Tensor Analysis & Geometry

Spherical Coordinates

$x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, $x^3 = r\cos\theta$, $x^4 = ct$

3-dimensional line element: $ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$

Christoffel Symbols

Christoffel Symbol of the first kind: $\Gamma_{rmn} = [mn, r] = \frac{1}{2}\left(\frac{\partial g_{rm}}{\partial x^n} + \frac{\partial g_{rn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r}\right).$

Christoffel Symbol of the second kind: $\Gamma^z_{mn} = \left\{ {z \atop mn} \right\} = g^{rz}\Gamma_{mnr}$.

Derivation of the Riemann Curvature Tensor

In general, a second order differentiation on a covariant vector is independent of the order in which it is carried out, i.e.:

$$\frac{\partial^2 V_i}{\partial x^j \partial x^k} = \frac{\partial^2 V_i}{\partial x^k \partial x^j} .$$

However, the presence of Christoffel symbols can have an effect on this statement. We investigate this by first finding the general second derivatives for both permutations of the differentiating parameters:

$$\left(V_{i;j}\right)_{;k} = V_{i,jk} - \Gamma^r_{ik} V_{r,j} - \Gamma^r_{jk} V_{i,r} .$$

But $V_{i;j} = V_{i,j} - \Gamma^s_{ij} V_s$,

$$\therefore \left(V_{i:j}\right)_{:k} = V_{i,jk} - \frac{\partial \Gamma^s_{ij}}{\partial x^k} V_s - \Gamma^s_{ij} V_{s,k} - \Gamma^r_{ik}\left[V_{r,j} - \Gamma^s_{rj} V_s\right] - \Gamma^r_{jk}\left[V_{i,r} - \Gamma^s_{ir} V_s\right]$$

$$\Leftrightarrow \left(V_{i:j}\right)_{:k} = V_{i,jk} - \frac{\partial \Gamma^s_{ij}}{\partial x^k} V_s - \Gamma^s_{ij} V_{s,k} - \Gamma^r_{ik} V_{r,j} + \Gamma^r_{ik}\Gamma^s_{rj} V_s - \Gamma^r_{jk} V_{i,r} + \Gamma^r_{jk}\Gamma^s_{ir} V_s$$

Now we interchange *j* and *k* (which is the other possible way of determining this second derivative):

$$\left(V_{i:k}\right)_{:j} = V_{i,kj} - \frac{\partial \Gamma^s_{ik}}{\partial x^j} V_s - \Gamma^s_{ik} V_{s,j} - \Gamma^r_{ij} V_{r,k} + \Gamma^r_{ij}\Gamma^s_{rk} V_s - \Gamma^r_{kj} V_{i,r} + \Gamma^r_{kj}\Gamma^s_{ir} V_s .$$

We now find the difference between these two. On the RHS, the first, third, fourth, sixth, and seventh terms cancel out, thus giving the result:

$$V_{i:jk} - V_{i:kj} = -\frac{\partial \Gamma^s_{ij}}{\partial x^k} V_s + \Gamma^r_{ik}\Gamma^s_{rj} V_s + \frac{\partial \Gamma^s_{ik}}{\partial x^j} V_s - \Gamma^r_{ij}\Gamma^s_{rk} V_s = \left[\frac{\partial \Gamma^s_{ik}}{\partial x^j} - \frac{\partial \Gamma^s_{ij}}{\partial x^k} + \Gamma^r_{ik}\Gamma^s_{rj} - \Gamma^r_{ij}\Gamma^s_{rk}\right] V_s .$$

We define the Riemann (or Riemann-Christoffel) Curvature tensor by:

$$R^s_{ijk} = \frac{\partial \Gamma^s_{ik}}{\partial x^j} - \frac{\partial \Gamma^s_{ij}}{\partial x^k} + \Gamma^r_{ik}\Gamma^s_{rj} - \Gamma^r_{ij}\Gamma^s_{rk} .$$

The difference between the covariant derivatives can thus be written as $V_{i:jk} - V_{i:kj} = R^s_{ijk} V_s$. The Riemann tensor used in this equation is called the Riemann curvature tensor of the second kind. The curvature tensor of the first kind is defined as:

$$R_{ijkl} = g_{ir} R^r_{jkl} .$$

Symmetry Properties:

