, then

$$\textbf{angle} = \frac{\textbf{arc}}{\textbf{radius}}.$$

That is, if θ, s, and r are the measures, respectively, of the angle, arc, and radius (Fig. 7),

$$\boldsymbol{\theta = s \div r,}$$

Solving this for s and then for r,

$$\boldsymbol{s = r\theta,}$$

and

$$\boldsymbol{r = s \div \theta.}$$

These are the simplest geometrical relations between the angle at the center of a circle, the intercepted arc, and the radius. They are of frequent use in mathematics and its applications, and should be remembered.

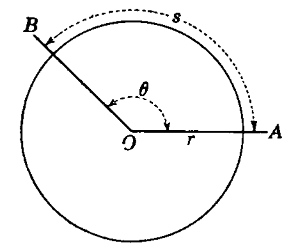

FIG. 7.

Example 1.—The diameter of a graduated circle is 10 ft., and the graduations are 5′ of arc apart; find the length of arc between the graduations in fractions of an inch to three decimal places.

Solution.—By formula, $s = r\theta$.

From the example, $r = 12 \times 5 = 60$ in.,

and $\theta = 0.01745 \times \frac{5}{60} = 0.00145$ radian.

Substituting in the formula, $s = 60 \times 0.00145 = 0.087$.

∴ length of 5′ arc is 0.087 in.

Example 2.—A train is traveling on a circular curve of $\frac{1}{2}$-mile radius at the rate of 30 miles per hour. Through what angle would the train turn in 45 sec.?

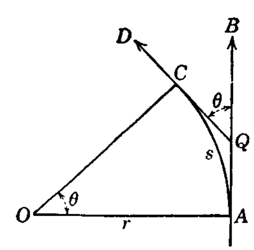

FIG. 8.

Solution.—When at the position A (Fig. 8), the train is moving in the direction AB. After 45 sec. it has reached C, and is then moving in the direction CD. It has then turned through the angle BQC.

But $\angle BQC = \angle AOC = \theta$. Why?

The train travels the arc $s = \frac{3}{8}$ mile in 45 sec.

To find value of θ, use formula

$$\theta = s \div r.$$

$$\therefore \theta = \tfrac{3}{8} \div \tfrac{1}{2} = 0.75 \text{ radian} = 42° 58′ 19″.$$

9. Area of circular sector.—In Fig. 9, the area BOC, bounded by two radii and an arc of a circle, is a sector. In geometry it is shown that *the area of a sector of a circle equals one-half the arc length times the radius.*

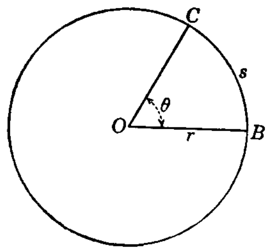

Fig. 9.

That is, $A = \frac{1}{2}rs.$

But $s = r\theta.$

Hence, $A = \frac{1}{2}r^2\theta.$

Example.—Find the area of the sector of a circle having a radius 8 ft. if the central angle is 40°.

Solution.—

$$40° = 40 \times 0.01745 = 0.698 \text{ radian.}$$

Using the formula $A = \frac{1}{2}r^2\theta,$

$$A = \frac{1}{2} \times 8^2 \times 0.698 = 22.34.$$

$\therefore$ area of sector $= 22.34$ sq. ft.

ORAL EXERCISES

1. How many radians are there in the central angle intercepting an arc of 20 in. on a circle of 5-in. radius?

2. The minute hand of a clock is 4 in. long. Find the distance moved by the outer end when the hand has turned through 3 radians. When it has moved 20 min.

3. A wheel revolves with an angular velocity of 8 radians per second. Find the linear velocity of a point on the circumference if the radius is 6 ft.

4. The velocity of the rim of a flywheel is 75 ft. per second. Find the angular velocity in radians per second if the wheel is 8 ft. in diameter.

5. A pulley carrying a belt is revolving with an angular velocity of 10 radians per second. Find the velocity of the belt if the pulley is 5 ft. in diameter.

6. An angle of 3 mils will intercept what length of arc at 1000 yd.?

7. A freight car 30 ft. in length at right angles to the line of sight intercepts an angle of 2 mils. What is its distance from the observer?

8. A train is traveling on a circular curve of $\frac{1}{2}$-mile radius at the rate of 30 miles an hour. Through what angle does it turn in 15 sec.?

9. A belt traveling 60 ft. per second runs on a pulley 3 ft. in diameter. What is the angular velocity of the pulley in radians per second?

