2. Laplace equation in 2D — *harmonic functions from analytic functions*

In two dimensions the Laplace equation takes the form

$$\boxed{u_{xx} + u_{yy} = 0,} \tag{1}$$

and any solution in a region Ω of the x-y plane is a *harmonic function* in Ω. All the general properties outlined in our discussion of the Laplace equation ($\rightarrow$ *ref*) still hold, including the maximum principle, the mean value principle, and the equivalence with minimisation of a Dirichlet integral. One thing that changes is that the fundamental solution, instead of being a power of the radius $r = (x^2 + y^2)^{1/2}$, now takes the form $\log r$. That is to say, $u(x, y) = \log r$ is a harmonic function in the x-y plane except for the singularity at the origin.

However, something very special happens in the two-dimensional case. Suppose we view the x-y plane as the complex plane $\mathbb{C}$ by introducing the complex variable $z = x + \mathrm{i}y$. Then a function $u(x, y)$ is a function $u(z)$, and here is the remarkable result: *if $u(z)$ is harmonic, it is the real part of an analytic function $f(z)$.* (This statement will be qualified below (2).) If we write $f(z) = u(z) + \mathrm{i}v(z)$, then $v(z)$ is harmonic too, and is known as the *conjugate harmonic function* of u. Thus potential theory in the plane becomes part of the magnificent subject of complex analysis, and to find examples of harmonic functions in the plane, we need look no further than the real (or imaginary) parts of analytic functions such as e^z, $\sin z$, or $z + 3z^4$. To analyse such functions, powerful tools such as power series, contour integrals, and conformal mapping can be brought to bear.

For example, Figure 1 shows the solution to Laplace's equation on a strip-like domain with boundary conditions $u = 0$ on the left, $u = 1$ on the right, and $\partial u/\partial n = 0$ (Neumann condition) along the top and bottom. The curves plotted represent lines of constant values $u = 1/8, 2/8, \ldots, 7/8$. These results were obtained by mapping the strip to a rectangle numerically by a Schwarz–Christoffel transformation, then plotting pre-images of straight lines in the rectangle. The justification for this procedure is as follows. Suppose $f(z)$ is a conformal map from the given domain to a rectangle whose left and right sides have real parts 0 and 1, respectively. Then $f(z)$ is an analytic function, so $\Re f(z)$ is harmonic, and it is evident that it satisfies the prescribed boundary conditions. Figure 2 shows the solution to a similar problem posed in an annulus with spikes, with $u = 0$ on the inner boundary and $u = 1$ on the outer boundary.

The relationship between harmonic and analytic functions can be derived as follows. If $f(z) = u(z) + \mathrm{i}v(z)$ is analytic, then u and v satisfy the *Cauchy–Riemann equations* ($\rightarrow$ *ref*),

$$u_x = v_y, \qquad u_y = -v_x. \tag{2}$$

Conversely, given a harmonic function $u(z)$, integration of the Cauchy–Riemann equations determines a conjugate harmonic function $v(z)$. We see immediately that $v(z)$ is not uniquely defined, as it can be modified by an arbitrary additive constant. A more profound complication is that if the domain Ω is not simply connected, then v may have to be *multivalued*. For example, the function $u(z) = \log r = \log|z|$ is well defined and harmonic throughout the punctured complex plane $\mathbb{C}\backslash\{0\}$. Its conjugate harmonic function $\arg z$, however, is multivalued, increasing by 2π every time one traverses a closed curve around the origin. Complex analysts are well familiar with such behaviour, and indeed, it is the starting point of the field of *Riemann surfaces*.

Perhaps the most basic boundary value problem associated with the Laplace equation in 2D is the *Dirichlet Problem on the unit disk:* find a harmonic function u in $\{z : |z| < 1\}$ with prescribed boundary data $u(z) = f(z)$ on the unit circle $\{z : |z| = 1\}$. According to the mean value principle for harmonic functions, the value of u for $z = 0$ is the mean of $f(z)$ over the unit circle. For general points $z = z_0$ in the unit disk, one can apply a conformal map $w = w(z)$ of the unit disk to itself (a *Möbius transformation*) that takes z_0 to 0; $u(z_0)$ is then the mean of the transformed boundary data $f(w(z))$. This is one way to derive the explicit result known as *Poisson's formula*,

$$u(z_0) \;=\; \frac{1}{2\pi}\int_{|z|=1} \frac{1 - |z_0|^2}{|z - z_0|^2}\, f(z)\, |\mathrm{d}z|. \tag{3}$$

Note that this formula expresses $u(z_0)$ as a weighted mean of the values $f(z)$ on the unit circle.

Poisson's formula also solves the Dirichlet problem in other simply connected regions Ω of the plane, at least in principle. All one must do is reduce Ω to the unit disk by a conformal map, which is always possible according to the Riemann mapping theorem, although this is not always an efficient solution method in practice.

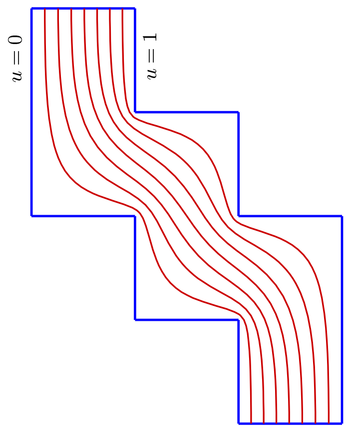

Fig. 1: Solution of the Laplace equation in a region conformally equivalent to a rectangle

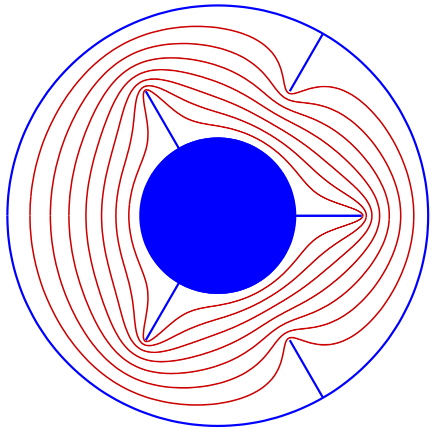

Fig. 2: Solution in a region conformally equivalent to an annulus

References

L. Ahlfors, *Complex analysis,* 3rd ed., McGraw-Hill, 1979.

R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces,* Interscience, 1950.

T. A. Driscoll and L. N. Trefethen, *Schwarz–Christoffel mapping,* Cambridge University Press, to appear.

P. Henrici, *Applied and computational complex analysis,* v. 1–3, Wiley, 1974, 1977, 1986.

T. Needham, *Visual complex analysis,* Clarendon Press, 1997.

G. Springer, *Introduction to Riemann surfaces,* Chelsea, 1981.

M. Tsuji, *Potential theory in modern function theory,* Dover, 1959.