First skew symmetry $R_{ijkl} = -R_{jikl}$

Second skew symmetry $R_{ijkl} = -R_{ijlk}$

Block symmetry $\quad R_{ijkl} = R_{klij}$

Bianchi's identity $\quad R_{ijkl} + R_{iklj} + R_{iljk} = 0$

The Ricci Tensor

The Ricci tensor of the first kind is simply a contraction of the Riemann tensor:

$$R_{ij} = R^k_{ijk} \, .$$

The last index can be raised to yield the Ricci tensor of the second kind:

$$R_i^j = g^{ik} R_{ki} \, .$$

If this tensor is finally contracted by letting I = j, we get the Ricci curvature scalar. If it is zero, the space is flat. From the first of the two equations above the Ricci tensor of the first kind can be calculated directly by:

$$R_{ij} = R^k_{ijk} = \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^r_{ik}\Gamma^k_{rj} - \Gamma^r_{ij}\Gamma^k_{rk} \, .$$

Transformation Of A Geodesic From Parameter u To v, Where v = f(u)

Given a particular geodesic in terms of a parameter *u*, in this section the geodesic equation will be transformed so that it is in terms of a new parameter *v*.

Start with $\frac{D}{du}\left(\frac{dx^a}{du}\right) = \frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$.

Substitute $\frac{dx^a}{du} = \frac{dx^a}{dv}\frac{dv}{du}$,

then $\frac{D}{du}\left(\frac{\partial x^a}{\partial v}\frac{dv}{du}\right) = \frac{\partial x^a}{\partial u \partial v}\frac{dv}{du} + \Gamma^a_{bc}\frac{\partial x^b}{\partial v}\frac{dv}{du}\frac{dx^c}{du} + \frac{\partial x^a}{\partial v}\frac{d^2 v}{du^2} = 0$

$$\Leftrightarrow \frac{\partial x^a}{\partial u \partial v}\frac{dv}{du} + \Gamma^a_{bc}\frac{\partial x^b}{\partial v}\frac{dx^c}{du}\frac{dv}{du} = -\frac{\partial x^a}{\partial v}\frac{d^2 v}{du^2}$$

$$\Leftrightarrow \frac{\partial x^a}{\partial u \partial v}\frac{dv}{du}\left(\frac{du}{dv}\frac{du}{dv}\right) + \Gamma^a_{bc}\frac{\partial x^b}{\partial v}\frac{dx^c}{du}\frac{dv}{du}\left(\frac{du}{dv}\frac{du}{dv}\right) = -\frac{\partial x^a}{\partial v}\frac{d^2 v}{du^2}\left(\frac{du}{dv}\frac{du}{dv}\right)$$

$$\Leftrightarrow \frac{\partial x^a}{\partial v^2} + \Gamma^a_{bc}\frac{\partial x^b}{\partial v}\frac{dx^c}{dv} = -\frac{\partial x^a}{\partial v}\frac{d^2 v}{du^2}\left(\frac{du}{dv}\right)^2 .$$

Since the LHS is now in terms of *v*, partial differentiation can be replaced by normal differentiation:

$$\Leftrightarrow \frac{dx^a}{dv^2} + \Gamma^a_{bc}\frac{dx^b}{dv}\frac{dx^c}{dv} = -\frac{dx^a}{dv}\frac{d^2 v}{du^2} \Big/ \left(\frac{dv}{du}\right)^2 .$$

By letting $\lambda = -\frac{d^2 v}{du^2} \Big/ \left(\frac{dv}{du}\right)^2$, we arrive at the final equation:

$$\frac{dx^a}{dv^2} + \Gamma^a_{bc}\frac{dx^b}{dv}\frac{dx^c}{dv} = \lambda \frac{dx^a}{dv} .$$

General Relativity

The Metric Tensor for Special Relativity

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Einstein's Law of Gravitation

Simply stated, Einstein's law of gravitation is:

$R_{\mu\nu} = 0$.

This condition holds when the local space is completely devoid of all forms of matter and energy.

Derivation of the Schwarzchild Solution

In this section the line-element solution to the field equations for a quasi-static gravitational field produced by a spherical body will be derived.

We start by setting up a general line element employing spherical coordinates:

$$c^2 d\tau^2 = Ac^2 dt^2 - Bdr^2 - Wr^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Before we continue, a few assumptions need to be made:

- The space is asymptotically flat. This means that $A = B \to 1$ as $r \to \infty$.
- The gravitational field only affects time and radial distance, so W = 1.