10. A circular target at 3000 yd. subtends an angle of 1 mil at the eye. How large is the target?

WRITTEN EXERCISES

1. The diameter of the drive wheels of a locomotive is 72 in. Find the number of revolutions per minute they make when the engine is going 45 miles per hour. *Ans.* 210.08 r.p.m.

2. A flywheel is revolving at the rate of 456 r.p.m. What angle does a radius of the wheel generate in 1 sec.? Express in degrees and radians. How many π radians are generated in 2.5 sec.? *Ans.* 2736°; 47.752 radians; 38.

3. A flywheel 6 ft. in diameter is revolving at an angular velocity of 30 radians per second. Find the rim velocity in miles per hour. *Ans.* 61.36 miles per hour.

4. The angular velocity of a flywheel is 10π radians per second. Find the circumferential velocity in feet per second if the radius of the wheel is 6 ft. *Ans.* 188.5 ft. per second.

5. A wheel is revolving at an angular velocity of $\frac{5\pi}{3}$ radians per second. Find the number of revolutions per minute. Per hour. *Ans.* 50 r.p.m.; 3000 r.p.h.

6. In a circle of 9-in. radius, how long an arc will have an angle at the center of 2.5 radians? An angle of 155° 36′? *Ans.* 22.5 in.; 24.44 in.

7. An automobile wheel 2.5 ft. in outside diameter rolls along a road, the axle moving at the rate of 45 miles per hour; find the angular velocity in π radians per second. *Ans.* 16.81 π radians.

8. Chicago is at north latitude 41° 59′. Use 3960 miles as the radius of the earth and find the distance from Chicago to the equator. *Ans.* 2901.7 miles.

9. Use 3960 miles as the radius of the earth and find the length in feet of 1″ of arc of the equator. *Ans.* 101.37 ft.

10. A train of cars is running at the rate of 35 miles per hour on a curve of 1000 ft. radius. Find its angular velocity in radians per minute. *Ans.* 3.08 radians per minute.

11. Find the length of arc which at 1 mile will subtend an angle of 1′. An angle of 1″. *Ans.* 1.536 ft.; 0.0253 ft.

12. The radius of the earth's orbit around the sun, which is about 92,700,000 miles, subtends at the star Sirius an angle of about 0.4″. Find the approximate distance of Sirius from the earth. *Ans.* 48 (10^{12}) miles.

13. Assume that the earth moves around the sun in a circle of 93,000,000-mile radius. Find its rate per second, using $365\frac{1}{4}$ days for a revolution. *Ans.* 18.5 miles per second.

14. The earth revolves on its axis once in 24 hours. Use 3960 miles for the radius and find the velocity of a point on the equator in feet per second. Find the angular velocity in radians per hour. In seconds of angle per second of time. *Ans.* 1520.6 ft. per second; 0.262 radian per hour.

15. The circumferential speed generally advised by makers of emery wheels is 5500 ft. per minute. Find the angular velocity in radians per second for a wheel 16 in. in diameter. *Ans.* 137.5 radian per second.

16. Find the area of a circular sector in a circle of 12 in. radius, if the angle is π radians. If 135°. If 5 radians. *Ans.* 226.2 sq. in.; 169.7 sq. in.

17. The perimeter of a sector of a circle is equal to two-thirds the circumference of the circle. Find the angle of the sector in circular measure and in sexagesimal measure. *Ans.* 2.1888 radians; 125° 24.5′.

10. General angles.—In Fig. 10, the angle XOP_1 is 30°; or if the angle is thought of as formed by one complete revolution and 30°, it is 390°; if by two complete revolutions and 30°, it is 750°. So an angle having OX for initial side and OP_1 for terminal side is $30°, 360° + 30°, 2 \times 360° + 30°$, or, in general, $n \times 360° + 30°$, where n takes the values $0, 1, 2, 3, \cdots$, that is, n is any integer, zero included.

In radian measure this is $2n\pi + \frac{1}{6}\pi$.

The expression $n \times 360° + 30°$, or $2n\pi + \frac{1}{6}\pi$, is called the **general measure** of all the angles having OX as initial side and OP_1 as terminal side.