We can immediately define the metric tensor:

$$g_{00} = Ac^2,\ g_{11} = -B,\ g_{22} = -r^2 \text{ and } g_{33} = -r^2\sin^2\theta .$$

Since we are dealing with the empty space surrounding the body, the Ricci tensor needs to equal zero. With this in mind, the derivation begins. We first calculate the Christoffel symbols. Note that since a Christoffel symbol of the second kind is defined as

$$\Gamma^z_{mn} = \begin{Bmatrix} z \\ mn \end{Bmatrix} = g^{rz}\Gamma_{mnr}$$

we need only calculate them for values when $r = z$.

$$\Gamma^0_{10} = g^{00}\Gamma_{100} = \tfrac{1}{2}A^{-1}c^{-2}\left[g_{00,1} + g_{10,0} - g_{10,0}\right] = \tfrac{1}{2}A^{-1}A'$$

$$\Gamma^1_{00} = g^{11}\Gamma_{001} = -\tfrac{1}{2}B^{-1}\left[g_{10,0} + g_{01,0} - g_{00,1}\right] = \tfrac{1}{2}B^{-1}g_{00,1} = \tfrac{1}{2}B^{-1}A'c^2$$

$$\Gamma^1_{11} = g^{11}\Gamma_{111} = -\tfrac{1}{2}B^{-1}\left[g_{11,1} + g_{11,1} - g_{11,1}\right] = -\tfrac{1}{2}B^{-1}g_{11,1} = \tfrac{1}{2}B^{-1}B'$$

$$\Gamma^1_{22} = g^{11}\Gamma_{221} = -\tfrac{1}{2}B^{-1}\left[g_{12,2} + g_{21,2} - g_{22,1}\right] = -\tfrac{1}{2}B^{-1}g_{22,1} = -\tfrac{1}{2}B^{-1}\cdot(2r) = -rB^{-1}$$

$$\Gamma^1_{33} = g^{11}\Gamma_{331} = -\tfrac{1}{2}B^{-1}\left[g_{13,3} + g_{31,3} - g_{33,1}\right] = -\tfrac{1}{2}B^{-1}\left(-g_{33,1}\right) = -\tfrac{1}{2}B^{-1}\left(2r\sin^2\theta\right) = -rB^{-1}\sin^2\theta$$

$$\Gamma^2_{21} = g^{22}\Gamma_{212} = -\tfrac{1}{2}r^{-2}\left[g_{21,2} + g_{22,1} - g_{21,2}\right] = -\tfrac{1}{2}r^{-2}\cdot(-2r) = r^{-1}$$

$$\Gamma^2_{33} = g^{22}\Gamma_{332} = -\tfrac{1}{2}r^{-2}\left[g_{23,3} + g_{32,3} - g_{33,2}\right] = -\tfrac{1}{2}r^{-2}\cdot 2r^2\cos\theta\sin\theta = -\cos\theta\sin\theta$$

$$\Gamma^3_{13} = g^{33}\Gamma_{133} = -\tfrac{1}{2}r^{-2}\sin^{-2}\theta\left[g_{33,1} + g_{13,3} - g_{13,3}\right] = -\tfrac{1}{2}r^{-2}\sin^{-2}\theta\cdot\left(-2r\sin^2\theta\right) = r^{-1}$$

$$\Gamma^3_{23} = g^{33}\Gamma_{233} = -\tfrac{1}{2}r^{-2}\sin^{-2}\theta\left[g_{33,2} + g_{23,3} - g_{23,3}\right] = -\tfrac{1}{2}r^{-2}\sin^{-2}\theta\cdot\left(-2r^2\cos\theta\sin\theta\right) = \cot\theta$$

We now solve the field equations:

$$R_{00} = \left[\Gamma^0_{00,0} - \Gamma^0_{00,0} + \left(\Gamma^0_{00}\Gamma^0_{00} - \Gamma^0_{00}\Gamma^0_{00}\right) + \left(\Gamma^1_{00}\Gamma^0_{10} - \Gamma^1_{00}\Gamma^0_{10}\right) + \left(\Gamma^2_{00}\Gamma^0_{20} - \Gamma^2_{00}\Gamma^0_{20}\right) + \left(\Gamma^3_{00}\Gamma^0_{30} - \Gamma^3_{00}\Gamma^0_{30}\right)\right]$$

$$+ \left[\Gamma^1_{01,0} \underline{\underline{- \Gamma^1_{00,1}}} + \left(\underline{\underline{\Gamma^0_{01}\Gamma^1_{00}}} - \Gamma^0_{00}\Gamma^1_{01}\right) + \left(\Gamma^1_{01}\Gamma^1_{10} \underline{\underline{- \Gamma^1_{00}\Gamma^1_{11}}}\right) + \left(\Gamma^2_{01}\Gamma^1_{20} - \Gamma^2_{00}\Gamma^1_{21}\right) + \left(\Gamma^3_{01}\Gamma^1_{30} - \Gamma^3_{00}\Gamma^1_{31}\right)\right]$$

$$+ \left[\Gamma^2_{02,0} - \Gamma^2_{00,2} + \left(\Gamma^0_{02}\Gamma^2_{00} - \Gamma^0_{00}\Gamma^2_{02}\right) + \left(\Gamma^1_{02}\Gamma^2_{10} \underline{\underline{- \Gamma^1_{00}\Gamma^2_{12}}}\right) + \left(\Gamma^2_{02}\Gamma^2_{20} - \Gamma^2_{00}\Gamma^2_{22}\right) + \left(\Gamma^3_{02}\Gamma^2_{30} - \Gamma^3_{00}\Gamma^2_{32}\right)\right]$$

$$+ \left[\Gamma^3_{03,0} - \Gamma^3_{00,3} + \left(\Gamma^0_{03}\Gamma^3_{00} - \Gamma^0_{00}\Gamma^3_{03}\right) + \left(\Gamma^1_{03}\Gamma^3_{10} \underline{\underline{- \Gamma^1_{00}\Gamma^3_{13}}}\right) + \left(\Gamma^2_{03}\Gamma^3_{20} - \Gamma^2_{00}\Gamma^3_{23}\right) + \left(\Gamma^3_{03}\Gamma^3_{30} - \Gamma^3_{00}\Gamma^3_{33}\right)\right]$$

$$\begin{aligned}
R_{11} = \Big[&\Gamma^0_{10,1} - \Gamma^0_{11,0} + \left(\Gamma^0_{10}\Gamma^0_{01} - \Gamma^0_{11}\Gamma^0_{00}\right) + \left(\Gamma^1_{10}\Gamma^0_{11} - \Gamma^1_{11}\Gamma^0_{10}\right) + \left(\Gamma^2_{10}\Gamma^0_{21} - \Gamma^2_{11}\Gamma^0_{20}\right) + \left(\Gamma^3_{10}\Gamma^0_{31} - \Gamma^3_{11}\Gamma^0_{30}\right)\Big] \\
+ \Big[&\Gamma^1_{11,1} - \Gamma^1_{11,1} + \left(\Gamma^0_{11}\Gamma^1_{01} - \Gamma^0_{11}\Gamma^1_{01}\right) + \left(\Gamma^1_{11}\Gamma^1_{11} - \Gamma^1_{11}\Gamma^1_{11}\right) + \left(\Gamma^2_{11}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{21}\right) + \left(\Gamma^3_{11}\Gamma^1_{31} - \Gamma^3_{11}\Gamma^1_{31}\right)\Big] \\
+ \Big[&\Gamma^2_{12,1} - \Gamma^2_{11,2} + \left(\Gamma^0_{12}\Gamma^2_{01} - \Gamma^0_{11}\Gamma^2_{02}\right) + \left(\Gamma^1_{12}\Gamma^2_{11} - \Gamma^1_{11}\Gamma^2_{12}\right) + \left(\Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{11}\Gamma^2_{22}\right) + \left(\Gamma^3_{12}\Gamma^2_{31} - \Gamma^3_{11}\Gamma^2_{32}\right)\Big] \\
+ \Big[&\Gamma^3_{13,1} - \Gamma^3_{11,3} + \left(\Gamma^0_{13}\Gamma^3_{01} - \Gamma^0_{11}\Gamma^3_{03}\right) + \left(\Gamma^1_{13}\Gamma^3_{11} - \Gamma^1_{11}\Gamma^3_{13}\right) + \left(\Gamma^2_{13}\Gamma^3_{21} - \Gamma^2_{11}\Gamma^3_{23}\right) + \left(\Gamma^3_{13}\Gamma^3_{31} - \Gamma^3_{11}\Gamma^3_{33}\right)\Big]
\end{aligned}$$