If the angle XOP_2 is 30° less than 180°, then the general measure of the angles having OX as initial side and OP_2 as terminal side is an odd number times 180° less 30°; and may be written

$$(2n + 1)180° - 30°,$$

or $$(2n + 1)\pi - \tfrac{1}{6}\pi.$$

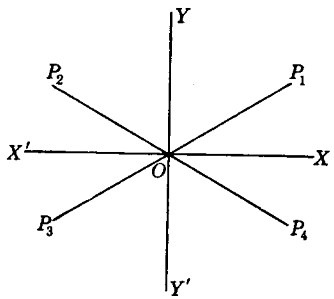

Fig. 10.

Similarly, $n\pi \pm \frac{1}{6}\pi$ means an integral number of times π is taken and then $\frac{1}{6}\pi$ is added or subtracted. This gives the terminal side in one of the four positions shown in Fig. 10 by OP_1, OP_2, OP_3, and OP_4.

It is evident that throughout this article n may have negative as well as positive values, and that any angle θ might be used instead of 30°, or $\frac{1}{6}\pi$.

EXERCISES

1. Use the same initial side for each and draw angles of 50°; 360° + 50°; $n \cdot 360° + 50°$.

2. Use the same initial side for each and draw angles of 40°; 180° + 40°; $2 \cdot 180° + 40°$; $3 \cdot 180° + 40°$; $n \cdot 180° + 40°$.

3. Use the same initial side for each and draw angles of 30°; 90° + 30°; $2 \cdot 90° + 30°$; $3 \cdot 90° + 30°$; $n \cdot 90° + 30°$.

4. Draw the terminal sides for all the angles whose general measure is $2n \cdot 90°$. For all the angles whose general measure is $(2n + 1)90°$.

5. Draw the following angles: $2n\pi$; $(2n + 1)\pi$; $(2n + 1)\frac{1}{2}\pi$; $(4n + 1)\frac{1}{2}\pi$ $(4n + 3)\frac{1}{2}\pi$.

6. Draw the following angles: $2n \times 180° \pm 60°$; $(2n + 1)180° \pm 60°$; $(2n + 1)\pi \pm \frac{1}{3}\pi$; $2n\pi + \frac{1}{3}\pi$; $(2n + 1)\frac{\pi}{2} \pm \frac{\pi}{3}$; $n\pi \pm \frac{1}{4}\pi$; $(4n + 1)\frac{\pi}{2} \pm \frac{\pi}{6}$; $(4n - 1)\frac{\pi}{2} \pm \frac{\pi}{6}$.

7. Give the general measure of all the angles having the lines that bisect the four quadrants as terminal sides. Those that have the lines that trisect the four quadrants as terminal sides.

COORDINATES

11. Directed lines and segments.—For certain purposes in trigonometry it is convenient to give a line a property not often used in plane geometry. This is the property of having *direction.*

In Fig. 11, RQ is a directed straight line if it is thought of as traced by a point moving without change of direction from R toward Q or from Q toward R. The direction is often shown by an arrow.

Let a fixed point O on RQ be taken as a point from which to measure distances. Choose a fixed length as a unit and lay it off on the line RQ beginning at O. The successive points located in this manner will be $1, 2, 3, 4, \cdots$ times the unit distance from O. These points may be thought of as representing the numbers, or the numbers may be thought of as representing the points.

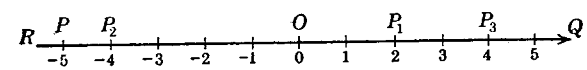

Fig. 11.

Since there are two directions from O in which the measurements may be made, it is evident that there are two points equally distant from O. Since there are both positive and negative numbers, we shall *agree* to represent the points to the *right* of O by positive numbers and those to the *left* by negative numbers.

Thus, a point 2 units to the right of O represents the number 2; and, conversely, the number 2 represents a point 2 units to the right of O. A point 4 units to the left of O represents the number -4; and, conversely, the number -4 represents a point 4 units to the left of O.

The point O from which the measurements are made is called the **origin.** It represents the number zero.

A **segment** of a line is a definite part of a directed line.

The segment of a line is read by giving its initial point and its terminal point. Thus, in Fig. 11, OP_1, OP_2, and P_1P_3 are segments. In the last, P_1 is the **initial point** and P_3 the **terminal point.**

The **value** of a segment is determined by its length and direction, and it is defined to be *the number which would represent the terminal point of the segment if the initial point were taken as origin.*

It follows from this definition that the value of a segment read in one direction is the negative of the value if read in the opposite direction.