$$\begin{aligned}
R_{22} = \Big[&\Gamma^0_{20,2} - \Gamma^0_{22,0} + \left(\Gamma^0_{20}\Gamma^0_{02} - \Gamma^0_{22}\Gamma^0_{00}\right) + \left(\Gamma^1_{20}\Gamma^0_{12} - \Gamma^1_{22}\Gamma^0_{10}\right) + \left(\Gamma^2_{20}\Gamma^0_{22} - \Gamma^2_{22}\Gamma^0_{20}\right) + \left(\Gamma^3_{20}\Gamma^0_{32} - \Gamma^3_{22}\Gamma^0_{30}\right)\Big] \\
+ \Big[&\Gamma^1_{21,2} - \Gamma^1_{22,1} + \left(\Gamma^0_{21}\Gamma^1_{02} - \Gamma^0_{22}\Gamma^1_{01}\right) + \left(\Gamma^1_{21}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^1_{11}\right) + \left(\Gamma^2_{21}\Gamma^1_{22} - \Gamma^2_{22}\Gamma^1_{21}\right) + \left(\Gamma^3_{21}\Gamma^1_{32} - \Gamma^3_{22}\Gamma^1_{31}\right)\Big] \\
+ \Big[&\Gamma^2_{22,2} - \Gamma^2_{22,2} + \left(\Gamma^0_{22}\Gamma^2_{02} - \Gamma^0_{22}\Gamma^2_{02}\right) + \left(\Gamma^1_{22}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{12}\right) + \left(\Gamma^2_{22}\Gamma^2_{22} - \Gamma^2_{22}\Gamma^2_{22}\right) + \left(\Gamma^3_{22}\Gamma^2_{32} - \Gamma^3_{22}\Gamma^2_{32}\right)\Big] \\
+ \Big[&\Gamma^3_{23,2} - \Gamma^3_{22,3} + \left(\Gamma^0_{23}\Gamma^3_{02} - \Gamma^0_{22}\Gamma^3_{03}\right) + \left(\Gamma^1_{23}\Gamma^3_{12} - \Gamma^1_{22}\Gamma^3_{13}\right) + \left(\Gamma^2_{23}\Gamma^3_{22} - \Gamma^2_{22}\Gamma^3_{23}\right) + \left(\Gamma^3_{23}\Gamma^3_{32} - \Gamma^3_{22}\Gamma^3_{33}\right)\Big]
\end{aligned}$$

$$\begin{aligned}
R_{33} = \Big[&\Gamma^0_{30,3} - \Gamma^0_{33,0} + \left(\Gamma^0_{30}\Gamma^0_{03} - \Gamma^0_{33}\Gamma^0_{00}\right) + \left(\Gamma^1_{30}\Gamma^0_{13} - \Gamma^1_{33}\Gamma^0_{10}\right) + \left(\Gamma^2_{30}\Gamma^0_{23} - \Gamma^2_{33}\Gamma^0_{20}\right) + \left(\Gamma^3_{30}\Gamma^0_{33} - \Gamma^3_{33}\Gamma^0_{30}\right)\Big] \\
+ \Big[&\Gamma^1_{31,3} - \Gamma^1_{33,1} + \left(\Gamma^0_{31}\Gamma^1_{03} - \Gamma^0_{33}\Gamma^1_{01}\right) + \left(\Gamma^1_{31}\Gamma^1_{13} - \Gamma^1_{33}\Gamma^1_{11}\right) + \left(\Gamma^2_{31}\Gamma^1_{23} - \Gamma^2_{33}\Gamma^1_{21}\right) + \left(\Gamma^3_{31}\Gamma^1_{33} - \Gamma^3_{33}\Gamma^1_{31}\right)\Big] \\
+ \Big[&\Gamma^2_{32,3} - \Gamma^2_{33,2} + \left(\Gamma^0_{32}\Gamma^2_{03} - \Gamma^0_{33}\Gamma^2_{02}\right) + \left(\Gamma^1_{32}\Gamma^2_{13} - \Gamma^1_{33}\Gamma^2_{12}\right) + \left(\Gamma^2_{32}\Gamma^2_{23} - \Gamma^2_{33}\Gamma^2_{22}\right) + \left(\Gamma^3_{32}\Gamma^2_{33} - \Gamma^3_{33}\Gamma^2_{32}\right)\Big] \\
+ \Big[&\Gamma^3_{33,3} - \Gamma^3_{33,3} + \left(\Gamma^0_{33}\Gamma^3_{03} - \Gamma^0_{33}\Gamma^3_{03}\right) + \left(\Gamma^1_{33}\Gamma^3_{13} - \Gamma^1_{33}\Gamma^3_{13}\right) + \left(\Gamma^2_{33}\Gamma^3_{23} - \Gamma^2_{33}\Gamma^3_{23}\right) + \left(\Gamma^3_{33}\Gamma^3_{33} - \Gamma^3_{33}\Gamma^3_{33}\right)\Big]
\end{aligned}$$

We are left with

$$\begin{aligned}
R_{00} &= -\Gamma^1_{00,1} + \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{11} - \Gamma^1_{00}\Gamma^2_{12} - \Gamma^1_{00}\Gamma^3_{13}, \\
R_{11} &= \Gamma^0_{10,1} + \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{12,1} - \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{21} + \Gamma^3_{13,1} - \Gamma^1_{11}\Gamma^3_{13} + \Gamma^3_{13}\Gamma^3_{31}, \\
R_{22} &= -\Gamma^1_{22}\Gamma^0_{10} - \Gamma^1_{22,1} - \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^3_{23,2} - \Gamma^1_{22}\Gamma^3_{13} + \Gamma^3_{23}\Gamma^3_{32}, \\
R_{33} &= -\Gamma^1_{33}\Gamma^0_{10} - \Gamma^1_{33,1} - \Gamma^1_{33}\Gamma^1_{11} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^2_{33,2} - \Gamma^1_{33}\Gamma^2_{12} + \Gamma^3_{32}\Gamma^2_{33}.
\end{aligned}$$

These equations must equal zero, thus after substitution, we have:

$$R_{00} = \left[\frac{B'A'}{4B} - \frac{A''}{2} + \frac{A'A'}{4A} - \frac{A'}{r}\right]\frac{1}{B}, \qquad (1)$$

$$R_{11} = -\frac{A'^2}{4A} + \frac{A''}{2} - \frac{A'B'}{4B} - \frac{B'A}{Br}, \qquad (2)$$

$$R_{22} = \frac{A'r}{2AB} - \frac{B'r}{2B^2} + \frac{1}{B} - 1, \qquad (3)$$

$$R_{33} = \left[\frac{A'r}{2AB} - \frac{B'r}{2B^2} + \frac{1}{B} - 1\right]\sin^2\theta = R_{22}\sin^2\theta\ . \qquad (4)$$

Equation 2 becomes:

$$0 = -\frac{A'^2}{4A} + \frac{A''}{2} - \frac{A'B'}{4B} - \frac{B'A}{Br} \Rightarrow \frac{B'A}{Br} = -\frac{A'^2}{4A} + \frac{A''}{2} - \frac{A'B'}{4B}.$$

This can be substituted into equation 1 to give:

$$0 = \left[-\frac{B'A}{Br} - \frac{A'}{r}\right]\frac{1}{B} \Rightarrow \frac{B'A}{Br} = -\frac{A'}{r} \Rightarrow \frac{B'}{B} = -\frac{A'}{A}.$$

This can also be written as:

$$\frac{dB}{Bdr} = -\frac{dA}{Adr} \Leftrightarrow \frac{dB}{B} = -\frac{dA}{A} \Rightarrow B = \frac{1}{A} \Rightarrow AB = 1,$$

upon solving the simple differential equation. We use this solution to simplify equation 3:

$$0 = \frac{A'r}{2} + \frac{A'r}{2A} + A - 1 \Leftrightarrow 1 = A'r + A = A'r + Ar' = (Ar)'$$

We now integrate:

$$\int dr = \int \frac{d}{dr}(Ar)dr \Rightarrow r + k = Ar,$$

where k is an integration constant. The equation can be rearranged to find A:

$$A = 1 + \frac{k}{r}.$$

Since $B = A^{-1}$,

$$B = \left(1 + \frac{k}{r}\right)^{-1}.$$

The value of k is the last thing to obtain. In the next section on the approximation of Newtonian gravitation, $g_{00} = 1 + h_{00}$. It can immediately be seen that the h_{00} is equivalent to k. In the Newtonian approximation, it is required that the Newtonian gravitational potential $V = \frac{1}{2}c^2 h_{00}$. Using the Newtonian potential $V = -GM/r$, this gives a value of $k = -2GM/rc^2$. By substituting A and B back into the original line-element equation at the start of this section, we have the Schwarzchild solution:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Utilising Geodesic Equation to Find GR Approximation of Newtonian Gravity

A particle travels through spacetime along a geodesic, given by the equation:

$$\frac{dx^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0.$$

τ is the time experienced relative to the particle. The equation simply states that relative to a free particle, it experiences no net acceleration (though other objects appear to accelerate if a gravitational field is present).

We wish to determine the motion of the particle relative to coordinate time, denoted by t. The equation would give the path of the particle in accordance to what other observers would see if they thought they were in a gravitational field.