In Fig. 12, taking O as origin, the values of the segments are as follows:

$$OP_1 = 3,\ OP_3 = 8,\ OP_5 = -5,\ P_2P_3 = 3,\ P_3P_1 = -5.$$
$$P_4P_6 = -6,\ P_6P_5 = 3,\ P_1P_2 = -P_2P_1 = 2.$$

Two segments are equal if they have the same direction and the same length, that is, the same value.

If two segments are so placed that the initial point of the second is on the terminal point of the first, the **sum of the two segments**

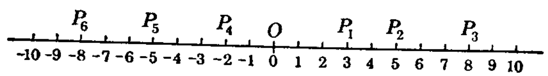

FIG. 12.

is the segment having as initial point the initial point of the first, and as terminal point the terminal point of the second.

The segments are subtracted by reversing the direction of the subtrahend and adding.

Thus, in Fig. 12,

$$P_5P_4 + P_4P_1 = P_5P_1 = 8.$$
$$P_2P_4 + P_4P_6 = P_2P_6 = -13.$$
$$P_1P_3 - P_2P_3 = P_1P_3 + P_3P_2 = P_1P_2 = 2.$$
$$P_2P_3 - P_1P_3 = P_2P_3 + P_3P_1 = P_2P_1 = -2.$$

12. Rectangular coordinates.—Let $X'X$ and $Y'Y$ (Fig. 13) be two fixed directed straight lines, perpendicular to each other and intersecting at the point O. Choose the positive direction towards the right, when parallel to $X'X$; and upwards, when parallel to $Y'Y$. Hence the negative directions are towards the left, and downwards.

The two lines $X'X$ and $Y'Y$ divide the plane into four quadrants, numbered as in **Art. 3.**

Any point P_1 in the plane is located by the segments NP_1 and MP_1 drawn parallel to $X'X$ and $Y'Y$ respectively, for the values of these segments tell how far and in what direction P_1 is from the two lines $X'X$ and $Y'Y$.

It is evident that for any point in the plane there is *one pair of values and only one;* and, conversely, for every pair of values there is *one point and only one.*

The value of the segment NP_1 or OM is called the **abscissa** of the point P_1, and is usually represented by x. The value of the segment MP_1 or ON is called the **ordinate** of the point P_1, and is usually represented by y. Taken together the abscissa x and the ordinate y are called the **coordinates** of the point P_1. They are written, for brevity, within parentheses and separated by a comma, the abscissa always being first, as (x, y).

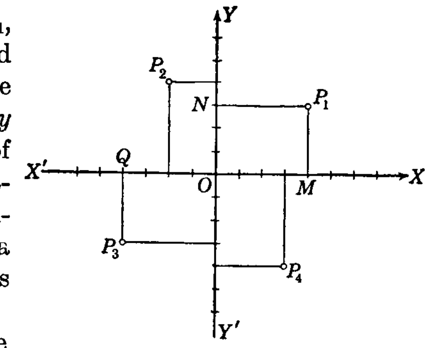

FIG. 13.

The line $X'X$ is called the **axis of abscissas** or the **x-axis.** The line $Y'Y$ is called the **axis of ordinates** or the **y-axis.** Together, these lines are called the **coordinate axes.**

It is evident that, in the first quadrant, both coordinates are positive; in the second quadrant, the abscissa is negative and the ordinate is positive; in the third quadrant, both coordinates are negative; and, in the fourth quadrant, the abscissa is positive and the ordinate is negative. This is shown in the following table:

Quadrant	I	II	III	IV
Abscissa	+	−	−	+
Ordinate	+	+	−	−

Thus, in Fig. 13, P_1, P_2, P_3, and P_4 are, respectively, the points $(4, 3)$, $(-2, 4)$, $(-4, -3)$, and $(3, -4)$. The points M, O, N, and Q are, respectively, $(4, 0)$, $(0, 0)$, $(0, 3)$, and $(-4, 0)$.

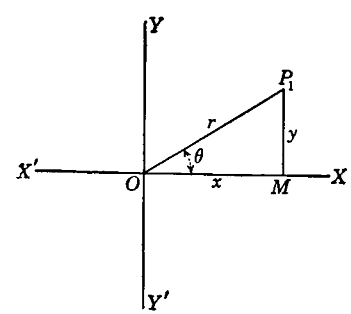

FIG. 14.

13. Polar coordinates.—The point P_1 (Fig. 14) can also be located if the angle θ and the length of the line OP_1 are known. The line OP_1 is called the **radius vector** and is usually represented by r. Since r denotes the distance of the point P_1 from O, it is always considered positive.