With the above said, the following equation can be immediately written, transforming from proper time to coordinate time:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{dt}\frac{dx^\sigma}{dt} = \left(\frac{d^2t}{d\tau^2}\Big/\left(\frac{dt}{d\tau^2}\right)^2\right)\frac{dx^\mu}{dt}.$$

First, we expand and consider its spatial components:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + 2\Gamma^i_{0k}\frac{dx^0}{dt}\frac{dx^k}{dt} + \Gamma^i_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} = \left(\frac{d^2t}{d\tau^2}\Big/\left(\frac{dt}{d\tau^2}\right)^2\right)\frac{dx^i}{dt}.$$

One of the Christoffel symbols has a coefficient of two since $\Gamma^i_{j0}\frac{dx^j}{dt}\frac{dx^0}{dt} = \Gamma^i_{0k}\frac{dx^0}{dt}\frac{dx^k}{dt}$. We simplify the equation:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + 2\Gamma^i_{0k}c\frac{dx^k}{dt} + c^2\Gamma^i_{00} = \left(\frac{d^2t}{d\tau^2}\Big/\left(\frac{dt}{d\tau^2}\right)^2\right)\frac{dx^i}{dt}.$$

We assume that the gravitational field is quasi-static, i.e. that it doesn't change with respect to time. Therefore, any derivatives of the metric tensor with respect to time can be left out. Now we evaluate the connection coefficients:

$\Gamma^i_{jk} = \frac{1}{2} g^{i\rho}\Gamma_{\rho jk} = \frac{g^{i\rho}}{2}\left(\frac{\partial g_{\rho j}}{\partial x^k} + \frac{\partial g_{\rho k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\rho}\right)$. These derivatives are quite small, and can be neglected.[1]

$$2\Gamma^i_{0k} = 2\frac{g^{i\rho}}{2}\left(\frac{\partial g_{\rho 0}}{\partial x^k} + \frac{\partial g_{\rho k}}{\partial x^0} - \frac{\partial g_{0k}}{\partial x^\rho}\right) \approx \left(\eta^{i\rho} + h^{i\rho}\right)\left(\frac{\partial h_{\rho 0}}{\partial x^k} + \frac{\partial h_{\rho k}}{\partial x^0} - \frac{\partial h_{0k}}{\partial x^\rho}\right)$$

$\approx \left(\eta^{i\rho} + h^{i\rho}\right)\left(\frac{\partial h_{0\rho}}{\partial x^k} + \frac{\partial h_{\rho k}}{\partial x^0} - \frac{\partial h_{0k}}{\partial x^\rho}\right) \approx -\delta^{is}\left(\frac{\partial h_{0s}}{\partial x^k} - \frac{\partial h_{0k}}{\partial x^s}\right)$ on neglecting terms involving $\frac{\partial h_{\mu\nu}}{\partial x^0}$.

$$\Gamma^i_{00} = \frac{g^{i\kappa}}{2}\left(\frac{\partial g_{\kappa 0}}{\partial x^0} + \frac{\partial g_{0\kappa}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\kappa}\right) \approx \frac{\delta^{iz}}{2}\frac{\partial h_{00}}{\partial x^z}.$$

Now we need to evaluate the RHS of the equation. We start by first looking at the following line element:

$c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu \Leftrightarrow \left(\frac{d\tau}{dt}\right)^2 = \frac{1}{c^2}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = \frac{1}{c^2}\left(1 + h_{\mu\nu}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$. Neglecting terms involving the spatial components, which are small in comparison to the temporal components, we get:

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 + h_{00}\right) \Rightarrow \frac{d\tau}{dt} = \left(1 + h_{00}\right)^{1/2} \approx \left(1 + \tfrac{1}{2}h_{00}\right).$$

Now $\frac{d^2\tau}{dt^2} = \frac{dh_{00}}{dt} = c\frac{dh_{00}}{dx^0}$, so the RHS of our major equation becomes:

$$\frac{dx^i}{dt}\left[\frac{d^2 t}{d\tau^2}\Big/\left(\frac{dt}{d\tau^2}\right)^2\right] \approx \frac{c\frac{dh_{00}}{dx^0}}{1 + \frac{1}{2}h_{00}}\frac{dx^i}{dt} \approx c\frac{dh_{00}}{dx^0}\left(1 - \tfrac{1}{2}h_{00}\right)\frac{dx^i}{dt}.$$

This is negligible since it involves temporal derivatives of the gravitational field.

By plugging everything into our equation, we get:

$$\frac{d^2 x^i}{dt^2} - \delta^{is}\left(\frac{\partial h_{0s}}{\partial x^k} - \frac{\partial h_{0k}}{\partial x^s}\right)\frac{dx^k}{dt} + c^2\frac{\delta^{iz}}{2}\frac{\partial h_{00}}{\partial x^z} = 0.$$

Through multiplying by mass *m* and rearranging the equation, we get:

$$m\frac{d^2 x^i}{dt^2} = -mc^2\frac{\delta^{iz}}{2}\frac{\partial h_{00}}{\partial x^z} + m\delta^{is}\left(\frac{\partial h_{0s}}{\partial x^k} - \frac{\partial h_{0k}}{\partial x^s}\right)\frac{dx^k}{dt}.$$

The term on the left is the force that the particle appears to experience. The first term on the right is some kind of potential of the field, since its temporal component is involved. The last term, which involves perpendicular velocities, is indicative of some sort of Coriolis force. We are not interested in the Coriolis effects, so we shall assume that we are in a non-rotating frame, so we get:

$$m\frac{d^2 x^i}{dt^2} = -mc^2\frac{\delta^{iz}}{2}\frac{\partial h_{00}}{\partial x^z}.$$

If we denote a potential by $V = \frac{1}{2}c^2 h_{00}$, then the equation simply becomes $F = -m\nabla V$, or $\frac{d^2 x^i}{dt^2} = -\delta^{iz}\frac{\partial V}{\partial x^z}$. We want the metric tensor to be flat when there is no gravitational field present. The

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[1] The metric tensor $g^{\mu\nu} \approx \eta^{\mu\nu} + h^{\mu\nu}$, where $\eta^{\mu\nu}$ is the familiar metric tensor of Special Relativity, and $h^{\mu\nu}$ are small terms which include the action of any gravitational fields which may be present, and are small in comparison to the $\eta^{\mu\nu}$ in weak gravitational fields, as opposed to the awesome sucking power of a black hole!

equation for the potential leads us to the expression $g_{00} = \eta_{00} + h_{00} = 1 + 2V/c^2$. To finally get the actual expression for the potential in terms of mass, we need to use the line element from the Schwarzchild solution. The temporal component gives $V = -GM/r$. With this expression, we can easily obtain an approximation of the gravitational force:

$$F = -m\nabla V = -GMm/r^2 \ .$$

Field Equations in the Presence of Matter: The Poisson Approximation

Let us write the equation:

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} = \kappa T^{\mu\nu} ,$$

or, more compactly as:

$$G^{\mu\nu} = \kappa T^{\mu\nu} ,$$

where G is the Einstein tensor.

As a test for General Relativity, at velocities which are small in comparison to the speed of light, there must be an approximation to Poisson's equation: $\nabla^2 V = 4\pi G\rho$. To achieve this requires the weak field approximation by leaving out negligible terms in the Ricci tensor.[2]

$$G^{\mu\nu} = \kappa T^{\mu\nu}$$
$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu}$$
$$g_{\mu\nu} R^{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\mu\nu} R = \kappa T^{\mu\nu} g_{\mu\nu}$$
$$R - 2R = \kappa T^{\mu\nu} g_{\mu\nu}$$
$$\therefore R = -\kappa T^{\mu\nu} g_{\mu\nu}$$

We substitute this back into our original equation:

$$R^{\mu\nu} = \kappa T^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} \kappa T^{\mu\nu} g_{\mu\nu}$$

We are approximating that the material energy tensor has a negligible value in for all μ, ν except when $\mu = \nu = 0$, so we get:

$$R^{00} = \kappa T^{00} - \tfrac{1}{2} g^{00} \kappa T^{00} g_{00} = \tfrac{1}{2} \kappa T^{00} .$$

We now make the Ricci tensor covariant:

$$g_{00} g_{00} R^{00} = \tfrac{1}{2} \kappa T^{00} g_{00} g_{00} \Leftrightarrow R_{00} = \tfrac{1}{2} \kappa T_{00}$$

Weak field approximation.

$$R_{\mu\nu} \approx \tfrac{1}{2}\left(g_{\alpha\alpha,\mu\nu} - g_{\nu\alpha,\mu\alpha} - g_{\mu\alpha,\nu\alpha} + g_{\mu\nu,\alpha\alpha}\right) \approx \tfrac{1}{2} g_{\mu\nu,\alpha\alpha}$$
$$\therefore R_{00} \approx \tfrac{1}{2} g_{00,\alpha\alpha} \approx \tfrac{1}{2} \nabla^2 g_{00} = \frac{1}{c^2} \nabla^2 V$$
$$\therefore \frac{1}{c^2} \nabla^2 V = \tfrac{1}{2} \kappa T_{00} = \tfrac{1}{2} \kappa \rho v_0 v_0$$

The particles are traveling at the travelling at the speed of light through time, so we get:

$$\nabla^2 V = \tfrac{1}{2} \kappa \rho c^4$$

This must equal Poisson's equation, stated earlier

This gives a value $\kappa = \dfrac{8\pi G}{c^4}$.

[2] Note that this particular calculation would be shorter if we took the Einstein tensor and the material energy tensor to be covariant as opposed to contravariant, but due to the actual form of the material energy tensor, I prefer it to be contravariant.